A Novel Symmetry in Sigma models

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Abstract

A class of non-linear sigma models possessing a new symmetry is identified. The same symmetry is also present in Chern-Simons theories. This hints at a possible topological origin for this class of sigma models. The non-linear sigma models obtained by non-Abelian duality are a particular case in this class.

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1. **Introduction**

Non-linear sigma models in two dimensions possess remarkable features due to their rich symmetries. The symmetry properties a sigma model can have depend very much on the form of the metric and the torsion tensors. The most studied non-linear sigma model is the Wess-Zumino-Novikov-Witten (WZNW) model. This model enjoys an extra symmetry, generating a current algebra [1], precisely when the metric and the torsion tensors take the very specific forms

\[
G_{ij} = e_i^a e_j^b \eta_{ab} = \bar{e}_i^a \bar{e}_j^b \eta_{ab} \\
H = -\frac{1}{3} \text{tr} (e \wedge e \wedge e) = \frac{1}{3} \text{tr} (\bar{e} \wedge \bar{e} \wedge \bar{e}) ,
\]

(1)

where we have introduced the differential forms on the group manifold [2]

\[
e = T_a e_i^a d\phi^i = g^{-1} dg \\
\bar{e} = T_a \bar{e}_i^a d\phi^i = -dgg^{-1} .
\]

(2)

The torsion tensor \( H = \frac{1}{3!} H_{ijk} d\phi^i \wedge d\phi^j \wedge d\phi^k \) is related to the antisymmetric tensor field \( B = \frac{1}{2!} B_{ij} d\phi^i \wedge d\phi^j \) by the usual relation \( H = \frac{1}{2} dB \). Here \( \eta_{ab} \) is the invariant bi-linear form of the Lie algebra generated by \( T_a \) and \( g \) is a group element parametrised by \( \phi^i \).

In fact one can ask whether there exist other forms for \( G_{ij} \) and \( H_{ijk} \) which would lead to other forms of current algebras. This question was answered in refs.[3, 4, 5, 6, 7] where a generalisation of the above expressions for the metric and the torsion were found. The WZNW model is then just a particular case of this generalisation.

We explore, in this paper, the possibility of finding other non-linear sigma models that might have further symmetries depending on the forms of \( G_{ij} \) and \( B_{ij} \). Indeed, we identify a class of sigma models which have a very interesting symmetry. Remarkably, the same symmetry appears in Chern-Simons theories in three dimensions. This hints at a possible connection between the two theories.

2. **The new symmetry**

Consider the action for a general bosonic two-dimensional non-linear sigma model

\[
S(\varphi) = \int d^2 x \sqrt{\gamma} \left( \gamma^{\mu\nu} G_{ij} (\varphi) \partial_\mu \varphi^i \partial_\nu \varphi^j + \bar{e}^{\mu\nu} B_{ij} (\varphi) \partial_\mu \varphi^i \partial_\nu \varphi^j \right) .
\]

(3)
In this equation $\gamma_{\mu\nu}$ is the metric on the two-dimensional world sheet, $\gamma$ is its determinant and $\tilde{\epsilon}^{\mu\nu} = \frac{1}{\sqrt{\gamma}} \epsilon^{\mu\nu}$ is the alternating tensor. This action can be written as

$$S(\varphi) = \int d^2 x \sqrt{\gamma} \left( \tilde{\epsilon}^{\mu\nu} \eta_{ij} A_{\mu} A_{\nu} \right),$$

where we have introduced the gauge field-like quantity $A_{\mu}^i$

$$A_{\mu}^i = R_{\mu\nu} \eta_{jk} \tilde{\epsilon}^{\nu\alpha} \partial_{\alpha} \varphi^k,$$

$$R_{\mu\nu}^i = \eta^{ik} \eta^{j\ell} (\gamma_{\mu\nu} G_{kl} + \tilde{\epsilon}_{\mu\nu} B_{kl})$$

with $\eta_{ij}$ a symmetric field-independent metric whose inverse is $\eta^{ij}$. Suppose now that $\eta_{ij}$ is the invariant bi-linear form of a Lie algebra whose structure constants we denote by $f_{ijk}$ (which means that $\eta_{ij} f^j_{ik} + \eta_{kj} f^j_{il} = 0$).

