Curvature expansion for the background–induced gluodynamics string

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Abstract

Using cumulant expansion for an averaged Wilson loop we derive an action of the gluodynamics string in the form of a series in powers of the correlation length of the vacuum. In the lowest orders it contains the Nambu–Goto term and the rigidity term with the coupling constants computed from the bilocal correlator of gluonic fields. Some higher derivative corrections are calculated.
1. Introduction

One of the most exciting questions of the modern quantum field theory is a possible relationship between the large-distance behaviour of the confining phase of gluodynamics and string theory (for a review see for example\textsuperscript{1,2}). The aim of this letter is to derive an effective action for the gluodynamics string using the averaged Wilson loop expressed in terms of field correlators\textsuperscript{3,4,5,6}. In what follows we keep for simplicity only the lowest – bilocal correlators, which are believed to be dominant according to lattice data\textsuperscript{5}, and briefly discuss the effect of higher correlators. The bilocal correlator may be parametrized in the following way\textsuperscript{3,4,5,6}:

\[ < F_{\mu\nu}(x)\Phi(x, x')F_{\lambda\rho}(x', x) > = \frac{1}{N_c} \left\{ (\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda})D\left(\frac{(x - x')^2}{T_g^2}\right) + \right. \]
\[ \left. \frac{1}{2} \left[ \frac{\partial}{\partial x_\mu}((x - x')_\lambda\delta_{\nu\rho} - (x - x')_\rho\delta_{\nu\lambda}) + \frac{\partial}{\partial x_\nu}((x - x')_\rho\delta_{\mu\lambda} - (x - x')_\lambda\delta_{\mu\rho}) \right]D_1\left(\frac{(x - x')^2}{T_g^2}\right) \right\}. \] \hspace{1cm} (1)

Here \( T_g \) is the correlation length of the vacuum, which is small in the comparison with the size \( r \) of a Wilson loop in the confining regime\textsuperscript{7}.

In order to derive an action for the gluodynamics string, induced by the nonperturbative background fields, we shall consider the expression for an averaged Wilson loop using the nonabelian Stokes theorem\textsuperscript{3}

\[ < W(C) > = tr \exp\left( -\frac{g^2}{2} \int_S d\sigma_{\mu\nu}(x)\int_S d\sigma_{\lambda\rho}(x') < F_{\mu\nu}(x, x')F_{\lambda\rho}(x', x) > \right). \] \hspace{1cm} (2)

At this point one should make clear of the notion of the surface \( S \) entering the integral in (2). In gluodynamics the Wilson loop average depends on the contour \( C \) and should not depend on the shape of the surface \( S \) bounded by this contour, and therefore one usually considers \( S \) to be the surface of minimal area. In what follows we generalize this definition, taking \( S \) to be any surface, since our aim is to derive an effective string action for the case when the surface, swept by this string, is arbitrary. Such an effective action contains all geometrical characteristics of the surface in question, and one can always specify the surface \( S \) in the final expression.

To derive this action we shall expand the integral in (2) in powers of \( T_g^2 \). Such an expansion yields in the lowest, second, order in \( T_g \) the usual Nambu-Goto term with the string tension proportional to the surface integral of the function \( D \) in agreement with\textsuperscript{5}, while in the order \( T_g^4 \) there arises the rigidity term\textsuperscript{8,9} with the inverse bare coupling constant proportional to the first moment of the function \( 2D_1 - D \). The sign of this coupling constant is connected with the type of dual superconductor, describing the nonperturbative gluodynamics vacuum\textsuperscript{10}, and hence we may quote the following result of the next section: if \( \int d^2 z z^2(2D_1(z^2) - D(z^2)) < 0 \) than this is a type-II dual superconductor (in the Abelian case the stability of the classical Abrikosov-Nielsen-Olesen strings is ensured only in the case when this type of superconductor is realized\textsuperscript{10}).

The main results of the letter and possible future developments are discussed in the Conclusion.

In the Appendix, as an example of geometrical structures arising in higher orders in \( T_g^2 \), we present some of the higher derivative terms in the order \( T_g^6 \). The corresponding inverse bare coupling constants are proportional to the second moment of the function \( D + D_1 \). Therefore the effective action of the gluodynamics string, generated by the nonperturbative confining background
fields, has the form of a series in powers of \( \frac{T_2}{r} \), corresponding to the series in powers of the scalar curvature of the manifold and more complicated geometrical structures.

