Estimate for the $0^{++}$ glueball mass in QCD

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Abstract

We obtain accurate result for the lightest glueball mass of QCD in 3 dimensions from lattice Hamiltonian field theory. Using the dimensional reduction argument, a good approximation for confining theories, we suggest that the $0^{++}$ glueball mass in 3+1 dimensional QCD be about 1.71 GeV.

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The spectroscopy of QCD in the pure gauge sector, i.e., the glueball masses attracts considerable attention. In the quenched approximation, these glueball are non $q\bar{q}$ gluonic bound states formed by strong self-interactions of the gluons, and their masses vary from about 1.4 Gev to 2.5 Gev. The flux tube models, bag models, sum rules and lattice techniques have been used to extract the glueball masses, among which lattice QCD seems to give more reliable estimates. Experimentally, a lot of glueball candidates such as $\iota(1440)$, $f_0(1520)$, $\theta/f_J(1720)$ and $\xi(2230)$, produced in the $J/\psi$ radiative decays [1, 2, 3] are within this range. The difficulty in experimental identification of a glueball comes from the complexity in determining the quantum numbers $J^{PC}$ of these particles.

The Monte Carlo simulation of lattice QCD on the Euclidean lattice has become a powerful and conventional method in spectrum calculations. Concerning the lightest glueball $0^{++}$, numerical data of 15 years ago, on rather small lattices with poor signal to noise ratio suggested $M(0^{++}) \approx 1$ Gev. The value of $M(0^{++})$ seems increasing with the lattice volume. Most recently, more accurate calculations by the IBM group [4] on much larger lattices, higher statistics and better algorithm gave $M(0^{++}) \approx 1.740 \pm 0.071$ Gev, where the infinite volume extrapolation has been made. For a review of the current status, see ref. [5].

Here we would discuss an alternative way [6, 7, 8] to extract the glueball masses and wavefunctions by solving the Schrödinger equation [8]

$$H|F\rangle = \epsilon_F |F\rangle.$$ (1)

Whereas our goal is to do concrete computations [9] in four dimensional QCD, in this paper we would like to discuss quantitatively the properties of three dimensional QCD, an interesting and relevant but much simple theory. We will then use the idea of dimensional reduction [10, 11, 12] to estimate the $0^{++}$ glueball mass in four dimensional QCD.

With the standard notations ($g$ being the lattice gauge coupling, $a$ the lattice spacing, $U_l = \exp(igaA_l)$ the gauge link, $E_l$ the color-electric field on the link $l$, $U_p$ the plaquette variable), the lattice Hamiltonian is given by

$$H = \frac{g^2}{2a} \sum_l E_l^\alpha E_l^\alpha - \frac{1}{ag^2} \sum_p Tr(U_p + U_p^\dagger - 2)$$ (2)

is the lattice version of the Yang-Mills Hamiltonian $\int d^{D-1}x(e^2 \vec{E} \cdot \vec{E} + e^{-2} \vec{B} \cdot \vec{B})/2$ with $e$ the continuum gauge coupling, $\vec{E}$ and $\vec{B}$ the color electric and
magnetic fields respectively. The glueball wavefunction in (1) is

$$|F\rangle = |F(U)\rangle - \frac{\langle\Omega|F(U)|\Omega\rangle}{\langle\Omega|\Omega\rangle} |\Omega\rangle$$

(3)

created by the gluonic operator $F(U)$ with the given quantum number $J^{PC}$ acting on the vacuum $|\Omega\rangle$. The vacuum wavefunction $|\Omega\rangle$ satisfies $H|\Omega\rangle = \epsilon_{\Omega}|\Omega\rangle$. An estimate of the glueball mass is then $M_{J^{PC}} = \Delta \epsilon = \epsilon_F - \epsilon_{\Omega}$.

