Abstract

A quantum system consisting of two subsystems is separable if its density matrix can be written as $\rho = \sum_A w_A \rho'_A \otimes \rho''_A$, where $\rho'_A$ and $\rho''_A$ are density matrices for the two subsystems. In this Letter, it is shown that a necessary condition for separability is that a matrix, obtained by partial transposition of $\rho$, has only non-negative eigenvalues. This criterion is stronger than the Bell inequality.

PACS: 03.65.Bz

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A striking quantum phenomenon is the *inseparability* of composite quantum systems: if the latter are suitably prepared, the results of tests performed on their subsystems cannot be assumed to depend only on local properties of the subsystems, even if these are far apart from each other [1]. In particular, if the state of a composite system is *pure*, quantum inseparability occurs whenever that pure state is not factorable into a direct product of pure states of the subsystems [2, 3].

For *mixed* quantum states, the situation is more complicated. A sufficient criterion for separability of a composite system is that its density matrix be of the form

$$\rho = \sum_A w_A \rho'_A \otimes \rho''_A,$$

where the positive weights $w_A$ satisfy $\sum w_A = 1$, and where $\rho'_A$ and $\rho''_A$ are density matrices for the two subsystems. The statistical properties of such a composite system are just like those of a classical mixture of uncorrelated subsystems (for example, the Bell inequality is always satisfied). In this Letter, I shall derive a simple algebraic test, which is a necessary condition for the existence of the decomposition (1). It is not, however, a sufficient condition (just as the Bell inequality is a necessary condition, but not a sufficient one, for the existence of a local hidden variable model reproducing the statistical properties of correlated quantum systems).

The derivation of this separability condition for density matrices requires writing them, as well as various vectors, explicitly with all their indices (the Dirac bra-ket notation is not adequate for this problem). For example, a state vector $\psi$ will be represented by its components $\psi_{m\mu}$. Latin indices (running from 1 to $M$) refer to the first subsystem, and Greek indices (from 1 to $N$), to the second one (the two subsystems may have different dimensions). A normalized state vector satisfies

$$\sum_{m\mu} |\psi_{m\mu}|^2 = 1.$$  

A summation symbol $\sum$ without indices will henceforth mean a sum over any indices that appear more than once in the expression following that symbol. With these notations, Eq. (1) becomes

$$\rho_{m\mu,n\nu} = \sum_{m\mu} w_A \rho'_{Amn} \rho''_{A\mu\nu}.$$
The problem thus is: given all the matrix elements $\rho_{m\mu,n\nu}$, is it possible to find positive weights $w_A$ and non-negative matrices $\rho'_{Amn}$ and $\rho''_{A\mu\nu}$ such that Eq. (3) holds?

Note that, for any $\psi$,

$$\sum \rho_{m\mu,n\nu} \psi^*_{m\mu} \psi_{n\nu} \geq 0,$$

(4)

because $\rho$ is a non-negative matrix. It will now be shown that, if (3) holds, we also have

$$\sum \rho'_{n\mu,m\nu} \psi^*_{n\mu} \psi_{m\nu} \geq 0.$$

(5)

This follows from the fact that any density matrix can be written as [4]

$$\rho_{nm} = \sum u^*_{bn} u_{bm},$$

(6)

and likewise

$$\rho_{\mu\nu} = \sum v^*_\beta \nu v_{\beta\nu},$$

(7)

where the indices $b$ and $\beta$ do not refer to Hilbert space dimensions, but are enumerators, like $A$ in Eq. (1). The vectors $u_{bm}$ and $v_{\beta\nu}$ are not normalized. The left hand side of Eq. (5) thus becomes

$$\sum w_A u^*_{Abn} u_{Abm} v^*_{A\beta\mu} v_{A\beta\nu} \psi^*_{m\mu} \psi_{n\nu} = \sum_{A\beta\mu\nu} w_A \left| \sum_{n\nu} u^*_{Abn} v_{A\beta\nu} \psi_{n\nu} \right|^2 \geq 0.$$

(8)

It is convenient to define a new matrix,

$$\sigma_{m\mu,n\nu} \equiv \rho_{n\mu,m\nu}.$$  

(9)

(The Latin indices of $\rho$ have been transposed, but not the Greek ones.) This matrix is Hermitian, and all its eigenvalues are real. It follows from Eqs. (3) and (5) that none of the eigenvalues of $\sigma$ is negative. This is a necessary condition for Eq. (1) to hold.

As an example, consider a pair of spin-$\frac{1}{2}$ particles in a Werner state [5] (an impure singlet), consisting of a singlet fraction $x$ and a random fraction $(1-x)$. (Note that the “random fraction” $(1-x)$ also includes singlets, mixed in equal proportions with the three triplet components.) We have

$$\rho_{m\mu,n\nu} = x S_{m\mu,n\nu} + (1-x) \delta_{mn} \delta_{\mu\nu}/4,$$

(10)
where the density matrix for a pure singlet is given by

\[ S_{01,01} = S_{10,10} = -S_{01,10} = -S_{10,01} = \frac{1}{2}, \tag{11} \]

and all the other components of \( S \) vanish. (The indices 0 and 1 refer to any two orthogonal states, such as “up” and “down.”) A straightforward calculation shows that \( \sigma \) has three eigenvalues equal to \((1 + x)/4\), and the fourth eigenvalue is \((1 - 3x)/4\). This lowest eigenvalue is positive if \( x < \frac{1}{3} \), and the separability criterion is then fulfilled. It is indeed known that if \( x < \frac{1}{3} \) it is possible to write \( \rho \) as a mixture of unentangled product states [6]. In this particular case, it happens that the necessary condition for separability (\( \sigma \) has no negative eigenvalue) is also a sufficient one.

It is interesting to compare this result with the Bell inequality, in the form given by Clauser, Horne, Shimony, and Holt [7]. That inequality holds for \( x < 1/\sqrt{2} \simeq 0.7 \). It is therefore considerably weaker than the separability test given above. However, it is also possible to perform collective tests, involving several Werner pairs, instead of testing them one by one. In that case, the critical value of \( x \) decreases, as more pairs are tested simultaneously [8], and the value \( x = \frac{1}{3} \) may be approached, asymptotically, for an infinite number of pairs.

On the other hand, with the new criterion given here, there is no need of collective tests: if \( \sigma \) has no negative eigenvalue for a composite system, there will still be no negative eigenvalue if several systems are combined together. This is because the combined \( \rho \) is simply given by \((\rho \otimes \rho \otimes \ldots)\), and likewise the combined \( \sigma \) is given by \((\sigma \otimes \sigma \otimes \ldots)\). Therefore, if the set of eigenvalues of \( \sigma \) (for a single composite system) is \( \{X_A\} \) and none of these is negative, the eigenvalues of \((\sigma \otimes \sigma \otimes \ldots)\) are all the products \( X_A X_B \ldots \), and again none is negative.

This result strongly suggests that the necessary condition derived above, \( X_A \geq 0 \), also is a sufficient condition. However, I have not been able to formally prove that result, nor to find any counterexample. This issue remains an open problem. Another open problem is the derivation of nonlocal effects directly from the existence of a negative eigenvalue \( X_A \), without the tediousness of collective tests of the Bell inequality.

This work was supported by the Gerard Swope Fund and the Fund for Encouragement of Research.