2. An action up to the order \( T_g^4 \)

Let us rewrite the correlator (1) in the form, which is more convenient for the future calculations:

\[
\langle F_{\mu\nu}(x)\Phi(x, x')F_{\lambda\rho}(x')\Phi(x', x) \rangle = \frac{1}{N_c} \left\{ (\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda}) \left[ D\left( \frac{(x - x')^2}{T_g^2} \right) + \right. \right.
\]

\[
+ D_1\left( \frac{(x - x')^2}{T_g^2} \right) \left. \right] + \frac{1}{T_g^2} \left[ (x - x')_\mu(x - x')_\lambda \delta_{\nu\rho} - (x - x')_\mu(x - x')_\rho \delta_{\nu\lambda} + (x - x')_\nu(x - x')_\rho \delta_{\mu\lambda} - 
\]

\[
-(x - x')_\nu(x - x')_\lambda \delta_{\mu\rho} \rangle D_1\left( \frac{(x - x')^2}{T_g^2} \right) \left. \right],
\]

where \( D_1' \) denotes the derivative of the function \( D_1 \) by the argument.

Our first goal is to compute the integral

\[
J \equiv \int d\sigma_{\mu\nu}(x) \int d\sigma_{\lambda\rho}(x')(\delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda}) \left[ D\left( \frac{(x - x')^2}{T_g^2} \right) + D_1\left( \frac{(x - x')^2}{T_g^2} \right) \right].
\]

Here \( d\sigma_{\mu\nu}(x) = \sqrt{g(\xi)}t_{\mu\nu}(\xi) d^2\xi \), \( t_{\mu\nu} = \frac{1}{\sqrt{g}}e^{ab}(\partial_\mu x_\rho)(\partial_\nu x_\rho) \) is the extrinsic curvature of the string world sheet, \( t_{\mu\nu}^2 = 2 \), \( g_{ab} = (\partial_a x_\mu)(\partial_b x_\nu) \) is the induced metric tensor, \( g = det \parallel g_{ab} \parallel, \partial_a \equiv \frac{\partial}{\partial x^a}, a, b = 1, 2 \). Expanding \( \sqrt{g(\xi')} \), \( t_{\lambda\rho}(\xi') \), \( x' - x \) and \( D\left( \frac{(x - x')^2}{T_g^2} \right) + D_1\left( \frac{(x - x')^2}{T_g^2} \right) \), where \( x' \equiv x(\xi') \), systematically in powers of \( \frac{T_2}{r} \), passing from the ordinary derivatives to the covariant ones (without torsion) via the familiar Gauss-Weingarten formulae \( D_a D_b x_\mu = \partial_a \partial_b x_\mu - \Gamma^c_{ab} \partial_c x_\mu = K^i_{ab} x_{i\mu} \), \( n_{ij} n_{j\mu} = \delta_{ij}, \ n_{ij} \partial_a x_\mu = 0; i, j = 1, 2 \), where \( \Gamma^c_{ab} \) is a Christoffel symbol, \( K^i_{ab} \) is the second fundamental form of the manifold, \( n_{ij} \) are the unit normals to the sheet, one gets in the conformal gauge \( g_{ab} = e^{2\delta} \delta_{ab} \)

\[
J = T_g^2 \int d^2\sqrt{g} \left[ 4 M_0 - T_g^2 M_1 + \frac{1}{4} g^{ab}(\partial_a t_{\mu\nu})(\partial_b t_{\mu\nu}) \right] + O\left( \frac{T_g^6 < F^2 >}{r^2} \right),
\]

where \( M_0 \equiv \int d^2 z (D(z^2) + D_1(z^2)) \), \( M_1 \equiv \int d^2 z z^2 (D(z^2) + D(z^2)) \), \( < F^2 > \equiv \)[:

\[
\equiv tr < F_{\mu\nu}(0)F_{\mu\nu}(0) >.
\]

Here we omitted the full derivative terms of the form \( \int d^2\xi \sqrt{g} R \), where \( R \) is a scalar curvature of the manifold, and used the formula \( (D_a D^b x_\mu)(D_b D^c x_\mu) = g^{ab}(\partial_a t_{\mu\nu})(\partial_b t_{\mu\nu}) \). The estimate for the neglected terms may be easily obtained if one assumes that the string world sheet is not much crumpled, so that the induced metric is a smooth function, which means that the typical values of \( \xi \) are of the order of \( r \).

