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OF THE CAR ALGEBRA

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AMS Classification (1991 revision) : 46L55, 58F06, 32H02, 14E25, 81R30.

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Determinants, Pfaffians, and Quasi-free Representations of the CAR Algebra

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Abstract

In this paper we apply the theory of quasi-free states of CAR algebras and Bogolubov automorphisms to give an alternative C^* -algebraic construction of the Determinant and the Pfaffian line bundles discussed by Pressley and Segal and by Borthwick. The basic property of the Pfaffian of being the holomorphic square root of the determinant line bundle (after restriction from the Hilbert space Grassmannian to the Siegel manifold, or isotropic Grassmannian, consisting of all complex structures on an associated Hilbert space) is derived from a Fock -anti-Fock correspondence and an application of the Powers-Størmer purification procedure. A Borel-Weil type description of the Spin Representation is obtained, via a Shale-Stinespring implementation of Bogolubov automorphisms.

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1. Introduction

In this paper we give an alternative construction of the Determinant and the Pfaffian line bundles over the Hilbert space Grassmannian, discussed by Pressley and Segal ([23]) and by Borthwick ([3]), in a C^* -algebraic fashion.

The Pfaffian line bundle is the holomorphic square root of the Determinant line bundle, after restriction of the latter, defined on the Grassmannian of the complexification of a Hilbert space (complex, but regarded as a real one), to the Siegel manifold (isotropic Grassmannian) consisting of all complex structures associated to the Hilbert space. Its dual carries the so-called Spin Representation ([23],[26],[3],[27],[8],[5],[14],[32]) and admits a Borel-Weil-Bott description ([4]) (a "coherent state" representation, in physicists' language ([18],[20])). More precisely, the Siegel manifold is homogeneous with respect to the action of the (restricted) rotation group, and the Spin Representation is a projective unitary representation thereof acting on the holomorphic sections of the dual of the Pfaffian bundle. In physical terms, the passage from Det to Pf amounts to considering a Majorana theory (a particle is identified with its antiparticle) instead of a Dirac one (where the two are regarded as distinct), cf. also [32]. We shall employ the theory of quasi-free states of CAR algebras ([21],[22],[12]), see §2,3,4, which has already proved to be very useful in providing the explicit Plücker type equations governing the projective embedding of the Hilbert space Grassmannian ([30]).

The layout of the paper is the following: the basic property of the Pfaffian will be derived from a Fock - anti-Fock correspondence (Proposition 2.1), which establishes a canonical C^* -algebra isomorphism between $A(K)$ and $A(K')$, with K' the dual of the one-particle space K and (a special instance of) the Powers-Størmer purification argument ([22]), and has a vivid geometric counterpart in terms of a (generalized) Segre embedding of a product of two "small" grassmannians into a "big" one (Theorems 6.4 and 7.1). The construction of the Determinant line bundle outlined in §5 is equivalent to the Pressley-Segal and Borthwick ones and it is based on ideas in [30], recalled in §4. The line bundle structure of the Pfaffian (§3 and §7) will be exhibited by realizing various CAR algebra representations in the same Hilbert space, related to each other by Bogolubov automorphisms ([2],[8],[10],[1]) induced by the action of the restricted rotation group. These are unitarily implemented, by a crucial result of Shale and Stinespring ([27])). This device realizes at the same time the Spin Representation. In §8 we derive an explicit expression of Calabi's diastasis function on the Siegel manifold and point out the relation to coherent state quantization.

2. The CAR algebra and its gauge invariant quasi free states

We briefly review some basic facts concerning the CAR algebra and the theory of quasi-free states. We refer to [21],[22] and [12] (and also to [30]) for a detailed treatment.

Given a complex, separable Hilbert space K (the *one-particle* space) the CAR (Canonical Anticommutation Relations) algebra $A(K)$ is the unital C^* -algebra generated

by *creation* operators $a(f)^*$ (depending linearly on $f \in K$) and their adjoints $a(f)$ (*annihilation operators*, depending antilinearly on $f \in K$) fulfilling the following relations:

$$(2.1) \quad \begin{aligned} a(f)^* a(g) + a(g) a(f)^* &= \langle f, g \rangle I, \\ a(f) a(g) + a(g) a(f) &= 0, \end{aligned}$$

with $f, g \in K$, $\langle, \rangle$ denoting the scalar product in K , depending linearly on the first factor, and I being the identity of $A(K)$ (slightly different conventions are often used in the literature).

In our present context, we shall consider a restricted class of *states* on $A(K)$, the so called *pure, gauge invariant, quasi-free states* ω_E associated to a hermitian projection operator on K ($E^* = E$, $E^2 = E$). Any such state is determined through its *N-point functions* ($N = n+m$) (see [22]):

$$(2.2) \quad \omega_E(a(f_1)^* a(f_2)^* \dots a(f_n)^* a(g_1) a(g_2) \dots a(g_m)) = \delta_{nm} \det(\langle f_i, E g_j \rangle)$$

$$(\omega_E(I) = 1).$$

For such a state, the N -point functions are completely determined by the 2-point functions, via the CAR. Hence, the 2-point functions already determine the state ([22]). Any state ω_E induces a (cyclic, faithful, *irreducible*) Gelfand-Naimark-Segal (GNS) representation (π_E, H_E, ξ_E) (see e.g. [19], [21], [31]), which is unique up to unitary equivalence. We recall the basic GNS formula (adapted to the present context)

$$(2.3) \quad \omega_E(a) = \langle \pi_E(a) \xi_E, \xi_E \rangle_{H_E}, \quad \forall a \in A(K)$$

If $E = 0$, we have the so-called Fock state, giving rise to the Fock representation of the CAR (which is explicitly reviewed in [12]). If $E = I$, (with I the identity operator in K) we get the so-called anti-Fock representation.

More concretely, the Fock vector $\xi_0 \in H_0$ is characterized by the property

$$(2.4) \quad \pi_0(a(f)) \xi_0 = 0, \quad \forall f \in K.$$

The corresponding characterization of the anti-Fock vector $\xi_1 \in H_1$ is

$$(2.5) \quad \pi_1(a(f))^* \xi_1 = 0, \quad \forall f \in K.$$

Essentially one has an infinite dimensional generalization of the exterior algebra $\Lambda(K)$ in finite dimension (cf. [12]): the Fock state in $A(K)$ is $z \cdot 1$ ($z = e^{ia}$, $a \in \mathbb{R}$, 1 is the unit in $\Lambda(K)$, which is the GNS space; the anti-Fock state corresponds to the top exterior power: this will be important for the sequel).

A special case of the Powers-Størmer theorem ([22]) states that two representations (π_E, H_E, ξ_E) and (π_F, H_F, ξ_F) (corresponding to hermitian projections E and F) are unitarily equivalent if and only if $E - F$ is a Hilbert-Schmidt operator. In particular, if K is infinite dimensional, the Fock and the anti-Fock representations (of the *same* algebra $A(K)$) are *not* equivalent).

Let K' denote the dual of K ; we have $\langle f', g' \rangle_{K'} = \langle g, f \rangle_K$
 ($f \mapsto f'$ denotes the natural *antilinear* duality operator).

Recall, for future use, that given two irreducible GNS representations (of the CAR algebra), they can be realized on the *same* space if and only if they are unitarily equivalent; this being the case, the corresponding states are of course *vector states* corresponding to (cyclic) vectors in the common representation space. This will be crucial for our construction of the Determinant and Pfaffian line bundles.

