Ambiguities in QED:
Renormalons versus Triviality

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Abstract

We point out that, contrary to what is believed to hold for QCD, renormalons are genuine in QED; i.e. the ambiguities which come with them do not require cancellation by hypothetical non-perturbative contributions. They are just the ambiguities characteristic of any trivial —and thus effective— theory. If QED remained an isolated theory up to an energy close to its triviality scale, these ambiguities would surely hint at new physics. This not being so, the renormalon ambiguities in QED lead to no new physics, not even to non-perturbative contributions within QED itself.

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Exactly as for $\lambda \phi^4$ there are two perturbative quantum-field theoretical frameworks for dealing with QED. If the approximation one uses is triviality insensitive, such as finite-order perturbation theory, one can take the continuum limit by removing the UV cutoff after having renormalized the theory order by order in perturbation theory. This procedure leads to quantitatively extremely successful results, the paradigm being the lepton anomalous magnetic moments. If the approximation used is triviality sensitive, such as a summed-up infinite order perturbative computation, one has to consider the theory as an effective one, with an UV cutoff which cannot be removed. If one insists in removing it, triviality makes the theory uninteresting.

In 1952 Dyson[1] gave a physically motivated argument leading to the conclusion that in QED renormalized perturbation theory has zero radius of convergence. Nowadays one believes that the series is not only divergent, but also non Borel-summable. As stressed by Parisi[2] this pathology is related to the existence of the Landau pole[3], or, in other words, to the fact that perturbation theory cannot be trusted at large momenta. The perturbative predictions are thus ambiguous. (To be sure Dyson himself pointed out that due to the smallness of $\alpha$ these ambiguities would not prevent practical calculations being made with the series, to an accuracy far beyond current or even future experimental requirements.) These ambiguities can also be seen to emerge as singularities of the Borel transform of the perturbative series, called renormalons. They have the structure of zero momentum insertions of local operators[2].

Renormalons were also introduced by 't Hooft to further the understanding of QCD[4]. Unlike QED, where renormalons are only of UV origin, there are both IR and UV renormalons in QCD. The first ones lead to singularities, and thus ambiguities, in perturbation theory, which call for some non-perturbative contributions[5]. (For instance, for heavy quarks, the usefulness of the notion of a pole mass[6] has been questioned by the unavoidable ambiguity of order $\Lambda_{QCD}$ which goes with it[7].) The second ones lead to ambiguities in matrix elements of local operators. It is now believed that renormalons appear because the usual (and useful) separation into coefficient functions and matrix elements (short and long distance contributions, respectively) is not a clear cut one when regulators without a hard cut-off are used. When a complete treatment is possible, IR and UV renormalons eventually cancel, leaving an unambiguous prediction for physical quantities[8]. These are good news for QCD. Unfortunately, it is not so easy to pin down the diagrams which in perturbation theory lead to singularities in the Borel plane, let alone to use these families of diagrams to somehow “estimate” the natural size of non-perturbative corrections. In practice one uses QED plus what is called
“naive non-abelianization” to evaluate them; but this approximation is somewhat sus-
pect, as there is no known expansion leading to such a simplification. Furthermore, one
often makes the logical leap that the ambiguities in the non-perturbative contributions
are of the order of the non-perturbative contributions themselves.

In QED everything is different, and this is what we want to point out here. There is
a well defined expansion which leads to renormalons, namely the expansion in inverse
powers of the number of leptons. The renormalon ambiguities appear because one
integrates over all momenta, being cavalier with respect to the effective character of
the theory (e.g. by using dimensional regularization). If, on the contrary, one treats
the theory as an effective one, with an unremovable UV cutoff, the ambiguities in summed-
up renormalized perturbation theory are the ones we expect in any effective theory. As
we will see in what follows, using this expansion one can stay well within the framework
of convergent perturbation theory and so the issue of non-perturbative contributions
which cancel the ambiguities simply does not pose itself. Contrary to QCD, the QED
ambiguities are genuine but irrelevant. No non-perturbative or indeed any physics can
be learned from them.

An effective field theory with an unremovable dimensionful UV cutoff $\Lambda$ may provide
accurate predictions, but not infinitely accurate ones[9], as some dependence on $\Lambda$
remains. We shall consider the large number of leptons limit of QED, $N_l \to \infty$, all of
them with the same mass $m$ for simplicity. QED is then basically solvable[10]. At the
leading order in the expansion the sum of all diagrams is equivalent to substituting the
coupling constant by the running coupling constant, which is given by

$$\alpha(-q^2) = \frac{\alpha_0(\Lambda^2)}{1 + \Pi_0(-q^2, m^2, \Lambda^2)}$$

(1)

$\Pi_0$ is the usual one-loop photon bare self-energy. For $\Lambda^2 \gg -q^2 \gg m^2$, eq. 1 reduces
to

$$\alpha(-q^2) \simeq \frac{\alpha_0(\Lambda^2)}{1 - \frac{\alpha_0(\Lambda^2)}{3\pi} \log \frac{-q^2}{\Lambda^2}}$$