We would like to investigate under which conditions the action (4) has a symmetry of the form

$$\delta \varphi^i = f^i_{jk} \xi^j F_{\mu\nu}^{ik} \tilde{\epsilon}^{\mu\nu},$$

where $\xi^j(x)$ is the infinitesimal gauge parameter and $F_{\mu\nu}^{ik} = \partial_{\mu} A_{\nu}^i - \partial_{\nu} A_{\mu}^i + f_{jkl} A_{\mu}^j A_{\nu}^k$ is the field strength of the gauge field $A_{\mu}^i$ as given by (5). The transformation is suggested by the form of the action (4). The same kind of symmetry was identified in the context of non-Abelian duality in sigma models [8] and in non-Abelian gauge theories [9].

We found that the action is invariant, up to a total derivative, provided that the metric $G_{ij}$ and the antisymmetric tensor $B_{ij}$ satisfy

$$\partial_k R_{\mu\nu}^{ij} = \eta_{kl} f^l_{mn} R_{\mu\rho}^{m\alpha} R_{\nu\beta}^{n\alpha} \tilde{\epsilon}^{\alpha\beta}.$$  

This condition can of course be expressed explicitly as two separate conditions on $G_{ij}$ and $B_{ij}$

$$\partial_k G_{ij} = f^l_{km} \eta^{mn} (G_{li} B_{nj} - G_{nj} B_{li})$$

$$\partial_k B_{ij} = -f^l_{km} \eta^{mn} (G_{li} G_{nj} + B_{li} B_{nj})$$

This shows the special geometry of this class of sigma models. In particular, the Riemann tensor and the torsion will be given in closed forms as products of $G_{ij}$, $G^{ij}$ and $B_{ij}$.

Under these conditions, the equations of motion of the non-linear sigma model lead to

$$\tilde{\epsilon}^{\mu\nu} F_{\mu\nu}^{ik} = 0.$$  

Therefore the above transformation vanishes on-shell. As seen later, this equation can also be thought of as deriving from a Chern-Simons theory.
It is straightforward to find the unique solution to the symmetry invariance condition in (7). In order to do this, we denote by $\tilde{R}^{\mu\nu}_{ij}$ the inverse of $R^{ij}_{\mu\nu}$ (that is, $R^{ij}_{\mu\nu}\tilde{R}^{\nu\sigma}_{jk} = \delta^\sigma_\mu\delta^i_j$). Equation (7) is then cast into the first order differential equation

$$\partial_k \tilde{R}^{\mu\nu}_{ij} = -\eta_{kl} f^l_{ij} \varepsilon^{\mu\nu}$$

whose general solution is given by

$$\tilde{R}^{\mu\nu}_{ij} = -\left[ N^{\mu\nu}_{ij} + \varepsilon^{\mu\nu} \eta_{kl} f^l_{ij} \varphi^k \right],$$

where $N^{\mu\nu}_{ij} = N_{ji}^{\nu\mu}$ is any field-independent matrix, and in general $N^{\mu\nu}_{ij} = \gamma^{\mu\nu} A^i_j + \varepsilon^{\mu\nu} C_{ij}$.

The action can be cast into a form which is familiar in the context of non-Abelian duality [10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. By extracting $\partial^\mu \varphi^i$ from (5) and eliminating it in (4), one finds, after some straightforward manipulations, the following action

$$S(\varphi, A) = \int d^2 x \sqrt{\gamma} \left( N^{\mu\nu}_{ij} A^i_\mu A^j_\nu + \varepsilon^{\mu\nu} \eta_{ij} F_{\mu\nu} \varphi^j \right),$$

where we have ignored a total derivative. If one treats $A^i_\mu$ and $\varphi^i$ as independent variables then the equations of motion for $A^i_\mu$ are precisely those in (5). Indeed, this action is obtained (when $A^i_\mu$ and $\varphi^i$ are independent) by performing a non-Abelian duality transformation on the following action

$$S(g) = \int d^2 x \sqrt{\gamma} N^{\mu\nu}_{ij} \eta^{ik} \eta^{jl} \text{tr} \left( T_k g^{-1} \partial^\mu g \right) \text{tr} \left( T_l g^{-1} \partial^\nu g \right),$$

where $T_i$ are the generators of the Lie algebra $[T_i, T_j] = f^k_{ij} T_k$, $g$ is a Lie group element and tr is the invariant bi-linear form $\text{tr} (XY) = \delta_{ij} X^i Y^j$. This action is invariant under the global transformation $g \rightarrow hg$. The non-Abelian dual theory is obtained by gauging this symmetry and at the same time restricting the gauge field strength to vanish [20]. We therefore obtain the action

$$S(g, \varphi, A) = \int d^2 x \sqrt{\gamma} \left( N^{\mu\nu}_{ij} \eta^{ik} \eta^{jl} \text{tr} \left( T_k g^{-1} D^\mu g \right) \text{tr} \left( T_l g^{-1} D^\nu g \right) + \varepsilon^{\mu\nu} \text{tr} (\varphi F_{\mu\nu}) \right).$$