We must also review (a particular case of the) Powers-Størmer "purification map" ([22]). The state ω_E on $A(K) = A(K \oplus 0)$ is the restriction of the state $\omega_{E \oplus I'}$ on $A(K \oplus K')$, where $F = I - E$, and F' is the corresponding adjoint operator on K' . The space $(E \oplus F')(K \oplus K')$ is the *isotropic* subspace in $K \oplus K'$ corresponding to EK (with respect to the complex bilinear form induced by the scalar product in $K \oplus K'$, given by evaluation, see below). This has a natural geometric counterpart in terms of grassmannians (see [23] and below).

For the sequel we need the following simple but crucial

Proposition 2.1. i) There exists a *canonical* complex linear C^* -algebra isomorphism χ between $A(K)$ and the $A(K')$ induced by the map $a(f) \mapsto a(f')^*$.
 ii) One has $\omega_{F'} \circ \chi = \omega_E$.
 iii) χ induces a map (denoted by the same symbol) which intertwines the representations (π_E, H_E, ξ_E) and $(\pi_{F'}, H_{F'}, \xi_{F'})$ of $A(K)$ and $A(K')$.

In particular, χ intertwines the respective Fock and anti-Fock representations. We shall call it *Fock - anti-Fock correspondence*.

Proof. i) is clear. Furthermore, the following computation involving 2-point functions

$$\begin{aligned} \omega_{F'}(\chi(a(f)^* a(g))) &:= \omega_{F'}(a(f') a(g')^*) = \omega_{(I-E)'}(\langle g', f' \rangle_{K'} I - a(g')^* a(f')) = \\ &= \langle g', E' f' \rangle_{K'} = \langle f, E g \rangle_K = \omega_E(a(f)^* a(g)) \end{aligned}$$

easily entails ii), by the above remarks.

iii) is then immediate from GNS theory ([19],[31]). $\square$

Remark. In finite dimension, this map is essentially Hodge star combined with duality. Also observe that this setting is basically the mathematical counterpart of Dirac's "hole" theory: a state containing no particles is interpreted as being completely filled with antiparticles.

3. Clifford Algebra and Bogolubov Transformations.

In this section we give a short account of Clifford algebras and Bogolubov transformations tailored to our purposes. We refer to [2],[17],[8],[26],[27],[10],[1], for a detailed treatment. Let H be a complex (separable) Hilbert space, and let $A(H)$ denote its CAR algebra. Denoting by $H_{\mathbb{R}}$ the space H considered as a *real* Hilbert space, with scalar product $(\cdot, \cdot)$, the Clifford algebra $C(H_{\mathbb{R}})$ is the real subalgebra of $A(H)$ generated by the operators (field operators, in physics)

$$(3.1) \quad \phi(f) := 1/\sqrt{2} [a(f) + a(f)^*] \quad , \quad f \in H .$$

The correspondence $f \rightarrow \phi(f)$ is *real* linear.

One immediately gets the corresponding anticommutation relations:

$$(3.2) \quad [\phi(f), \phi(g)]_+ = \operatorname{Re} \langle f, g \rangle \equiv \langle f, g \rangle_{H_{\mathbb{R}}} \equiv (f, g) .$$

Let $(H_{\mathbb{R}}, J) = H$ be $H_{\mathbb{R}}$ equipped with the given complex structure J , namely, $J \in B(H_{\mathbb{R}})$, $J^2 = -I$, the identity on $H_{\mathbb{R}}$. Recall that J is extended complex linearly to $(H_{\mathbb{R}})_{\mathbb{C}} = H_{\mathbb{C}}$, and one has $\langle f, g \rangle = (f, g) + i (f, Jg) = \langle f, g \rangle_J$, if we specify the complex structure involved.

Conversely, given the Clifford algebra $C(H_{\mathbb{R}})$ and the complex structure J , one can *complexify* it, and reconstruct the CAR algebra $A(H) = A((H_{\mathbb{R}}, J))$ via the map

$$(3.3) \quad \begin{aligned} a(f) &= 1/\sqrt{2} [\phi(f) + i \phi(Jf)] \\ a(f)^* &= 1/\sqrt{2} [\phi(f) - i \phi(Jf)] \end{aligned}$$

inverting (3.1) (upon recalling that, in H , $Jf = if$).

Now let $O \in O(H)$ (the *orthogonal group* of the real Hilbert space H).

In general, a rotation O implements the passage to a *new* complex structure on $H_{\mathbb{R}}$.

The action of O yields $J_O = O^{-1}JO$, namely

$$(3.4) \quad \langle f, g \rangle_{J_O} = \langle Of, Og \rangle_J$$

The various complex structures are then parametrized by the homogeneous space $\mathfrak{Z}(H) = O(H)/U(H)$ (*Siegel manifold*). We also notice, for later purposes and in order to make contact with [23], that $\mathfrak{Z}(H)$ can be identified with the manifold consisting of

closed subspaces Y of a fixed $H_{\mathbb{C}}$ such that Y is *isotropic* for B , and $Y \oplus \overline{Y} = H_{\mathbb{C}}$, where B is the complex bilinear form extending the *real* scalar product in H . Any rotation O induces an automorphism of $C(H)$ via the definition

$$(3.5) \quad \alpha_O \phi(f) := \phi(Of),$$

which can be extended to a $*$ -automorphism of the CAR algebra $A(H)$ (*Bogolubov $*$ -automorphism*) denoted by the same symbol), upon setting

$$(3.6) \quad \begin{aligned} \alpha_O a(f) &:= 1/\sqrt{2} [\alpha_O \phi(f) + i \alpha_O \phi(Jf)] = \\ &= 1/\sqrt{2} [\phi(Of) + i \phi(OJf)] = \\ &= 1/\sqrt{2} [\phi(Of) + i \phi(J_{O^{-1}} O f)]. \end{aligned}$$

If our algebras act on a concrete representation Hilbert space S (typically it will be the Fock or anti-Fock representation space) one wishes to investigate conditions ensuring the *unitary implementation* of the Bogolubov $*$ -automorphisms. Namely, one requires (in the given representation space S)

$$(3.7) \quad \alpha_O \phi(f) = \phi(O f) = O^{\sim} \phi(f) O^{\sim -1}$$

and, more generally

$$(3.7') \quad \alpha_O a(f) = O^{\sim} a(f) O^{\sim -1}$$

with $O^{\sim} \in U(S)$. The answer is provided by an important theorem of Shale and Stinespring ([27]), which we formulate in a different but equivalent way for our purposes (cf. the proof of Remark 2, §2.7 of [27]; see also [17], and below)

the Bogolubov $$ -automorphism α_O is unitarily implemented in the Fock representation space if and only if $[O, J]$ is Hilbert-Schmidt.*

In §7 we shall rephrase this by saying that α_O is unitarily implemented in a representation space if and only if $O \in O_{\text{res}}(H)$. The map $O \rightarrow O^{\sim}$ will then yield a (projective) unitary representation of $O_{\text{res}}(H)$. The space S will be the GNS space pertaining to an (anti)-Fock state (the space of *spinors*) and the representation involved will be the Spin Representation.