Using the relations \( t_{\mu\lambda} t_{\nu\lambda} = g^{ab}(\partial_a x_\mu)(\partial_b x_\nu), (g^{ab} g^{cd} + g^{ac} g^{bd} + g^{ad} g^{bc})(\partial_a t_{\mu\nu})(\partial_b t_{\mu\nu}) \), \( (\partial_a x_\lambda)(\partial_b x_\nu) = K^i_{a\mu} K^i_{b\nu} - R \) and omitting the full derivative terms one can in analogous way compute the integral

\[
\frac{1}{T_g^2} \int d\sigma_{\mu\nu}(x) \int d\sigma_{\lambda\rho}(x')(x - x')_\mu(x - x')_\lambda \delta_{\nu\rho} - (x - x')_\mu(x - x')_\rho \delta_{\nu\lambda} +
\]

3
\[(x - x')_\nu (x - x')_\rho \delta_{\mu \lambda} - (x - x')_\nu (x - x')_\lambda \delta_{\mu \rho}] D_1^i \left( \frac{(x - x')^2}{T_g^2} \right),\]

which occurs to be equal to

\[T_g^2 \int d^2 \xi \sqrt{g} \left[ -4M_0^{(1)} + T_g^2 \frac{3M_1^{(1)}}{4} - g^{ab}(\partial_a t_{\mu \nu})(\partial_b t_{\mu \nu}) \right] + O \left( \frac{T_g^6 < F^2 >}{r^2} \right),\]

where \(M_0^{(1)} \equiv \int d^2 z D_1(z^2)\), \(M_1^{(1)} \equiv \int d^2 z z^2 D_1(z^2)\).

During the derivation of the formulae (3) and (4) we exploited the fact that for any odd \(n \int d^2 \xi \xi^1 ... \xi^n D(\xi^2) = \int d^2 \xi \xi^1 ... \xi^n D_1(\xi^2) = 0\).

Combining together (3) and (4) we finally obtain the effective action of the gluodynamics string, induced by the nonperturbative background fields, in the approximation when all the correlators higher than bilocal are neglected:

\[S_{biloc.} = \frac{g^2}{2} \left[ \sigma \int d^2 \xi \sqrt{g} + \frac{1}{\alpha_0} \int d^2 \xi \sqrt{g} g^{ab}(\partial_a t_{\mu \nu})(\partial_b t_{\mu \nu}) + O \left( \frac{T_g^6 < F^2 >}{r^2} \right) \right],\]

where

\[\sigma \equiv 4T_g^2 \int d^2 z D(z^2)\]

is a string tension (which agrees with\(^5\)) and

\[\frac{1}{\alpha_0} \equiv \frac{1}{4} T_g^4 \int d^2 z z^2 (2D_1(z^2) - D(z^2))\]

is an inverse bare coupling constant of the rigidity term.

Hence we proved the statement, announced in the Introduction, namely when \(\int d^2 z z^2 (2D_1(z^2) - D(z^2)) < 0\), the nonperturbative Euclidean gluodynamics vacuum may be considered as a type–II dual superconductor, which in the Abelian Higgs Model case implies that the Londons’ penetration depth of magnetic field is larger than the correlation radius of the Higgs field fluctuations, and the classical Abrikosov-Nielsen-Olesen strings are stable\(^10\). In particular one concludes that when \(D > 2D_1\) everywhere, the confining regime of an averaged Wilson loop is realized according to the dual Meissner effect mechanism\(^11\) with the string tension given by the formula (6).

In the higher orders in \(\frac{T_g}{r}\) more and more complicated geometrical structures, containing higher covariant derivatives, arise in the string action. The inverse bare coupling constants of these terms are linear combinations of higher moments of the functions \(D\) and \(D_1\). One can therefore establish some type of correspondence between the expansion of the string action in powers of \(\frac{T_g}{r}\) and a multipole expansion in two-dimensional gravity. A generic \(n\)-th (\(n \geq 2\)) term of the string action is proportional to some linear combination of the \((n-1)\)-th moments of the functions \(D\) and \(D_1\), which are of the order of \(T_g^{2n} < F^2 >\) and to the \((2n - 2)\)-th derivative of the induced metric. Integrating over \(d^2 \xi\) one obtains an estimate \(< F^2 > r^4 (\frac{T_g}{r})^{2n}\). As an example we present some geometrical structures, arising in the string action in the order \(T_g^6\), in the Appendix.