(2)

In the above expressions $\alpha_0(\Lambda^2)$ is the bare coupling, or, if one wishes, the running
coupling constant defined at scale of the cutoff. The number of leptons has been
absorbed in the coupling, and is not explicitly written. The fine structure constant is
the zero energy limit of $\alpha(-q^2)$

$$\alpha \equiv \alpha(0) \simeq \frac{\alpha_0(\Lambda^2)}{1 - \frac{\alpha_0(\Lambda^2)}{3\pi} \log \frac{m^2}{\Lambda^2}}$$

(3)
The running coupling constant in the large $N_l$ limit has two interesting features. The first one is triviality: $\alpha(-q^2)$ cannot be independent of $\Lambda$ unless it is zero. Indeed, from eq. 2 it is clear that, irrespective of $\alpha_0(\Lambda^2)$,

$$\lim_{\Lambda \to \infty} \alpha(-q^2) = 0$$

(4)

In other words, there is no strict scaling region for the interacting theory. For theories for which the sign in the denominator is the opposite there is no obstacle in taking $\Lambda \to \infty$ while $\alpha(-q^2)$ is kept fixed. It is enough to ensure that $\alpha_0(\Lambda^2)$ goes to zero adequately. This leads to a theory which interacts even when the UV cutoff is removed, thanks to asymptotic freedom. And it leads to the existence of renormalization group invariants, such as $\Lambda_{QCD}$. On the contrary, there are no genuine renormalization group invariants in large $N_l$ QED, except those corresponding to a trivial free theory, such as the electron mass.

There is however a limited scaling region. It is given by $\Lambda < \Lambda_S$, where

$$\Lambda_S \simeq m \exp \frac{3\pi}{2\alpha}$$

(5)

This stems from requiring that eq. 3 is invertible, i.e. that an $\alpha_0(\Lambda^2)$ exists such that $\alpha$ is $\Lambda$-independent. Indeed for $\Lambda < \Lambda_S$

$$\alpha_0(\Lambda) \simeq \frac{\alpha}{1 + \frac{\alpha}{3\pi} \log \frac{m^2}{\Lambda^2}}$$

(6)

At $\Lambda = \Lambda_S$ $\alpha_0(\Lambda^2)$ diverges. This can only be avoided by lowering $\alpha$ from its physical value $\alpha = 1/137$. This again leads to triviality, but in a way in which one never uses the relation between $\alpha$ and $\alpha_0(\Lambda^2)$ beyond the radius of convergence of the corresponding perturbative series, while eq. 4 uses eq. 2 beyond the radius of convergence. Therefore in the large $N_l$ limit the relation given by eq. 6 is an exact one, free from any hypothetical non-perturbative corrections. From eq. 3 one also sees that massless QED ($m = 0$) implies $\alpha = 0$, but $\alpha(-q^2 \neq 0) \neq 0$. In other words, in massless QED the electron is completely screened at long distances.

As long as $\Lambda < \Lambda_S$ one can rewrite eq. 2 in the form of renormalized perturbation theory. For $-q^2 \gg m^2$

$$\alpha(-q^2) \simeq \frac{\alpha}{1 - \frac{\alpha}{3\pi} \left( \log \frac{-q^2}{m^2} - \frac{5}{3} + \mathcal{O}\left(\frac{m^2}{q^2}\right) \right)}$$

(7)

Although this expression looks $\Lambda$-independent, it is, for fixed $\alpha$, valid only if $\Lambda < \Lambda_S$. At this point one should mention that the actual value of $\Lambda_S$ is regulator dependent.
Changing the regularization scheme may modify the constant pieces accompanying the logs in eqs. 2, 3 (which we have not written up) and thus the relation between $\Lambda_S$, $m$ and $\alpha$ given by eq. 5, but only by subleading corrections. Fixing unambiguously these corrections would require considering the relation between $\alpha$ and $\alpha_0(\Lambda^2)$ at the next-to-leading order in the large $N_l$ expansion. Such a relation is available in the literature (see the third reference in [10]). It should be mentioned that the next-to-leading beta function in the large $N_l$ expansion is given by a series in $\alpha$ with a finite radius of convergence.

The second remarkable feature of the running coupling constant is the existence of the Landau pole. The running coupling constant blows up at high energies. From eq. 7 one finds the Landau pole at $\sqrt{-q^2} = \Lambda_L \simeq \Lambda_S$. Since the difference between $\Lambda_L$ and $\Lambda_S$ is regulator dependent, we can take $\Lambda_L = \Lambda_S$ as the effective theory is anyway quantitatively reliable only for $-q^2 \ll \Lambda^2$ and the regulator ambiguities are thus small. For exactly the same reasons the effective theory will never reach the Landau pole in its range of validity.