Now consider the Fock state ω on $C(H)$. It is characterized by its 2-point functions which, after a straightforward computation read

$$(3.8) \quad \omega(\phi(f)\phi(g)) = \langle g, f \rangle_J,$$

this showing its dependence on the complex structure (actually, on its class in $\mathfrak{S}(\mathbf{H})$). In other words ω is *not* invariant under Bogolubov transformations, and upon varying the complex structure via the action of the orthogonal group, one obtains a *family* of Fock states $\{\omega \circ \alpha_O\}$ parametrized by $[O] \in \mathfrak{S}(\mathbf{H})$. They exhaust the possible Fock states by [2]. In view of the theorem of Shale and Stinespring quoted above, if we take a *restricted* Siegel manifold (that is, $O_{\text{res}}(\mathbf{H})$ -homogeneous, see §6), one can realize all Fock states as *vector states* on a *single* representation space (by the GNS theory) and their corresponding lines give rise to a $O_{\text{res}}(\mathbf{H})$ -homogeneous line bundle over the restricted Siegel manifold. This is the Pfaffian. This portrait will be implemented rigorously in §7. It is close in spirit to the finite dimensional treatment outlined in [8]. The Spin Representation is then displayed in a Borel-Weil-Bott fashion ([4],[23],[3]).

We remark that the characterization given by Manuceau and Verbeure in [17] that two Fock representations associated to two different complex structures J and K are unitarily equivalent if and only if $\|J-K\|$ ($\|\cdot\|$ = modulus of an operator) is Hilbert-Schmidt fits the Shale-Stinespring description since $\|J-K\| = \text{H.S.}$ is equivalent to saying $J - K = \text{H.S.}$ and any two complex structures are related by a rotation O : $K = O J O^{-1}$, so $J - K = J - O J O^{-1} = J O O^{-1} - O J O^{-1} = [J, O] O^{-1}$ (see also §6).

4. The CAR algebraic approach to the Hilbert space Grassmannian

Our basic object is the *Hilbert space Grassmannian* $\text{Gr}(\mathbf{K})$ (see [23], and [30] as well for complete details).

Let $(\mathbf{K}, F)$ be a complex separable Hilbert space endowed with a *polarization* F , i.e. $\mathbf{K} = \mathbf{K}_+ \oplus \mathbf{K}_-$, with $F = E_+ - E_-$, with $E_{\pm}$ denoting the orthogonal projection onto $\mathbf{K}_{\pm}$ (so $F^2 = I$, the identity on $\mathbf{K}$); we have $F = 2E_+ - I$. We assume that both E_+ and E_- are *infinite dimensional*. Let us denote by $L^2(\mathbf{K})$ the space of Hilbert-Schmidt operators on $\mathbf{K}$ and let E_W denote orthogonal projection onto $W \in \text{Gr}(\mathbf{K})$. Then:

$\text{Gr}(\mathbf{K})$ consists precisely of the closed subspaces of $\mathbf{K}$ such that $E_+ - E_W \in L^2(\mathbf{K})$ (see e.g. [30], Prop.2.1). A more precise notation would be $\text{Gr}(\mathbf{K}, \mathbf{K}_+)$; we shall switch to it every time some danger of confusion arises.

The *restricted general linear group* $GL_{\text{res}}(\mathbf{K})$ ($:= \{g \in GL(\mathbf{K}) \mid [F, g] \in L^2(\mathbf{K})\}$) acts transitively on $\text{Gr}(\mathbf{K})$, and so does $U_{\text{res}}(\mathbf{K})$, which consists by definition of the unitary elements of $GL_{\text{res}}(\mathbf{K})$. The manifold $\text{Gr}(\mathbf{K})$ is indeed a $GL_{\text{res}}(\mathbf{K})$ -(and $U_{\text{res}}(\mathbf{K})$)-homogeneous *Kähler* manifold. Its Kähler form is described in [23] (see also [29]). The Grassmannian $\text{Gr}(\mathbf{K})$ is naturally embedded in a projective Hilbert space $P(\mathbf{H})$ ([23]), from which it inherits its Kähler structure. Its points are then labelled by the so-called *Plücker coordinates*.

Let us add a few details needed in the sequel. We shall take $\mathbf{K} = L^2(S^1, \mathbb{C})$ from now

on. Let $\mathbf{S}$ be the collection of all subset $S \subseteq \mathbb{Z}$ such that $S \cap \mathbb{N}$ and $\mathbb{N} - S$ are finite. Then $\mathbf{H} = l^2(\mathbf{S})$. Let K_S be the closed subspace of K spanned by the z^i , $i \in S$. Obviously $K_S \subseteq \text{Gr}(K)$. The set $S = \mathbb{N}$ gives $K_{\mathbb{N}} = K_+$. The collection of the K_S , $S \in \mathbf{S}$, (with their canonical basis $\{z^i\}$, $i \in S$, chosen) constitutes a natural *orthonormal* basis of $\mathbf{H}$. Upon varying S in $\mathbf{S}$, one obtains an atlas $U_S = \{W \in \text{Gr}(K) / \pi_S(W) \neq 0\}$ for $\text{Gr}(K)$, where $\pi_S(W)$ is a *Plücker coordinate*. We shall give below a formula for it. This construction can be easily understood within the framework of geometric quantization of Kähler manifolds and their *coherent states* in the sense of Rawnsley ([25],[6],[14],[15],[16],[18],[23],[28],[24],[29]). The geometric quantization (holomorphic) line bundle corresponding to the Kähler form ω is, in this case, the *dual* Det^* of the *determinant* line bundle Det . The Plücker coordinates naturally appear as *coherent states* in view of the Rawnsley - Kodaira construction (see also [23],[29]) and correspond to the states induced by the vectors K_S (when looked upon as holomorphic sections, à la Borel-Weil (cf.e.g.[23],[3])). The determinant line bundle realizes a (Kostant) central extension $U_{\text{res}}(K)^\sim$ of the group $U_{\text{res}}(K)$ governed by the Kähler structure of $\text{Gr}(K)$ ([23]).

Now, a crucial fact ([30] Theorems 3.1,2) is that the Hilbert space $\mathbf{H}$ is the carrier of a natural representation π of the CAR algebra unitarily equivalent to the GNS representation π_{l_+} . Let the symbol $\cong$ denote unitary equivalence. Specifically we have

$$(i) \quad \pi \cong \pi_{l_+} \cong \pi_{l_W} \quad \text{for any } W \in \text{Gr}(K).$$

(ii) the Plücker coordinates are given by the expression

$$(4.1) \quad \pi_S(W) = \langle W, K_S \rangle ,$$

(where $\langle, \rangle$ denotes the hermitian inner product on $\mathbf{H} = l^2(\mathbf{S})$) which vanishes if W and K_S have different *virtual dimensions* (cf.[23]).

With a slight abuse of language, we have identified a subspace W with the vector it defines in the representation space after choosing an orthonormal admissible basis for it (as in [30]). We give a few details for the sake of clarity and for future use.

We shall omit the symbol π for the representation in the discussion below.

Let us consider the canonical orthonormal basis $\{z^i\}$ of K , and let us specify the action of $a(z^i)^*$ on the orthonormal basis $\{K_S\}$ ($S \in \mathbf{S}$), of $\mathbf{H} = l^2(\mathbf{S})$. We have

$$(4.2) \quad a(z^i)^* K_S = \varepsilon(i, S) K_{S \cup \{i\}} ,$$

where $\varepsilon(i, S) = 0$ if and only if $i \in S$, and, if $i \notin S$, then $\varepsilon(i, S) = \pm 1$, depending on the sign of the permutation which places z^i in the "right" position in $K_{S \cup \{i\}}$.