The same effect will be due to the higher correlators. However there is one more parameter in the problem, which is assumed to be much less than 1 in the confining regime of the Wilson loop. This is the parameter of cumulant expansion\(^3\), \(g < F^2 > \frac{1}{2} T_g^2\), whose \(n\)-th power estimates the
upper limit of the $n$-th order cumulant $g^n \ll F_{\mu_1 \nu_1}(x_1) \Phi(x_1, x_2) F_{\mu_2 \nu_2}(x_2) \ldots F_{\mu_n \nu_n}(x_n) \Phi(x_n, x_1) \gg$.

One may conclude that the higher terms of cumulant expansion will contribute to $\sigma$ and $\alpha_0$ also (as well as to the coupling constants of the terms arising in higher orders of $T_{g^r}$), but their contributions will be suppressed by the additional powers of the parameter of cumulant expansion in comparison with the formulae (6) and (7) derived from the bilocal correlator.

Therefore we see that the Wilson loop average, written through the field correlators, while expanded in powers of $T_{g^r}$ gives rise to the expansion of the string effective action, which may be called curvature expansion.

Note that the formulae (3)-(7) (as well as (A.1)-(A.4) presented in the Appendix) were derived in the conformal gauge, while we dealt with the open string, sweeping the area inside the Wilson loop. It is known\(^1\) that in the case of a unit disc (onto which the string world sheet in our case may be unambiguously mapped) this gauge is accessible, but one should take into account diffeomorphisms reparametrizing the boundary. This reparametrization, defined modulo $SL(2, \mathbb{R})$ transformations, is determined by the original metric $g_{ab}(\xi)$, and thus if we quantize the Nambu-Goto term using the method suggested in\(^{12}\), in the functional integral over all the metrics one should take into account not only $\varphi$-integration, but also integration over all possible reparametrizations.

3. Conclusion

In this letter we used the Wilson loop average, written via the field correlators, to derive the effective action for the gluodynamics string in the approximation when all the correlators higher than bilocal are neglected. Such an action has the form of a series in powers of the vacuum correlation length and in the lowest orders is given by the formula (5), containing the Nambu-Goto term with the string tension, defined via (6), and the rigidity term with the inverse bare coupling constant (7) proportional to the linear combination of the first moments of the functions parametrizing the bilocal correlator. It is shown that in agreement with the ’t Hooft-Mandelstam mechanism of confinement the confining regime of the Wilson loop takes place in the case of type-II dual superconductor model of the gluodynamics vacuum, which in the Abelian case corresponds to the situation when stable Abrikosov-Nielsen-Olesen strings exist. Therefore the criterion of distinguishing of the confining and deconfining regimes (or the types of superconductor in the Abelian case), following from the bilocal correlator, is established. In general expansions of the Wilson loop average, expressed in terms of field correlators, in powers of the vacuum correlation length and in powers of the parameter of cumulant expansion produce the curvature expansion of the effective action of the gluodynamics string generated by the nonperturbative background fields.

However the approach suggested for derivation of this action leaves us with the conformal anomaly of the Nambu-Goto term if $D \neq 26$. One of the possible methods of its cancellation in $D = 4$ was suggested in\(^{13}\) for the case of the strings in the Abelian Higgs Model. In the framework of this method passing from the field variables to the collective string ones and computing the Jacobian, corresponding to such transformation, one gets from it the Polchinski-Strominger term\(^{14}\) in the action, which exactly cancels the conformal anomaly in $D = 4$. It should be emphasized that the Polchinski-Strominger terms do not arise during our derivation of the effective action (5), and hence one should think about the mechanisms of cancellation of the conformal anomaly in $D = 4$ possibly similar to one suggested in\(^{13}\). While in this letter we derived the effective action of the string, induced only by the nonperturbative confining fields, it seems natural to
try to disentangle this problem in a rather elegant way taking into account perturbative gluons’ contributions and reformulating the summation over the surfaces in the functional integral$^{1,12}$ in terms of the perturbative theory in the nonperturbative gluodynamics background$^{6}$. Within this approach the perturbative gluons and ghosts, propagating inside the Wilson loop, generate the string world sheet excitations. The investigation of this problem will be the topic of another publication.