In a series of papers [6, 7, 8], we developed a method for solving the lattice Schrödinger equation (1) in a new scheme which preserves the correct continuum behavior at any order of approximation. Physically, if one wants to well describe a glueball on the lattice, the size or the Compton length of a glueball, which is usually of the same order as that of a hadron, should be greater than the lattice spacing $a$. In other words, the low energy spectrum originates mainly from the long wavelength excitations. Our starting point is to embody this physical implication in the eigenvalue equation when solve it approximately. The philosophy is to have the correct long wavelength limit at any order of approximation. The advantage and reliability of such a method has been confirmed by the results of two-dimensional $\sigma$ models (in [13]), three-dimensional U(1) (in [14]), SU(2) (in [6, 15, 16]) and SU(3) (in [7, 8]) gauge theories: the results converge very rapidly, and even at very low truncation orders clear scaling windows for the vacuum wavefunction and mass gaps have been established. It is very exciting that for the $\sigma$ models, 2+1 D U(1), and 2+1 D SU(2) gauge theories, the vacuum wavefunction and the mass gap are in perfect agreement with the most recent Monte Carlo data. In our pioneering study of 2+1 D SU(3), we obtained the first estimates for the vacuum wavefunction [7] and the glueball masses [8]. Most of these results have been summarized in [17, 18].

We begin with recapitulating briefly our method. Suppose the ground state has the form $|\Omega\rangle = \exp[R(U)]|0\rangle$, with $|0\rangle$ being the fluxless bare vacuum and $R(U)$ being a linear combination of gauge invariant gluonic operators $G$. Substituting this, (2) and (3) into (1), we have the following exact eigenvalue equation for the operator $F(U)$

$$\sum_i \{[E_i, [E_i, F(U)]] + 2[E_i, F(U)][E_i, R(U)]]\} = \frac{2a\Delta \epsilon}{g^2} F(U).$$

(4)

In [6, 7, 8, 17, 18], we have illustrated how to obtain $R(U)$ and $F(U)$ by
expanding them in order of graphs (Wilson loops) $G_{n,i}(U)$:

$$F(U) = \sum_n F_n(U) = \sum_{n,i} f_{n,i} G_{n,i}(U),$$  \hspace{1cm} (5)

with $n$ being the order of the graphs. In practice, equation (4) has to be truncated to some finite order $N$

$$\sum_l \{[E_l, [E_l, \sum_n F_n(U)]] + 2 \sum_{n_1+n_2 \leq N} [E_l, F_{n_1}(U)][E_l, R_{n_2}(U)]\}$$

$$= \frac{2a\Delta\epsilon}{g^2} \sum_n F_n(U),$$  \hspace{1cm} (6)

from which the coefficients $f_{n,i}$ are determined.

The first term in (6) doesn’t create higher order graphs, while the second term generates new or higher order graphs of order $n_1 + n_2$. Therefore, one should carefully truncate the second term in this calculation. The essential feature of our approach is in the correct treatment of this second term. It has been generally proven [6] that in the long wavelength limit this term should behave as

$$[E_l, F_i(U)][E_l, R_j(U)] \propto a^6 \text{Tr}(\mathcal{D}\mathcal{F}_{\mu,\nu})^2.$$  \hspace{1cm} (7)

Not to violate this behavior, when the equation (6) is truncated to the $N$th order, all the graphs created by $[E_l, F_i(U)][E_l, R_j(U)]$ for $n_1 + n_2 > N$ must be discarded. For example, equation (6) truncated to $N = 2$ is

$$\sum_l \{[E_l, [E_l, F_1 + F_2]] + 2[E_l, F_1][E_l, R_1]\} = \frac{2a\Delta\epsilon}{g^2} (F_1 + F_2).$$  \hspace{1cm} (8)

For $N=3$, the truncated equation (6) is

$$\sum_l \{[E_l, [E_l, F_1 + F_2 + F_3]] + 2[E_l, F_1][E_l, R_1] + 2[E_l, F_1][E_l, R_2]$$

$$+ 2[E_l, F_2][E_l, R_1]\} = \frac{2a\Delta\epsilon}{g^2} (F_1 + F_2 + F_3).$$  \hspace{1cm} (9)
Higher order truncated eigenvalue equations satisfy the same rule. From an eigenvalue equation at order $N$, we derive a set of nonlinear algebraic equations for the coefficients $f_{n,i}$ of the operators. Solving these equations, we obtain not only the wavefunction of the glueball, but also the glueball mass at order $N$. (For the vacuum wavefunction, see ref. [7]). It is worth mentioning another advantage of such an approach: no group integration is necessary so that higher order calculations are feasible. The difference between different truncation orders is the estimate for the systematic error in the calculation.