Formula (2.1) easily entails, if $i \in S$,

$$(4.3) \quad a(z^i) K_S = \varepsilon(i, S - \{i\}) K_{S - \{i\}},$$

and zero if $i \notin S$. Actually $a(z^i)^*$ acts by "wedging" with z^i , and $a(z^i)$ by contraction with z^i (cf. the construction of the Fock representation given in [12], the discussion of finite dimensional grassmannians in [7], and the above remarks on the Fock representation). This easily leads to the desired representation π of the CAR algebra $A(K)$ on H .

Thus the vectors K_S can be usefully visualized as "semi-infinite wedge products" (relative to $\mathbb{N} \subset \mathbb{Z}$) of the z^i 's, $i \in S$, and physically represent states in which the one-particle modes z^i , $i \in S$, are occupied, whereas the modes z^i , $i \notin S$, are empty. Formula (4.1) is rigorously interpreted, via the GNS theory, as an infinite determinant, involving elements of orthonormal admissible bases (i.e., the basic finite dimensional exterior algebra picture persists). Hence (cf.[30]), the Plücker coordinates can be interpreted as suitable " ∞ -point functions", after replacing E_+ with E_S (projection onto K_S). See also the remarks after theorem 7.1.

5. The Determinant line bundle revisited

In this section we shall sketch "ex novo" a C^* -algebraic construction of the determinant line bundle, equivalent to Pressley - Segal's one, as we shall see, building on the ideas in [30] recalled in the previous section. We shall also compare our treatment with Borthwick's one. First we need a general lemma, which will be employed for the Pfaffian as well.

Lemma 5.1. Let G be a Banach Lie group and $\rho: G \rightarrow PGL(V)$ be a projective representation of G by bounded, linear, invertible operators on a complex topological vector space V . Let G/H be a G -homogeneous complex manifold. Furthermore let $E \subseteq G/H \times V$ be a closed analytic subset such that $\{gH\} \times E_{gH} = (\text{pr}_1|_{E_g})^{-1}(gH) \subseteq G/H \times V$ gives a complex vector subspace E_{gH} of V . Assume further that at the projective level we have, for $[v] \in P(E_{g'H}) \subseteq P(V)$,

$$(5.1) \quad g(g'H, [v]) = (g g'H, [\rho(g) v]),$$

and this is in $P(E_{gg'H})$.

Then E is biholomorphic to a holomorphic vector bundle on G/H with typical fibre E_{clt} .

Proof. We have the following commutative diagram

$$\begin{array}{ccc}
& \rho & \\
G & \rightarrow & \text{PGL}(V) \\
\pi^{\text{ind}} \uparrow & & \uparrow \pi \\
G^{\sim} & \xrightarrow{\rho^{\sim}} & \text{GL}(V)
\end{array}$$

where $G^{\sim} = \{(g, A) \in G \times \text{GL}(V) / \rho(g) = \pi(A)\}$ denotes the induced central $\mathbb{C}^*$ -extension of G . $G^{\sim}$ acts on G/H with isotropy $H^{\sim} = (\pi^{\text{ind}})^{-1}(H)$ and on V . It follows easily that the map

$$\begin{array}{ccc}
G^{\sim} \times_{H^{\sim}} E_{\text{ell}} & \xrightarrow{\quad} & E \\
& \cong & \\
& \downarrow & \\
G^{\sim}/H^{\sim} & \cong & G/H
\end{array}$$

defined by

$$(5.2) \quad [g^{\sim}, v] = [g^{\sim} (h^{\sim})^{-1}, \rho^{\sim}(h^{\sim})v] \rightarrow (g^{\sim} H^{\sim}, \rho^{\sim}(g^{\sim})v)$$

yields the desired isomorphism, by (5.1). $\square$

We now recall some standard objects of the theory of holomorphic line bundles for the sake of clarity. They are needed further on in this section.

Let $L \rightarrow X$ be a holomorphic line bundle over a complex manifold X , and let L^* be its dual bundle. The space of holomorphic sections of L^* will be denoted by $\Gamma_0(X, L^*)$. One has

$$(5.3) \quad \Gamma_0(X, L^*) \cong \{f: L \rightarrow \mathbb{C} / f \text{ is holomorphic and linear in each fibre}\}.$$

In our infinite dimensional setting extra care is needed, but the calculations involved characterize the "square integrable" holomorphic sections (cf.[23]). In our case all fibres will be complex lines in a fixed Hilbert space, and we shall confine ourselves to those sections which can be induced by a continuous linear functional on the Hilbert space, and hence, via the Riesz theorem, by a vector in it.

Explicitly we have the following maps, inverse to each other:

$$f \rightarrow \sigma_f, \quad \text{with } \sigma_f(x) := f|_{L_x} \in \text{Hom}(L_x, \mathbb{C}) \text{ and}$$

$$\sigma \rightarrow f_\sigma, \quad \text{with } f_\sigma(l_x) := \sigma(x)(l_x) \in \mathbb{C}, l_x \in L_x.$$

This yields - in absence of base points, where all sections vanish - a commutative diagram (setting $V = \Gamma_O(X, L^*)$)

$$\begin{array}{ccc} L - \{0\} & \rightarrow & V^* - \{0\} \\ \downarrow & & \downarrow \\ X & \rightarrow & P(V^*) \end{array}$$

by $l_x \rightarrow (f \rightarrow f(l_x))$ and $x \rightarrow (f \rightarrow f(l_x))$ for $l_x \in L_x - \{0\}$ or $\sigma \rightarrow \sigma(x)$.

This works for a "sufficiently big" submodule W of V as well: $X \rightarrow P(W^*)$.

Let us now fix the GNS-representation $(\pi_{E_+}, H_{E_+}, \xi_{E_+})$. Let us remove the suffix E_+ in order to ease notation. Any other GNS-representation corresponding to $Y \in \text{Gr}(K)$ is unitarily equivalent to it and *can thus be realized on H* (see §2). The GNS state becomes a vector state and gives rise to a complex line in H , which will be denoted by Det_Y . In view of $U_{\text{res}}(K)$ -equivariance, one has if $Y = UK_+$, $U \in U_{\text{res}}(K)$:

$$(5.4) \quad \text{Det}_Y = \text{Det}_{UK_+} = U \sim \text{Det}_{K_+} = U \sim \mathbb{C} \cdot \xi,$$

where $U \sim$ denotes its implementing unitary (up to a phase) in H (one gets a central extension of β by $U(1)$). More explicitly, $U \in U_{\text{res}}(K)$ induces a C^* -algebra automorphism β_U of $A(K)$ by $\beta_U(a(f)) = a(Uf)$. Since the representation $\pi \circ \beta_U$ is unitarily equivalent to π_{l_Y} , and $\pi \circ \beta_U$, as well as π , are realized on the same Hilbert space H , by Powers-Størmer we have a unitary $U \sim$ on H implementing U , i.e.

$$(5.5) \quad \pi \circ \beta_U = U \sim \pi \circ U \sim^{-1}.$$

This easily entails the following characterization of Det_Y , which obviously generalizes the finite dimensional one:

$$\text{Det}_Y = \{\xi \in H \mid \pi(a(y)^*)\xi = 0 \quad \forall y \in Y \quad \text{and} \quad \pi((a(y^\perp)))\xi = 0 \quad \forall y^\perp \in Y^\perp\}.$$

In quantum field theoretic terms this corresponds to the Dirac vacuum filled with the antiparticles relative to Y ("Dirac sea").