One more set of questions is connected with the rigidity term in (5). It is known$^{1,8}$, that in the generic case the rigid string has a crumpled world sheet, and its spectrum contains bosonic tachyon – the particle with imaginary mass. However as was discussed in$^{1,8}$, these problems disappear if the $\beta$-function has a zero at some value of the coupling constant, which may take place for example due to some $\theta$-terms in the action. This problem will be also treated elsewhere.

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Appendix. Some examples of geometrical structures arising in the action in the order $T_g^6$.

In this Appendix we present a part of the string action terms in the next order in $T_g$. Namely we demonstrate the geometrical structures arising from $J$ in the order $T_g^6$.

One can prove that $O\left(\frac{T_g^6}{g^2}\right)$ in (3) equals to

$$T_g^6 \int d^2\xi \sqrt{g} d^2\lambda \left\{ J_1 + J_2 + J_3 + J_4 + P(\lambda^2) \Lambda^{abcd} \left[ \frac{1}{6} (\partial_a \partial_b \partial_c \partial_d \sqrt{g}) - \frac{1}{2} (\partial_a t_{\mu\nu}) (\partial_b t_{\mu\nu}) (\partial_c \partial_d \sqrt{g}) + \right. \right.$$

$$+ \frac{1}{3} t_{\mu\nu} (\partial_a \partial_b \partial_c t_{\mu\nu}) (\partial_d \sqrt{g}) + \frac{1}{12} \sqrt{g} t_{\mu\nu} (\partial_a \partial_b \partial_c \partial_d t_{\mu\nu}) \right] + P'(\lambda^2) \Lambda^{abcdef} \left[ \frac{1}{3} (\partial_a \sqrt{g}) (2 (\partial_b \partial_c x_{\mu}) \cdot \right.$$

$$\cdot (\partial_d \partial_x f_{\mu}) + (\partial_b x_{\mu}) (\partial_c \partial_x f_{\mu}) + (\partial_b \partial_c \sqrt{g}) \left( \frac{1}{4} (\partial_x x_{\lambda}) (\partial_c \partial_f x_{\lambda}) + \frac{1}{3} (\partial_c x_{\lambda}) (\partial_d \partial_x f_{\lambda}) \right) \right.$$

$$+ \frac{2}{3} (\partial_a \partial_b \partial_c \sqrt{g}) (\partial_d x_{\mu}) (\partial_c \partial_f x_{\mu}) - \sqrt{g} (\partial_a t_{\mu\nu}) (\partial_b t_{\mu\nu}) \left( \frac{1}{4} (\partial_x x_{\lambda}) (\partial_c \partial_f x_{\lambda}) + \frac{1}{3} (\partial_c x_{\lambda}) (\partial_d \partial_x f_{\lambda}) \right) \right.$$

$$- (\partial_a t_{\mu\nu}) (\partial_b t_{\mu\nu}) (\partial_c \sqrt{g}) (\partial_d x_{\mu}) (\partial_c \partial_f x_{\mu}) + \frac{1}{2} \sqrt{g} t_{\mu\nu} (\partial_a \partial_b \partial_c \partial_d t_{\mu\nu}) (\partial_x x_{\lambda}) (\partial_d \partial_x f_{\lambda}) \right] + P''(\lambda^2) \Lambda^{abcdefij}.$$
Here \( P = D + D_1, \lambda^2 = g_{ab} \lambda^a \lambda^b, \Lambda_i^{\ldots i_n} = \lambda_i \ldots \lambda_i, \) and \( J_1, J_2, J_3, J_4 \) are the following terms, which do not contain derivatives of \( P \) higher than of the third order and do not depend explicitly on \( \sqrt{g_{\mu\nu}}, \sqrt{g_{a\mu\nu}} \) and derivatives of \( \sqrt{g} \):