The covariant derivative is $D^\mu g = \partial^\mu g + A^\mu g$ with $A^\mu \rightarrow h^{-1} A^\mu h - \partial^\mu hh^{-1}$, the gauge field is $A^\mu = T_i A^i_\mu$ and the Lagrange multiplier is $\varphi = T_i \varphi^i$ with $\varphi \rightarrow h \varphi h^{-1}$. The gauge invariance allows us to choose a gauge such that $g = 1$. In this gauge $S(g, \varphi, A)$ reproduces precisely $S(g, A)$ as given by (12).

The dual of the principal chiral model is a special case of this construction [21, 22, 23]. The dual of the chiral model is obtained when $A_{ij} = \eta_{ij}$ and $C_{ij} = 0$. 

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3. Generalisation

Another important case is obtained by splitting the field $\varphi^i$ of the previous non-linear sigma model as $\varphi^i = (X^a, Y^A)$ and restricting the transformation to the fields $X^a$ only. In this case, the non-linear sigma model action takes the form

$$S(X,Y) = \int d^2x \sqrt{\gamma} \left( \hat{e}^{\mu\nu} \eta_{ab} A^a_{\mu} \partial_{\nu} X^b + \hat{e}^{\mu\nu} \varepsilon_{\alpha\beta} M^A_{\mu\alpha} \eta_{ab} L_{AB} \partial_{\beta} Y^B \partial_{\nu} X^b + R_{AB} \partial_{\mu} Y^A \partial_{\nu} Y^B \right),$$  \hspace{1cm} (15)

The gauge field $A^a_{\mu}$ involves both $X^a$ and $Y^A$ through

$$A^a_{\mu} = R^{ab}_{\mu} \varepsilon^{a\alpha} \eta_{bc} \partial_{\alpha} X^c + R^{aA}_{\mu} \varepsilon^{a\alpha} L_{AB} \partial_{\alpha} Y^B.$$  \hspace{1cm} (16)

The different quantities introduced here can depend on both $X^a$ and $Y^A$ and are such that

$$R_{\mu\nu}^{ab} = \eta^{ac} \eta^{bd} [\gamma_{\mu\nu} G_{cd} + \hat{\epsilon}_{\mu\nu} B_{cd}],$$

$$R_{AB}^{\mu\nu} = [\gamma_{\mu\nu} G_{AB} + \hat{\epsilon}_{\mu\nu} B_{AB}],$$

$$R^{aA}_{\mu\nu} + M^{aA}_{\mu\nu} = 2 \eta^{ab} L_{AB} [\gamma_{\mu\nu} G_{bB} + \hat{\epsilon}_{\mu\nu} B_{bB}].$$  \hspace{1cm} (17)

The symmetric matrices $\eta_{ab}$ and $L_{AB}$ are field-independent and their inverses are, respectively, $\eta^{ab}$ and $L^{AB}$. We then suppose that $\eta_{ab}$ is a bi-linear invariant form of a Lie algebra with structure constants $f^a_{bc}$.

Let us find the conditions under which the sigma model (15) is invariant under

$$\delta X^a = f^{a}_{bc} \xi^b F^{c}_{\mu\nu} \varepsilon^{\mu\nu}, \quad \delta Y^A = 0.$$  \hspace{1cm} (18)

We find that the action remains invariant, up to a total derivative, when the following conditions are fulfilled

$$M^{aA}_{\mu\nu} = R^{aA}_{\mu\nu},$$

$$\partial_c R^{ad}_{\mu\alpha} = \eta_{ab} f^{b}_{er} R^{ae}_{\mu\sigma} R^{dr}_{\alpha\tau} \varepsilon^{\sigma\tau},$$

$$\partial_c R^{aA}_{\mu\alpha} = \eta_{ab} f^{b}_{er} R^{ae}_{\mu\sigma} R^{A}_{\alpha\tau} \varepsilon^{\sigma\tau},$$

$$\partial_c R^{\mu\alpha}_{AB} = \eta_{ab} f^{b}_{er} L_{AE} L_{BD} R^{E}_{\alpha\sigma} R^{D}_{\beta\tau} \varepsilon^{\beta\nu} \varepsilon^{\alpha\tau}.$$  \hspace{1cm} (19)