In view of the preceding lemma, this arrangement leads to a holomorphic line bundle over $\text{Gr}(K)$.

Since $\text{Det} \subseteq \text{Gr}(K) \times H$ and $\text{Det}_Y \subseteq H$, every vector in H defines a function $f^u: \text{Det} \rightarrow \mathbb{C}$ by $f^u((Y, v)) := \langle v, u \rangle_H$ ($v \in \text{Det}_Y \subseteq H$), namely, a holomorphic section of the dual Det^* . By virtue of the previous discussion, we shall restrict to sections of the above form. This yields a $\mathbb{C}$ -linear map $H^* \rightarrow \Gamma_O(X, \text{Det}^*)$ and therefore $H = (H^*)^*$ gives, as the dual of a section module, a map $\text{Pl}: \text{Gr}(K) \rightarrow P(H)$ (Pl standing for Plücker).

Let $\text{Det}^\sim$ denote for the moment the Pressley-Segal determinant line bundle, $\text{Pl}^\sim$ its (Pressley-Segal) Plücker embedding into $P(H^\sim)$, with $H^\sim = \mathbf{H} = l^2(\mathbb{S})$. Then we have

Theorem 5.2. With the above notation: $\text{Det} \cong \text{Det}^\sim$.

Proof. In view of theorem 3.2 of [30], see also §4, we have two unitarily equivalent irreducible representations of $A(K)$ on H and $H^\sim$, thereby yielding a unique projective unitary $\theta^\sim: P(H) \rightarrow P(H^\sim)$. In view of (4.1) one has - by a lengthy, but elementary calculation using the Hilbert bases of [23] and [30] - for any $Y \in \text{Gr}(K)$

$$(5.6) \quad \theta^\sim \text{Pl}(Y) = \text{Pl}^\sim(Y),$$

i.e. the commutative diagram

$$\begin{array}{ccc} & \text{Pl} & \\ \text{Gr}(K) & \rightarrow & P(H) \\ \text{Pl}^\sim \downarrow & & \downarrow \theta^\sim \\ & & P(H^\sim) \end{array}$$

Let $O(-1)_V$ denote the tautological line bundle over $P(V)$, for an arbitrary $\mathbb{C}$ -vector space V .

Since both maps are $U_{\text{res}}(K)$ -equivariant embeddings, we have

$$\text{Det}^\sim \cong (\text{Pl}^\sim)^*(O(-1)_{H^\sim}) = \text{Pl}^*(\theta^\sim)^*(O(-1)_{H^\sim}) = \text{Pl}^*(O(-1)_H) \cong \text{Det}. \quad \square$$

Remark. $\text{Det}^\sim$ and Det are equivalent not only as holomorphic line bundles, but also as $U_{\text{res}}(K)^\sim$ -bundles with hermitian metric (induced by the embeddings).

In [3], Borthwick has given explicitly the transition functions of Det . First we introduce further notation. Recall that $\text{Gr}(H, H_+)$ is a Hilbert manifold modelled on $L^2(H_+, H_-)$, the space of Hilbert-Schmidt operators from H_+ to H_- . Any $W \in \text{Gr}(H, H_+)$ can be viewed as a *graph* of an operator A in $L^2(H_S, H_S^\perp)$. If $W \in U_R \cap U_S$, then one finds the following expression for the transition functions g_{RS} of Det ([3]):

$$(5.7) \quad g_{RS}(A) = \det \{ \langle A e_k, e_j \rangle \}, \quad j \in R/S, k \in S/R$$

(one has indeed a finite dimensional square matrix; also recall that our scalar product is linear in the first variable)

We observe that they can also be easily obtained by recalling that Det is the pull-back to $\text{Gr}(H)$ of the tautological line bundle $\mathcal{O}(-1)_{\text{Pl}}$ under the Plücker embedding and $\text{Gr}(H)$ is described by the Plücker coordinates. Thus we obtain, by the above remark, or by direct computation with Plücker coordinates (it is enough to check (5.7) in the case $S = \mathbb{N}$, $A \in L^2(H_+, H_-)$) the following

Proposition 5.3. We have the equality

$$(5.8) \quad g_{RS}(A) = \pi_R(W) \pi_S(W)^{-1} = \langle W | K_R \rangle \langle W | K_S \rangle^{-1}. \quad \square$$

6. The restricted Siegel manifold

Let $H = H_+ \oplus H_-$ be a complex, polarized Hilbert space and $(H_{\mathbb{R}}, (\cdot, \cdot))$ be the underlying real Hilbert space. The complexification $H_{\mathbb{C}} := H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is canonically isomorphic to a sum of complex Hilbert spaces: $H_{\mathbb{C}} \cong H \oplus \bar{H}$, $\xi \otimes \lambda \rightarrow (\lambda \xi \oplus \bar{\lambda} \bar{\xi})$, where $\bar{H}$ is the "conjugate $\mathbb{C}$ -vector space" on the set H . Of course this decomposition can also be described in terms of the complexification $L^{\mathbb{C}}$ of the original multiplication by i on H : $L = (i)_{\mathbb{R}}$, namely $H_{\mathbb{C}} = \text{Eig}(L^{\mathbb{C}}, +i) \oplus \text{Eig}(L^{\mathbb{C}}, -i)$. Furthermore the conjugation (or "real structure") $\sigma: \xi \otimes \lambda \rightarrow \xi \otimes \bar{\lambda}$ of $H_{\mathbb{C}}$ exchanges H and $\bar{H}$ (and the $(\pm i)$ -eigenspaces, respectively). It allows for an easy description of the hermitian extension $\langle, \rangle$ of $(\cdot, \cdot)$ to $H_{\mathbb{C}}$ by means of the $\mathbb{C}$ -bilinear extension $B = (\cdot, \cdot)^{\mathbb{C}}$: $\langle u, v \rangle = B(u, \bar{v})$ for $u, v \in H_{\mathbb{C}}$. The polarization yields a real endomorphism $J = i E_+ - i E_-$ of $H_{\mathbb{R}}$ and the $\mathbb{C}$ -linear extension $J^{\mathbb{C}} \text{ to } H_{\mathbb{C}}$ has eigenspaces $\text{Eig}(J^{\mathbb{C}}, +i) = W$, $\text{Eig}(J^{\mathbb{C}}, -i) = \bar{W}$, respectively. Using $H_{\mathbb{C}} \cong H \oplus \bar{H}$, W corresponds to $H_+ \oplus \bar{H}_-$.

We consider $\text{Gr}(H_{\mathbb{C}}) := \text{Gr}(H_{\mathbb{C}}, H_+ \oplus \bar{H}_-) = \text{Gr}(H_{\mathbb{C}}, W)$ and its natural symmetry

groups $Gl_{res}(H_{\mathbb{C}})$ and $U_{res}(H_{\mathbb{C}})$. Furthermore we set

$$O_{res}(H_{\mathbb{C}}, B) := \{g \in Gl_{res}(H_{\mathbb{C}}) \mid g^* B = B\}$$

and its "real part"

$$O_{res}(H) := \{g \in O_{res}(H_{\mathbb{C}}, B) \mid g \circ \sigma = \sigma \circ g\}.$$

This group obviously equals $\{g \in U_{res}(H_{\mathbb{C}}) \mid g \circ \sigma = \sigma \circ g\}$ since $\langle u, v \rangle = B(u, \overline{v})$. In order to apply later the "Shale-Stinespring theory", it is useful to give the following characterization of this group.