For a non-abelian gauge theory, the element $A$ of the gauge group has to satisfy the uni-modular condition [7] (or Caley Hamilton relation [19]), which is for SU(3)

$$A_{ij}A_{kl}A_{mn} \epsilon_{jln} = \epsilon_{ikm}. \quad (10)$$

Multiplying it by $A_{pi}^\dagger$, and then summing over the $i$ index, it becomes

$$A_{kl}A_{mn} \epsilon_{pln} = A_{pi}^\dagger \epsilon_{ikm} \quad (11)$$

Multiplying it again by $\epsilon_{pqr}$ and summing it over the $p$ index, we obtain

$$A_{il}A_{kj} = A_{ij}A_{kl} - Tr A_{il}^\dagger (\delta_{ji} \delta_{kl} - \delta_{kj} \delta_{il}) - A_{kl}^\dagger (\delta_{il} \delta_{kj} - \delta_{ij} \delta_{kl}) + A_{kl}^\dagger \delta_{ij}. \quad (12)$$

These formulae are useful in classification of graphics. Because of these conditions, one should choose properly an independent set from the graphs generated by the second term in (6). Mathematically, any independent set chosen in this way can be used in the calculation.

Physically, the connected graphs represent more coherence and have less mixing with lower order graphs. It was shown in [16] that the use of connected set makes the convergence of the results much faster than the use of disconnected set [15] in a (2+1)-dimensional SU(2) model.

For the realistic gauge group SU(3), the complication is that not all the disconnected graphs can be transformed to the connected ones. However, we observed that if more disconnected graphs are transformed according to the uni-modular conditions (10), (11) or (12) into the connected ones [7], the scaling behavior was much better. We have also tested several sets of operators [20]. One of them is classified according to inverse of the graphs

$$G_{n,j}^I \propto Inv[G_{n,j}]$$

generated by the second term in (6) with the uni-modular
conditions taken into account. (For the definition of the inverse operator, see ref. [10]).

Since QCD is a super-renormalizable gauge theory, the renormalization requirements amounts to dimensional analysis. In the weak coupling region (for large \( \beta = 6/g^2 \)), because the renormalized charge \( e \) and the bare coupling are related by \( g^2 = e^2 a \), dimensional analysis tells us that the dimensionless masses \( aM_{JPC} \) should scale as

\[
\frac{aM_{JPC}}{g^2} \rightarrow \frac{M_{JPC}}{e^2} \approx \text{const.}, \tag{13}
\]

from which the continuum physical glueball masses \( M_{JPC} \) are extracted.

Using the techniques in [8, 20] and after a careful analysis of our results, we obtain the value for \( M(0^{++})/e^2 \) more accurate than the previous paper [8], and estimate the systematic error due to the finite \( N \) truncation to be less than 0.06. The validity of equation (13) extends from \( \beta = 5 \) to \( \beta = 12 \), and in this range

\[
\frac{M(0^{++})}{e^2} \approx 2.15 \pm 0.06, \tag{14}
\]

where the error denotes the systematic uncertainties due to the finite order truncation [in [8], we obtained \( M(0^{++})/e^2 \approx 2.1 \) for \( \beta \in [5, 8) \) at third order approximation using the connected graphs as an independent basis]. Our value for the lightest glueball can be compared with Samuel’s recent result [12] from the 2+1 D Hamiltonian QCD in the continuum: \( M(0^{++})/e^2 \approx 1.84 \pm 0.46 \).

One may also understand the relation between the glueball mass and the confinement scale from the vacuum wavefunction. The vacuum functional, which interpolates the strong and weak coupling regimes, are [21, 12]

\[
|\Omega\rangle = \exp\left\{ \frac{1}{2e^2} \int d^{D-1}x \text{ tr}[\mathcal{F}_{ij}(\mathcal{D}_k\mathcal{D}_k + \xi^{-2})^{-1/2}\mathcal{F}_{ij}] \right\}, \tag{15}
\]

with \( \mathcal{F} \) being the field strength tensor in spatial dimensions and \( \mathcal{D} \) the covariant derivative. The correlation length \( \xi \), with dimension of inverse mass, is proportional to \( e^{-2} \), i.e., the confinement scale in the vacuum. It is suggested in ref. [12] that \( \xi^{-1} \) might also be related to the constituent gluon mass and the lightest glueball mass. In the strong coupling limit or large \( N_c \) (number
of colors) limit, it reduces to the strong coupling wavefunction obtained by [22, 10],

$$|\Omega\rangle = \exp[-\mu_0 \int d^{D-1}x \, tr F^2].$$  \hspace{1cm} (16)