Notice that this set of equations cannot be obtained from (7) by simply splitting the field $\varphi^i$ as $(X^a, Y^A)$. The second equation of this set has the unique solution for the inverse of $R^{ab}_{\mu\nu}$, namely $\tilde{R}^{\mu\nu}_{ab}$, given by

$$\tilde{R}^{\mu\nu}_{ab} = - \left[ \eta^{\mu\nu} (Y) + \hat{\epsilon}^{\mu\nu} \eta_{cd} f^{c}_{ad} X^d \right]$$  \hspace{1cm} (20)
The general solution of the remaining last two equations is provided by

\[ R^a_{\mu \nu} = R^b_{\mu \alpha} W^a_{\nu \beta} (Y) \]
\[ R^\mu_{AB} = L_{AB} L_{BD} R^b_{\tau \rho} W^\tau E W^\rho D \epsilon^{\mu \sigma} \epsilon^{\nu \beta} + T^\mu_{AB} (Y) \]

with \( N^\mu_{ab} \). \( W^\mu_{\nu a} \) and \( T^\mu_{AB} \) any arbitrary functions which depend on the field \( Y^A \) only.

Subject to these conditions, the variation with respect to \( X^a \) of our action leads to the equations of motion \( \hat{\epsilon}^{\mu \nu} F^a_{\mu \nu} = 0 \), where \( F^a_{\mu \nu} \) is constructed from \( A^a_\mu \) in (16).

Similarly, by extracting \( \partial_\mu X^a \) from (16) and substituting in (15), we find (up to a total derivative)

\[ S (Y, X, A) = \int d^2 x \sqrt{\gamma} \left[ T^\mu_{AB} (Y) \partial_\mu Y^A \partial_\nu Y^B + N^\mu_{ab} (Y) A^a_\mu A^b_\nu + 2W^\mu_{a \alpha} (Y) L_{AB} \epsilon^{\alpha \nu} A^a_\mu \partial_\nu Y^B + \hat{\epsilon}^{\mu \nu} \eta_{ab} X^a F^b_{\mu \nu} \right] . \]

Again, the equations of motion for \( A^a_\mu \), if considered as an independent field, are precisely those in (16). The symmetry \( \delta X^a = f^a_{bc} \epsilon^b \epsilon^{c \nu} \hat{\epsilon}^{\mu \nu} \) is transparent in this case.

The same procedure can be applied here to find the non-Abelian dual theory. Consider now the action

\[ S (Y, g, A) = \int d^2 x \sqrt{\gamma} \left[ T^\mu_{AB} (Y) \partial_\mu Y^A \partial_\nu Y^B + N^\mu_{ab} (Y) \eta^{ac} \eta^{bd} \epsilon^g (T_c g^{-1} \partial_\mu g) \epsilon^g \partial_\nu g \right] + 2W^\mu_{a \alpha} (Y) L_{AB} \eta^{ab} \epsilon^{c \nu} \epsilon^g (T_b g^{-1} \partial_\mu g) \partial_\nu Y^B \]

which is invariant under the left symmetry \( g \rightarrow hg \). This symmetry can be gauged by the replacement \( \partial_\mu g \rightarrow D_\mu g = \partial_\mu g + A_\mu g \). The dual theory is obtained when the Lagrange multiplier term \( \int d^2 x \sqrt{\gamma} \epsilon^{\mu \nu} \epsilon^g (X F^\mu_\nu) \) is added. Choosing then a gauge such that \( g = 1 \) yields the action in (22).