Proposition 6.1. The restricted orthogonal group of H , $O_{res}(H)$, is the group given by $\{h^{\mathbb{C}} \mid h \in O(H_{\mathbb{R}})\}$ and $[h, J]$ is Hilbert-Schmidt.

Proof. The definition immediately implies that every g in $O_{res}(H)$ preserves $H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{R} = \text{Fix } \sigma \subset H_{\mathbb{C}}$ and the real Hilbert space structure on it. Thus $g = h^{\mathbb{C}}$ for an element $h \in O(H_{\mathbb{R}})$. The identity $-i [h, J]^{\mathbb{C}} = -i [h^{\mathbb{C}}, J^{\mathbb{C}}] = [g, E_W - E_{\overline{W}}]$ shows that $g \in Gl_{res}(H_{\mathbb{C}}, W)$ if and only if $[h, J]$ is Hilbert-Schmidt. $\square$

The group $U_{res}(H, H_+)$ is naturally injected in $O_{res}(H)$, since $[h, J] = L([h, E_+ - E_-])$ for $h \in U_{res}(H, H_+)$.

We define the restricted Siegel manifold $\mathfrak{S}_{res}(H)$ as follows (cf.[23]: our $\mathfrak{S}_{res}(H)$ is simply denoted $\mathfrak{S}(H)$ there)

$$\mathfrak{S}_{res}(H) := \{Y \in Gr(H_{\mathbb{C}}) \mid Y \text{ is isotropic for } B, \text{ and } Y \oplus \overline{Y} = H_{\mathbb{C}}\}.$$

We remark that the identity

$$(6.1) \quad (J_W - J_Y)^{\mathbb{C}} = 2i(E_W - E_Y)$$

immediately implies that $\mathfrak{S}_{res}(H)$ is the set of complex structures on H "comparable to J ", the complex structure given by the reference subspace $W \subset H_{\mathbb{C}}$, i.e.

$$\mathfrak{S}_{res}(H) = \{J \equiv J_Y \in \mathfrak{S}(H) \mid J_Y - J_W \text{ is Hilbert-Schmidt}\}.$$

Proposition 6.2. (cf.[23],[3]). The restricted Siegel manifold $\mathfrak{S}_{res}(H)$ is $O_{res}(H)$ -homogeneous. Indeed $\mathfrak{S}_{res}(H) \cong O_{res}(H)/U(W)$.

Proof. First observe that $O_{res}(H)$ maps $\mathfrak{S}_{res}(H)$ into itself. In fact, if $Y = OW$, with $O \in O_{res}(H)$, then (also cf.§3)

$$J_Y - J_W = O J_W O^{-1} - J_W = O J_W O^{-1} - J_W O O^{-1} = [O, J_W] O^{-1}$$

is Hilbert-Schmidt. Conversely, if $Y \in \mathfrak{S}_{\text{res}}(H)$, and $Y = O W$, with $O \in O(H)$ (and this is always true), then $J_Y - J_W$ Hilbert-Schmidt easily entails $[O, J_W]$ Hilbert-Schmidt so the map is onto as well. The isotropy group of J_W is clearly $U(W)$. $\square$

Remark. Observe that the above argument is closely related to the one used in [30] (see also §4) to characterize $\text{Gr}(K)$.

In general, to any $W_1 \in \text{Gr}(H, H_+)$ corresponds the *isotropic* subspace $W_1 \oplus (W_1^\perp)^\perp$ in $\text{Gr}(H_{\mathbb{C}}, W)$. This gives rise to an *embedding* of $\text{Gr}(H, H_+)$ into $\text{Gr}(H_{\mathbb{C}}, W)$ (actually, into $\mathfrak{S}_{\text{res}}(H)$), which may be viewed as a *geometric version* of (a special instance of) the Powers-Størmer purification argument, as we have already observed ([22]): E_+ goes to $E_+ \oplus \overline{E}_-$.

Let $Y \in \mathfrak{S}_{\text{res}}(H)$. We have the following

Lemma 6.3. There exists a *canonical* complex linear isomorphism between $A(Y)$ and the $A(\overline{Y})$ induced by the map $a(f) \rightarrow a(\overline{f})^*$ which intertwines the respective Fock and anti-Fock representations.

Proof. The proof is straightforward in view of Proposition 2.1, since $\overline{Y}$ can be identified, $\mathbb{C}$ -linearly, via B , with Y' , the dual of Y (also cf. [23]). $\square$

Let us draw some geometric consequences from the above discussion and Proposition 2.1. Let $\overline{H}$ denote the GNS space $H_{\overline{E}_-}$ carrying the GNS-representation of $A(\overline{H})$ induced by $\omega_{\overline{E}_-}$, where $\overline{E}_-$ is the projection onto $\overline{H}_-$.

Let S' denote the set of all $S' := \mathfrak{E} - S$.

Theorem 6.4. i) One has $\overline{H} = l^2(S')$ and $H_{E_+ \oplus \overline{E}_-} \cong \mathbf{H} \hat{\otimes} \overline{H}$ (topological tensor product) $\cong l^2(S \times S') \cong l^2(S) \hat{\otimes} l^2(S') = H_E \otimes H_{\overline{E}_-}$.

ii) The Fock - anti-Fock correspondence reads as follows, in terms of the natural orthonormal bases in $\mathbf{H}$ and $\overline{H}$ (with a slight abuse of notation)

$$\chi: H_S \rightarrow (H_S^\perp)^\perp = \overline{H}_{S'}.$$

iii) The Fock - anti-Fock correspondence gives rise to a Kähler isometry

$$(6.2) \quad \text{Gr}(H, H_+) \cong \text{Gr}(\overline{H}, \overline{H}_-)$$

(which allows identification of the two manifolds).

iv) The embedding $\text{Gr}(H) \rightarrow \text{Gr}(H_{\mathbb{C}})$ reads, in terms of Plücker coordinates:

$$(6.3) \quad \begin{aligned} \pi_{(S,T)}(W \oplus (W^{\perp})^{\overline{}}) &= \langle W \oplus (W^{\perp})^{\overline{}}, H_S \oplus (H_T^{\perp})^{\overline{}} \rangle = \\ &= \langle W, H_S \rangle \langle W, H_T \rangle = \pi_S(W) \pi_T(W) \end{aligned}$$

In particular

$$(6.3') \quad \pi_{(S,S)}(W \oplus (W^{\perp})^{\overline{}}) = \pi_S(W)^2.$$

Proof. i) and ii) are obvious.

iii) follows easily from i): in fact the Fock - anti-Fock correspondence maps W to $(W^{\perp})^{\overline{}}$ (complex linearly) and

$$(6.4) \quad \langle W, H_S \rangle = \langle (W^{\perp})^{\overline{}}, (H_S^{\perp})^{\overline{}} \rangle,$$

(in the corresponding representation spaces), whence the conclusion follows, because the Kähler structure of the Grassmannians is induced from the projective space one, and the latter is governed entirely by the Hilbert space structure.

iv) is then immediate from (4.1) and (6.4).