\[
J_1 = \frac{1}{3} \sqrt{g} P' (\lambda^2) \Lambda^{abcdijkl} \left( \frac{1}{2} (\partial_a \partial_b x_{\mu}) (\partial_c \partial_d x_{\nu}) + \frac{1}{3} (\partial_a \partial_b \partial_c \partial_d x_{\mu}) (\partial_e \partial_f x_{\mu}) + \frac{1}{5} (\partial_a x_{\mu}) (\partial_b \partial_c \partial_d \partial_e x_{\mu}) \right),
\]

\[
J_2 = 2 \sqrt{g} P'' (\lambda^2) \Lambda^{abcdijkl} \left( \frac{1}{4} (\partial_a \partial_b x_{\mu}) (\partial_c \partial_d x_{\mu}) + \frac{1}{3} (\partial_a x_{\mu}) (\partial_b \partial_c \partial_d x_{\mu}) \right).
\]

\[
J_3 = \frac{1}{3} \sqrt{g} P''' (\lambda^2) \Lambda^{abcdijkl} (\partial_a x_{\mu}) (\partial_b \partial_c x_{\mu}) (\partial_d \partial_e x_{\mu}) (\partial_f x_{\mu}).
\]

\[
J_4 = 2 \sqrt{g} P''' (\lambda^2) \Lambda^{abcdijkl} (\partial_a x_{\mu}) (\partial_b \partial_c x_{\mu}) (\partial_d \partial_e x_{\mu}) (\partial_f x_{\mu}).
\]

Using the standard rules of computation of higher covariant derivatives and the Gauss-Weingarten formulae it is possible to calculate the integrals \( \int d^2 \lambda J_1, \ldots, \int d^2 \lambda J_4 \). The results have the form:

\[
\int d^2 \lambda J_1 = -\frac{M_2}{48} \left\{ \frac{1}{3} \left[ (D_a D^2 x_{\mu}) (2 D^2 D^a + D^a D^a) x_{\mu} + (D_a D_b D^a D^b) x_{\mu} + \right] + \frac{1}{2} (D_a D^2 D^a x_{\mu} + D_a D_b D^a D^b x_{\mu}) \right\}.
\]

\[
\int d^2 \lambda J_2 = \frac{1}{4} (D_a D^2 D^a x_{\mu} + D_a D_b D^a D^b x_{\mu}).
\]

\[
\int d^2 \lambda J_3 = \frac{1}{3} (D_a D^2 D^a x_{\mu} + D_a D_b D^a D^b x_{\mu}).
\]

\[
\int d^2 \lambda J_4 = \frac{1}{2} (D_a D^2 D^a x_{\mu} + D_a D_b D^a D^b x_{\mu}).
\]
\begin{align*}
+ (\partial^a x_\mu)(D_c D_b D^c D^b + D_b D^2 D^b + D^2 D^2) x_\mu & - \frac{1}{30} (D_a D_b x_\mu)(D_c D^b x_\mu)(7(\partial^a \varphi) \partial^c \varphi + 13 \partial^a \partial^c \varphi) - \\
- \frac{1}{30} (D_a D_b x_\mu)(D^a D_c x_\mu)((\partial^b \varphi) \partial^c \varphi + 7 \partial^b \partial^c \varphi) - (D_a D_b x_\mu)(D_c D^a x_\mu) \left( \frac{9}{5} (\partial^b \varphi) \partial^c \varphi + \frac{2}{3} \partial^b \partial^c \varphi \right) - \\
- \frac{1}{3} (D_a D_b x_\mu)(D^2 x_\mu) \left( \frac{19}{5} (\partial^a \varphi) \partial^b \varphi + 2 \partial^a \partial^b \varphi \right) - \frac{1}{3} (D_a D_b x_\mu)(D^b D^a x_\mu) \left( 2 \partial^2 \varphi + \frac{89}{10} (\partial \varphi)^2 \right) - \\
- \frac{1}{3} (D_a D_b x_\mu)(D^a D^b x_\mu) \left( 2 \partial^2 \varphi + \frac{53}{10} (\partial \varphi)^2 \right) - \frac{1}{3} (D^2 x_\mu)(D^2 x_\mu) \left( 2 \partial^2 \varphi + 5 (\partial \varphi)^2 \right) + \\
+ \sqrt{g} \left( \frac{26}{15} (\partial_\alpha \partial_\beta \varphi)^2 + \frac{89}{20} (\partial_\alpha \partial_\beta \varphi)(\partial^a \varphi) \partial^b \varphi + \frac{23}{15} (\partial^2 \varphi)^2 + \\
+ \frac{13}{8} (\partial^2 \varphi)(\partial_\alpha \varphi)^2 + 5 (\partial^2 \varphi)(\partial_\alpha \varphi)^2 + \frac{9}{5} \partial^4 \varphi + \frac{5}{8} (\partial_\alpha \varphi)^2 (\partial_\beta \varphi)^2 \right) \right}, \tag{A.1}
\end{align*}