Let us now forgo the effective character of QED. Consider an observable like the anomalous magnetic moment, given in the large $N_l$ limit by[11]

$$a = \frac{\alpha}{\pi} \int_0^1 dx \frac{1-x}{1-\frac{\alpha}{\pi}f(x)}$$

where

$$f(x) = \frac{1}{3}[(1 - \frac{6}{x^2} + \frac{4}{x^3}) \log(1-x) - \frac{5}{3} - \frac{4}{x} + \frac{4}{x^2}]$$

with $-q^2/m^2 = x^2/(1-x)$. Thus, $-(\alpha/\pi)f(x)$ is the renormalized self-energy, and

$$\frac{\alpha}{1 - \frac{\alpha}{\pi}f(x)}$$

is nothing but $\alpha(-q^2)$. All divergences have been subtracted in perturbation theory and, indeed, at any finite order in $\alpha a$ is finite and unambiguous. (Up to terms of order higher than those actually evaluated. These are of course the usual perturbative ambiguities related to the choice of a renormalization scheme.) However, after summing up perturbation theory, the integrand in the above equation has obviously a singularity at the location of the Landau pole

$$x_L = 1 - e^{-\frac{3\pi}{2} - \frac{5}{3}} - 4e^{2(-\frac{3\pi}{2} - \frac{5}{3})} + \ldots$$

This singularity corresponds to the UV renormalon of QED[12]. The prescription to handle this singularity is of course ambiguous. For instance, following ref [12] one could
modify the integrand to
\[ a = \frac{\alpha}{\pi} \int_0^1 dx \frac{x_L - x}{1 - \frac{\alpha}{\pi} f(x)} \] (12)

The modification is non-analytic and invisible in perturbation theory, but cancels the singularity. But we could have taken another prescription, such as the principal part. This would also lead to a well defined —but different— result. The difference is of \( O((1 - x_L)^2) \). Do we learn anything about the possible non-perturbative structure of the theory from this? No.

In order to see this we have to remember that QED is an effective theory. Then eq. 8 should be written as
\[ a = \frac{\alpha}{\pi} \int_0^{1 - k^2 \Lambda^2/S} dx \frac{1 - x}{1 - \frac{\alpha}{\pi} f(x)}, \quad k \gg 1, \quad \log k \ll \frac{1}{\alpha} \] (13)

showing that the momentum is only integrated up to \(-q^2 = \Lambda^2 = \Lambda^2_S/k\), as befits an effective theory. Thus one never integrates over the location of the Landau pole and no ambiguities arise because of it. Furthermore, the perturbative series is used within its radius of convergence and hence there is no question of possible non-perturbative contributions. However the result is of course not infinitely accurate because the cutoff is arbitrary. The uncertainty is of the order
\[ \Delta a \simeq k^2 m^4 \Lambda^4_S \] (14)

(modulo logarithmic corrections). If one pushes the UV cutoff to its largest possible value, \( \Lambda_S \), the uncertainty becomes of the form
\[ \Delta a \simeq \exp \left( -\frac{6\pi}{\alpha} \right) \] (15)

exactly as the ambiguity due to the Landau pole. The ambiguity disappears when \( \Lambda \to \infty \), but then the anomalous magnetic moment \( a \) vanishes because \( \alpha \to 0 \).

Of course the result of the previous equation is purely perturbative and it is obtained without ever leaving the region of convergence of the perturbative series in the renormalized coupling. Because there are no non-perturbative contributions, the previous result does not lead to any non-perturbative insight into the theory. It is just a reflection of triviality.

Nothing prevents us from going to next order in the large number of leptons expansion[13]. One encounters subdiagrams which lead to the integral we have just been discussing and the Landau singularity is still there. These subdiagrams have also
to be cutoff at a scale $\Lambda^2$ as corresponds to the effective character of QED (or, more specifically, of large $N_l$ QED). The analysis goes through much as before.

A nice feature of QED is that the leading and next-to-leading contributions in the large number of leptons expansion can be worked out explicitly. This is unlike in QCD where no known systematic expansion displays the renormalon. It has been advocated that in QCD the analysis should mimic the one of QED but replacing in an ad-hoc manner $N_l/3$ by $\beta_1$, the first coefficient of the beta function or even by the beta function up to two loop order. The consequences of such a daring approach have been analyzed recently[14]. Nobody really knows whether such replacements are meaningful for QCD.

In conclusion, the renormalon ambiguities of QED are nothing but the ambiguities characteristic of a trivial, and thus effective, theory. If QED remains an isolated theory, they are there to stay. They will not be cancelled by any other ambiguities. They do not contain non-perturbative physics. They are just what characterizes triviality. Furthermore in the specific case of QED they are even of no use as harbingers of new physics, as at energies well below $\Lambda_S$ (where any ambiguity is ridiculously small) the theory is subsumed by the Weinberg Salam theory. This theory has its own Landau poles, but this is a different story. In the specific case of the lepton anomalous magnetic moment, a numerically important part originates from the hadronic contribution to the vacuum polarization $\Pi_0$, which has not been considered either.

These results are not expected to be different for real QED, away from the large number of leptons limit. They have been obtained within a well defined expansion and within the radius of convergence of perturbation theory. They also show that neither Landau poles nor renormalons are problems in need of solving.

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