Remarks. i) Formula (6.3') corresponds to the *square map* of Pressley and Segal (see page 243 in [23]); therefore, the latter map gives only *part* of the coordinates (this can be ascertained within their formalism as well). This will be even clearer after stating theorem 7.1 below.

ii) Formula (6.3') can also be interpreted in a natural way in terms of diastatic formulae ([29],[30],[7]). In particular it appears as a manifestation of the naturality of Calabi's diastasis function under pull-backs ([7]).

7. The Pfaffian and the Spin Representation

In this section we introduce the Pfaffian line bundle in a C^* -algebraic manner, the basic idea of the construction being already sketched in §3. Let us introduce the following intermediate notation (S', π') for the GNS representation of $C(H_{\mathbb{C}})$ on $H_{H_{\mathbb{C}}}^{C(H_{\mathbb{C}})}$, where $C(H_{\mathbb{C}}) = C(H_{\mathbb{C}}, B) = C(H_{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{R}$, and

$$\text{Pf}_{Y'} := \{\psi' \in S' \mid \pi'(\phi(y))\psi' = 0 \quad \forall y \in Y\}$$

for $Y \in \mathfrak{S}_{\text{res}}(H)$. We shall often attach a superscript to a representation space specifying the algebra it refers to, in order to avoid confusion, at the cost of burdening the notation a little bit. The implementation

$$O^\sim \circ \pi' \circ O^{\sim -1} = \pi' \circ \alpha_{\mathfrak{t}}$$

of an element O of $O_{\text{res}}(H)$ by Shale-Stinespring (see §3) shows

$$\text{Pf}_Y = O^\sim (\text{Pf}_W)$$

if $Y = OW$.

Combining the isomorphism of real vector spaces with complex structure $(H_{\mathbb{R}}, J) \rightarrow W, f \mapsto 1/2(f - iJ_W f)$ with the relation (3.3) between the a 's and the ϕ 's, we get a C^* -algebra isomorphism $\sigma_W: C(H_{\mathbb{C}}) \rightarrow A(W)$. We remark that, up to a factor, $u \in H_{\mathbb{C}} \subset C(H_{\mathbb{C}})$ is mapped to $a(E_W u)^* + a((E_{\overline{W}} u)^\sim)$ in $A(W)$. This mixing of creation and annihilation operators is called *Bogolubov transformation* (see [2] and [10]).

We denote the induced C^* -algebra automorphism $\sigma_W \circ \alpha_O \circ \sigma_W^{-1}$ again by α_O . Defining (S, π_{I_W}) as the anti-Fock GNS-representation of $A(W)$, acting on $H_{I_W}^{A(W)}$ and

$$\text{Pf}_Y := \{\psi \in S \mid \pi_{I_W}(\sigma_W(y))\psi = 0 \quad \forall y \in Y\}$$

for $Y \in \mathfrak{S}_{\text{res}}(H)$ we immediately observe that $\text{Pf}_W = \mathbb{C} \xi_{I_W}$, with ξ_{I_W} the GNS state vector of the representation π_{I_W} (anti-Fock vector) and $\text{Pf}_Y = O^\sim \text{Pf}_W$, where $O^\sim$ denotes as well the implementation of $O \in O_{\text{res}}(H)$ (such that $OW = Y$) on S . This $O^\sim$ obviously exists, since it corresponds to the $O^\sim$ given above for the isomorphic GNS representation S' of $C(H_{\mathbb{C}})$. Remark that we have a central extension $O_{\text{res}}(H)^\sim$ of $O_{\text{res}}(H)$ by $U(1)$ acting unitarily on S . Applying Lemma 5.1, we get a holomorphic line bundle Pf over $\mathfrak{S}_{\text{res}}(H)$.

We are now ready to state the following

Theorem 7.1 i) The Pfaffian line bundle is a *square root* of $\text{Det}^\dagger|_{\mathfrak{S}_{\text{res}}(H)}$.

ii) In particular, upon *restriction* to the "small" grassmannian $\text{Gr}(H)$ we have $\text{Pf} = \text{Det}$.

iii) We have a natural *Segre embedding* $S : \text{Gr}(H) \times \text{Gr}(H) \rightarrow \text{Gr}(H_{\mathbb{C}})$

defined by : $S((W_1, W_2)) = W_1 \oplus (W_2^\perp)^\perp$ (projective embeddings understood, cf.[9],[11])

The embedding is realized, à la Kodaira ([9],[11]) via the bundle $\text{Det}_{\text{Gr}(H)} \boxtimes \text{Det}_{\text{Gr}(H)} = \text{Pf}|_{\text{Gr}(H)} \boxtimes \text{Pf}|_{\text{Gr}(H)} = S^* \text{Det}_{\text{Gr}(H\mathbb{C})}$.

Proof. i) Recall that $\text{Pf}_W = \mathbb{C} \xi_{I_W} \subset H_{I_W}^{\Lambda(W)}$. The Fock - anti-Fock correspondence (see Prop.2.1. and Lemma 6.2) yields

$$\begin{aligned} \text{Pf}_W \otimes \text{Pf}_W &\cong \mathbb{C} \xi_{I_W} \otimes \mathbb{C} \xi_{I_W} \subset H_{I_W}^{\Lambda(W)} \hat{\otimes} H_{I_W}^{\Lambda(W)} \\ &\cong \mathbb{C} \xi_{I_W} \otimes \mathbb{C} \xi_{0\bar{W}} \subset H_{I_W}^{\Lambda(W)} \hat{\otimes} H_{\bar{W}}^{\Lambda(\bar{W})}, \end{aligned}$$

passing on the algebra level from $A(W) \hat{\otimes} A(W)$ to $A(W) \hat{\otimes} A(\bar{W})$. The latter algebra being canonically isomorphic to $A(H\mathbb{C})$ we finally get by the Powers-Størmer "purification map" (see §2)

$$\text{Pf}_W \otimes \text{Pf}_W \cong \mathbb{C} \xi_{I_W \oplus 0\bar{W}} = \mathbb{C} \xi_{E_W} \subset H_{E_W}^{\Lambda(H\mathbb{C})},$$

which is the line Det_W . Given $Y \in \mathfrak{Y}_{\text{res}}(H)$, there is $O \in O_{\text{res}}(H)$ such that $OW = Y$. Since $\text{Pf}_Y = O \sim \text{Pf}_W$, the claim i) follows by $O_{\text{res}}(H)$ -equivariance.

Given $W_1 \in \text{Gr}(H)$, realized as $Y = W_1 \oplus (W_1^\perp)^\perp \in \mathfrak{Y}_{\text{res}}(H)$, there is an element $U \in U_{\text{res}}(H)$ such that $W_1 = U H_+$ and the above map is equivariant with respect to the embedding of $U_{\text{res}}(H)$ into $O_{\text{res}}(H)$. Thus, it is enough to consider $W_1 = H_+$ in proving ii). The determinant line is $\text{Det}_{H_+} = \mathbb{C} \xi_{I_+} \subset H_{I_+}^{\Lambda(H)}$, whereas the Pfaffian line is $\text{Pf}_{H_+ \oplus \Pi_- = W} = \mathbb{C} \xi_{I_W} \subset H_{I_W}^{\Lambda(W)}$. Applying a Powers-Størmer purification and a Fock - anti-Fock correspondence we get

$$H_{I_W}^{\Lambda(W)} \cong H_{I_{E_+}}^{\Lambda(H_+)} \hat{\otimes} H_{I_{E_-}}^{\Lambda(\Pi_-)} \cong H_{E_+}^{\Lambda(H_+ \oplus \Pi_-)} = H_{I_+}^{\Lambda(H)},$$

identifying Pf_W with Det_{H_+} .