\begin{align*}
\int d^2 \lambda J_2 &= \frac{M_2}{16} \left\{ \frac{1}{72} \left[ (D_a D_b x_\mu)(D^2 x_\mu)((D^2 x_\nu)(D^a D^b + D^b D^a)x_\nu + 2(D^a D_c x_\nu + D_c D^a x_\nu)(D^b D^c + \\
+ D^c D^b) x_\nu + (D_a D_b x_\mu)(D_c D_d x_\mu)(2(D^c D^b x_\nu)D^d D^a x_\nu + 2(D^c D^a x_\nu)D^d D^b x_\nu + (D^a D^b x_\nu + D^b D^a x_\nu). \\
\cdot (D^c D^d + D^d D^c) x_\nu + (D^c D^a x_\nu)D^b D^d x_\nu + (D^c D^d x_\nu)D^b D^a x_\nu + (D^b D^c x_\nu)D^d D^a x_\nu + \\
+ (D^c D^b x_\nu)D^a D^d x_\nu + (D_a D_b x_\nu + D_b D_a x_\mu)(D_b D_c x_\mu + D_c D_b x_\mu)(D^2 x_\nu)D^c D^a x_\nu + \\
+ (D_a D_b x_\mu)(D_c D_d x_\nu + D_d D_c x_\nu)(D^b D^c x_\mu + D^c D^b x_\mu)(D^a D^d + D^d D^a) x_\nu + \\
+ (D^b D^c x_\nu + D^c D^b x_\nu)(D^a D^d + D^d D^a) x_\nu + \frac{1}{2} (D^2 x_\mu)(D^2 x_\mu)(D_a D_b x_\mu). \\
\cdot (D^a D^b + D^b D^a) x_\nu + (D_a D_b x_\mu)(D^a D^b x_\mu)(D_c D_d x_\mu) \left( \frac{1}{2} D^c D^d + D^d D^c \right) x_\nu + \frac{1}{2} (D_a D_b x_\mu)(D^b D^a x_\mu). \\
\cdot (D_c D_d x_\nu)(D^d D^c x_\nu) \right] - \frac{1}{9} \sqrt{g}(D^2 x_\mu)(D^2 x_\mu) \left( 2 \partial^2 \varphi + \frac{17}{2} (\partial_\alpha \varphi)^2 \right) - \sqrt{g}(D_a D_b x_\mu)(D^a D^b x_\mu) + \\
+ D^b D^a x_\mu) \left( \frac{2}{9} \partial^2 \varphi + \frac{5}{6} (\partial_\alpha \varphi)^2 \right) - \frac{1}{9} \sqrt{g} \left[ 8(D^2 x_\mu)(D_a D_b x_\mu)(\partial^a \partial^b \varphi + (\partial^a \varphi) \partial^b \varphi) + (\partial_\alpha \partial_\beta \varphi). \\
\cdot (7(D_c D^a x_\mu)D^c D^b x_\mu + 8(D^a D_c x_\mu)D^c D^b x_\mu + 4(D^a D_c x_\nu)D^b D^c x_\mu) + (\partial_\alpha \varphi)(\partial_\beta \varphi)((D_c D^a x_\mu)D^c D^a x_\mu + \\
+ 6(D_a D^a x_\mu)D^b D^a x_\mu + 2(D^a D_c x_\mu)D^b D^c x_\mu) + \frac{1}{3} g \left[ 8(\partial_\alpha \partial_\beta \varphi)^2 + 16(\partial_\alpha \partial_\beta \varphi)(\partial^a \varphi) \partial^b \varphi + \\
+ 4(\partial^2 \varphi)(\partial_\alpha \varphi)^2 + 4(\partial^2 \varphi)^2 + \frac{91}{8} (\partial_\alpha \varphi)^2 (\partial_\beta \varphi)^2 \right] \right\}, \tag{A.2}
\end{align*}