In the intermediate and weak coupling, it becomes [21, 6, 7]

$$|\Omega\rangle = \exp[-\frac{\mu_0}{e^2} \int d^{D-1}x \, tr F^2 - \frac{\mu_2}{e^6} \int d^{D-1}x \, tr (DF)^2],$$  \hspace{1cm} (17)

which is just our vacuum wavefunction for the long wavelength configurations [6, 7]. The correlation length has a relation with the coefficients $\mu_0$ and $\mu_2$:

$$\xi = \left(\frac{-2\mu_2}{\mu_0}\right)^{1/2}. \hspace{1cm} (18)$$

For 2+1 D SU(2), $\xi = 0.65/e^2$ (see refs. [21, 6]), while for 2+1 D SU(3), our result [7] is $\xi = 0.53/e^2$. If the glueball mass is proportional to the constituent gluon mass, from the difference of the scales between SU(2) and SU(3), one may also guess $M(0^{++})/e^2 \approx 2$, consistent with the result (14) from our practical calculation.

Combining the most recent Monte Carlo data [23] for the string tension $\sigma$ in QCD$_3$, which is $\sqrt{\sigma} = (0.554 \pm 0.004)e^2$, we obtain the ratio of the $0^{++}$ glueball mass over square root of the string tension in the continuum limit

$$\frac{M_{0^{++}}}{\sqrt{\sigma}} \approx 3.88 \pm 0.11. \hspace{1cm} (19)$$

Now we follow the argument of dimensional reduction [10, 11, 12]. In a confining theory in $D$ space-time dimensions with $2 < D \leq 4$, the function in the exponential of the vacuum functional acts like an action of an effective field theory in $D - 1$ dimensions. In other words, in computing vacuum expectation values, a confining theory in $D$ dimensions becomes a localized field theory in $d = D - 1$ dimensions. This can be exactly proven in the strong coupling or large $N_c$ limit. Because in this limit, the fixed time vacuum expectation value of a operator $O(U)$ in $D$ dimensions is

$$\frac{\langle \Omega | O(U) | \Omega \rangle}{\langle \Omega | \Omega \rangle} \rightarrow \frac{\int [dU] O(U) \exp[-2\mu_0 \int d^{D-1}x \, tr F^2]}{\int [dU] \exp[-2\mu_0 \int d^{D-1}x \, tr F^2]}, \hspace{1cm} (20)$$

\text{7}
corresponding to the path integral expression for $< O(U) >$ in $D - 1$ dimensional lattice field theory. It has been argued [12, 24, 25] that 3+1 D theory can still be approximated by its 2+1 D theory for long wavelength configurations in comparison to the confinement scale. According to this argument, $M(J^{PC})/\sqrt{\sigma}$ for the lightest glueball should be approximately the same for 2+1 and 3+1 dimensions. Since for SU(3) the number of color is larger, and the measured length $\xi$ in the vacuum functional (17) is smaller than that for SU(2), our speculation is that the approximation is better for SU(3) gauge theory. In fact, equation (19) is consistent with IBM data $M(0^{++})/\sqrt{\sigma} = 3.95$ from Monte Carlo simulation of 3+1 D lattice QCD, providing $\sqrt{\sigma} = 0.44$ Gev is used. From this world average value for the string tension and (19), we expect

$$M(0^{++}) = 1.71 \pm 0.05 \text{ GeV},$$

in nice agreement with the IBM data $M(0^{++}) = 1.740 \pm 0.071$ [4]. This favors $\theta/f_J(1710)$ as a candidate of the $0^{++}$ glueball.

In conclusion, using the eigenvalue equation method developed in [6, 7], we obtain accurate result for the lightest glueball mass in (2+1)-dimensional QCD, with systematic uncertainty under well control. We also use the idea dimensional reduction to extrapolate the results to QCD in 3+1 dimensions. Perfect agreement with the most accurate Monte Carlo data indicates that QCD$_3$ is not just a toy model for QCD$_4$, and their relation should be more deeply understood.

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