Now iii) is an easy consequence of Theorem 6.4. In terms of Plücker coordinates the Segre embedding reads (cf.(6.4))

$$\begin{aligned} (7.1) \quad \pi_{(S,T)}(W_1 \oplus (W_2^\perp)^\perp) &= \langle W_1 \oplus (W_2^\perp)^\perp, H_S \oplus (H_T^\perp)^\perp \rangle = \\ &= \langle W_1, H_S \rangle \langle W_2, H_T \rangle = \pi_S(W_1) \pi_T(W_2). \quad \square \end{aligned}$$

Remarks. i) The Pfaffian is *uniquely* defined as the square root of Det (after restriction to the Siegel manifold) (see [3],[23]).

ii) The previous argument can be perfectly adapted to the finite (even) dimensional situation as well .

iii) Without elaborating on it we would like to point out that - upon interpreting the Plücker coordinates as " ∞ -point functions" as in [30] - (6.4) (and (7.1)) can be seen as particular cases of tensor product state formulac.

iv) We finally observe that in Borthwick's treatment ([3]), the Pfaffian is given by taking the square root of the transition functions (5.4), after restriction to the Siegel manifold, by using the well known fact that the determinant of a finite-dimensional antisymmetric matrix, as a function of its entries, is the square of the Pfaffian polynomial (see e.g.[23]).

We are now in a position to define the representation we are looking for.

Definition 7.2. Let S be the GNS-representation space $H_{I_W}^{A(W)}$ of the CAR algebra $A(W)$ and $O_{\text{res}}(H) \sim$ the central $U(1)$ -extension of $O_{\text{res}}(H)$ implementing the Bogolubov transformations as unitary endomorphisms of S . This linear action of $O_{\text{res}}(H) \sim$ on S is called the " Spin^c -representation of $O_{\text{res}}(H)$ ". One cannot render it two-valued only, in contrast with the finite dimensional case ([23],[27],[3]).

Proposition 7.3. The submodule $S \subset \Gamma_O(\mathfrak{F}_{\text{res}}(H), \text{Pf}^*)$ is basepoint -free and thus yields a holomorphic, $O_{\text{res}}(H) \sim$ -equivariant embedding of $\mathfrak{F}_{\text{res}}(H)$ into $P(S^*)$.

Proof. The construction of the line bundle Pf as a subset of $\mathfrak{F}_{\text{res}}(H) \times S$ displays S as a base point-free module of sections of Pf^* . Equivariance follows from the very definition of $O_{\text{res}}(H) \sim$. $\square$

Remarks. i) The Pfaffian line bundle thus permits a description à la Borel-Weil-Bott of the infinite-dimensional Spin^c -representation.

ii) In our treatment, an explicit construction of the central extension $O_{\text{res}}(H) \sim$ is lacking; however, at the Lie algebra level, it is induced by the Kähler form of $\mathfrak{F}_{\text{res}}(H)$, which equals one-half of the Kähler form on $\text{Gr}(H_{\mathbb{C}})$. The fairly detailed realization of the group extension in [3], based on [23], employs the Jaffe-Lesniewski-Weismann theory of the relative Pfaffian (see [13]).

iii) We would like to stress that the C^* -algebraic approach to the Pfaffian and Determinant bundles uncovers the geometric nature of the Powers-Størmer and Shale-Stinespring theories. This shows the "naturalness" of the Hilbert-Schmidt operators with respect to more general Schatten ideals (cf.[3]).

8. Further remarks.

i) Denoting $H_{LW}^{\Lambda(H\mathbb{C})}$ by V and $H_{LW}^{\Lambda(W)}$ by S , we have a Segre map τ from $P(S^*) \times P(S^*)$ to $P(V^*)$ fulfilling $\tau \circ ("Pf" \times "Pf") = Pl$, where "Pf" denotes the projective embedding of $\mathfrak{S}_{res}(H)$ by means of the section module S of the line bundle Pf^* . Thus the induced Kähler structure ω^{Det} induced by Pl on $Gr(H_{\mathbb{C}})$ equals twice the structure ω^{Pf} induced by the map "Pf" on $\mathfrak{S}_{res}(H)$.

We recall the formula for the diastasis function D^{Det} given in [29]. For Y_1, Y_2 in a single coordinate chart (given by graphs) of $Gr(H_{\mathbb{C}}, W)$ one has:

$$(8.1) \quad D^{Det}(Y_1, Y_2) = 2 \| E_1 - E_2 \|_{HS(H_{\mathbb{C}})}^2$$

where E_j is the projector onto Y_j , for $j=1,2$. By naturality $D^{Pl}(Y_1, Y_2) = 1/2 D^{Det}(Y_1, Y_2)$ if $Y_j \in \mathfrak{S}_{res}(H)$, hence equation (6.1) yields immediately an explicit description of the diastasis function:

$$(8.2) \quad D^{Pl}(Y_1, Y_2) = 1/4 \| J_{Y_1} - J_{Y_2} \|_{HS(H_{\mathbb{R}})}^2.$$

ii) The formulae relating to scalar products of spinors in section 12.2 of [23] and its generalizations to infinite dimensions in [3] should be easily be interpreted in terms of coherent states, in the spirit of [29]. Thus, in accordance with the general coherent state philosophy (see e.g. [25],[20],[6],[17],[29]), a geometric quantization procedure without using a measure is possible.

iii) Since one of the motivations of this work was to elucidate sections 12.3 and 12.4 of [23], we were insisting on a polarization of the initial Hilbert space H in the set-up of §6. This allows for the consideration of $U_{res}(H)$ and the various Segre maps. The minimal data for the construction of the Pfaffian (and of course only these are used in our proof of $Pf^{\otimes 2} = Det$ on $\mathfrak{S}_{res}(H)$) are the following:

(*) $(H, (,))$ a real Hilbert space with a reference complex structure J .

They immediately yield $H_{\mathbb{C}}$ with the hermitian extension $\langle, \rangle$ of $(,)$, the conjugation

$\sigma: \xi \otimes \lambda \rightarrow \xi \otimes \overline{\lambda}$, the $\mathbb{C}$ -bilinear extension $B = (,)^{\mathbb{C}}$ of $(,)$, and the B -isotropic subspace $W = \text{Eig}(J^{\mathbb{C}}, +i)$. Furthermore $\overline{W} := \sigma(W) = W^{\perp}$. In this set-up $L = J$ (see the beginning of §6) and W cannot be identified with $H_+ \oplus \overline{H}_+$, since H is no

longer assumed to be polarized.

In the set-up (*) the reference complex structure might, of course, be equivalently replaced by a reference complex subspace W of $H_{\mathbb{C}}$ fulfilling $\sigma(W) = W^{\perp}$. The reference complex structure is then given by $J := (i(E_W - E_{\overline{W}}))|_H$.

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