\begin{align*}
\int d^2 \lambda J_3 &= \frac{M_2}{48} \sqrt{g} \left\{ (\partial_\alpha \varphi)[4(D_b D_c x_\mu)(D^a D^b D^c + D^b D^a D^c + D^c D^a D^b + D^a D^c D^b + D^b D^c D^a + \\
+ D^c D^b D^a) x_\mu + (D_a D_b x_\mu + D_b D_a x_\mu)(4D^2 D^b + 4D^a D^b D^b + D^b D^2) x_\mu + (D^2 x_\mu)(4D^2 D^a + 4D_c D^a D^c + \\
+ D^a D^2) x_\mu] + (\partial_\alpha x_\mu) \left[ 2(\partial_\beta \varphi) \left( D_c D^a D^b D^c + D_c D^b D^a D^c + D_c D^a D^b D^c + D^a D_c D^a D^c + \\
+ D^b D_c D^a D^c +
\end{align*}
\[ + D^a D_c D^b D^c + D_c D^a D^c D^b + D^2 D^a D^b + D^2 D^b D^a + \frac{5}{4} D^a D^b D^2 + \frac{5}{4} D^b D^a D^2 \] \] \) x_\mu + \\
\[ + (\partial^a \varphi) \left( 2D_b D^2 D^b + 2D_b D_c D^b D^c + \frac{1}{2} D^2 D^2 \right) x_\mu \] + \sqrt{g} \left[ 17 (\partial^2 \varphi) (\partial_a \varphi)^2 + 7 (\partial_a \varphi)^2 (\partial_b \varphi)^2 + \\
+ 9 (\partial_a \varphi) \partial^2 \partial_a \varphi + 12 (\partial_a \partial_b \varphi) (\partial^a \varphi) \partial^b \varphi \right] - 2 (\partial_a \varphi) (\partial_b \varphi) \left[ (D_b D_c x_\mu + D_c D_b x_\mu) (D^a D^c + D^c D^a) x_\mu + \\
+ (D^2 x_\mu) (D^a D^b + D^b D^a) x_\mu \right] - 2 (\partial_a \varphi)^2 \left[ (D_a D_b x_\mu + D_b D_a x_\mu) (D^a D^b + D^b D^a) x_\mu + 2 (D^2 x_\mu) D^2 x_\mu \right] \right) \right], \tag{A.3} \]
\[
\int d^2 \lambda J_A = M_2^2 \frac{1}{32} g \left\{ \frac{1}{3} (\partial_a \varphi) (\partial_b \varphi) \left[ 13 (D^a D^b x_\mu) D^2 x_\mu + 11 (D^a D_c x_\mu) D^c D^b x_\mu + 7 (D^a D_c x_\mu) D^b D^c x_\mu + \\
+ 7 (D_c D^a x_\mu) D^c D^b x_\mu \right] + \frac{1}{3} (\partial_a \varphi)^2 \left[ \frac{9}{2} (D^2 x_\mu) D^2 x_\mu + 4 (D_a D_b x_\mu) D^a D^b x_\mu + 2 (D_a D_b x_\mu) D^b D^a x_\mu \right] - \\
- 2 \sqrt{g} \left[ 12 (\partial_a \partial_b \varphi) (\partial^a \varphi) \partial^b \varphi + \frac{355}{24} (\partial_a \varphi)^2 (\partial_b \varphi)^2 + \frac{17}{3} (\partial^2 \varphi) (\partial_a \varphi)^2 \right] \right\}, \tag{A.4} \]
\]
where \( M_2 \equiv \int d^2 z (z^2) (D(z^2) + D_1(z^2)) \), \( D^2 \equiv D_i D^i \), \( \partial^2 \equiv \partial_i \partial^i \), \( \partial^4 \equiv \partial_i \partial^i \partial_j \partial^j \), \( (\partial_a \varphi)^2 \equiv (\partial_a \varphi)(\partial^a \varphi) \), \( (\partial_a \partial_b \varphi)^2 \equiv \partial_a \partial_b \partial^a \partial^b \varphi \).
References