Notice that in the above model (22) we have not assumed any transformation for the fields \( Y^A \). In fact these fields could transform when \( T^\mu_{AB} \), \( N^\mu_{ab} \) and \( W^\mu_{\nu a} \) are restricted to satisfy certain conditions as shown below. It is found that the theory in (22), when \( A^a_\mu \) is treated as an independent field, has the infinitesimal local gauge symmetry

\[ \delta Y^A = \lambda^a K^A_a (Y) \]
\[ \delta A^a_\mu = -\partial_\mu \lambda^a + f^a_{bc} \lambda^b A^c_\mu \]

provided that the two quantities \( T^\mu_{AB} \) and \( K^A_a \) satisfy

\[ \partial_E T^\mu_{AB} K^F_a + T^\mu_{EB} \partial_A K^F_a + T^\mu_{AE} \partial_B K^F_a = \hat{\epsilon}^{\mu \nu} (\partial_A V^B - \partial_B V^A) \]
\[ K^A_a \partial_A K^B_b - K^B_b \partial_A K^A_a = -f^c_{ab} K^B_c . \]
The second equation merely expresses the fact that the differential operators $K_a = -K^A_a \frac{\partial}{\partial Y^A}$ form a representation of the Lie algebra defined by $\eta_{ab}$ and $f^{c}_{ab}$. The first equation defines the new quantity $V_{Aa}$ which is required to satisfy

\[
\begin{align*}
\partial_D V_{Ab} K^D_c &+ \partial_A V_{Dc} K^D_b - \partial_D V_{Ac} K^D_b + V_{Db} \partial_A K^D_c = -f^d_{cb} V_{Ad} \\
V_{Aa} K^A_b + V_{Ab} K^A_a &= 0 .
\end{align*}
\]

(26)

The remaining two quantities $N^{\mu\nu}_{ab}$ and $W^{\mu A}_{\nu a}$ are then given by

\[
\begin{align*}
N^{\mu\nu}_{ab} &= T^{\mu\nu}_{AB} K^A_b K^B_a + \hat{\epsilon}^{\mu\nu} V_{Ab} K^A_a \\
W^{\mu A}_{\nu a} &= -L^{AE} \left( \hat{\epsilon}^{\alpha \nu} T^{\alpha \mu}_{EB} K^B_a + \delta^{\mu}_{\nu} V_{Ea} \right) .
\end{align*}
\]

(27)

By writing $T^{\mu\nu}_{AB} = \gamma^{\mu\nu} G_{AB}(Y) + \hat{\epsilon}^{\mu\nu} B_{AB}(Y)$, the equations (24)–(27) are precisely the equations needed to gauge the isometries of a general sigma model with metric $G_{AB}$ and antisymmetric tensor $B_{AB}$ [24, 25]. Hence the non-linear sigma model obtained through a non-Abelian duality procedure is a particular case of this general construction.

4. Conclusions

It is worth mentioning that a symmetry similar to the one identified for the sigma model exists in Chern-Simons theory. To see this, consider a Chern-Simons theory for some gauge group $G$

\[
I (A) = \int d^3 x \epsilon^{ijk} \left[ \text{tr} (A_i F_{jk}) - \frac{2}{3} \text{tr} (A_i A_j A_k) \right]
\]

(28)

where $i, j, \ldots = 1, 2, 3$ and $\epsilon^{123} = 1$. Let also $\mu, \nu, \ldots = 1, 2$ and $\epsilon^{\mu\nu}$ the corresponding alternating tensor, with $\epsilon^{12} = 1$. By splitting the three-dimensional indices, the Chern-Simons action can be written as

\[
I (A) = 2 \int d^3 x \epsilon^{\mu\nu} \left[ \text{tr} (A_3 F_{\mu\nu}) - \text{tr} (A_\mu \partial_3 A_\nu) - \text{tr} (\partial_\mu (A_\nu A_3)) \right]
\]

(29)

It is then clear, if we drop the total divergence term, that the Chern-Simons theory has a further symmetry given by

\[
A_3 \rightarrow A_3 + \epsilon^{\mu\nu} [\xi, F_{\mu\nu}] ,
\]

(30)

where $\xi$ is a local Lie algebra-valued function. This symmetry is of the form (6). Furthermore, varying the Chern-Simons action with respect to $A_3$ leads to an equation similar to (9). This hints at a deep connection between Chern-Simons theory and the class of sigma models we identified as having the new symmetry. We speculate that a non trivial compactification to two dimensions of the Chern-Simons theory would lead to our sigma models.
As mentioned earlier, the new symmetry vanishes on-shell. Therefore, at the classical level this symmetry has no effects. We expect, however, that this symmetry would play a crucial role at the quantum level. We will report elsewhere on the work in progress regarding the quantisation of these models. The methods designed for the quantisation of Chern-Simons theories are essential to this investigation [26].

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References


