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A PATH INTEGRAL FOR TURBULENCE IN INCOMPRESSIBLE FLUIDS

W. D. THACKER

*Max Planck Institute for Physics
Werner Heisenberg Institute
Foehringer Ring 6, 80805 Munich, Germany*

and

*Department of Theoretical Physics
University of Trieste
Trieste, Italy*

Swg621

ABSTRACT

In this paper a classical path integral is formulated for incompressible fluids which evolve according to the Navier-Stokes equation. The path integral propagates probability distributions deterministically on the space $\mathcal{G}_{vol}$ of solenoidal velocity fields. We construct a set of $ISp(2)$ charges associated with the geometry of $\mathcal{G}_{vol}$ and its Poisson structure, and a pair of supersymmetry charges connected with the Hamiltonian. These charges generate exact symmetries of the classical path integral when the viscosity is set equal to zero. When the effect of dissipation is included, the charges associated with the Poisson structure and the Hamiltonian are no longer conserved. Charges which generate Kolmogorov scaling and Galilean transformations are also constructed. The classical path integral is formulated in terms of vorticity as well.

★ On leave from Parks College, Saint Louis University, Cahokia, IL 62206 U.S.A. (address after Dec. 1, 1995).

1. INTRODUCTION

Turbulence^{[1][2]} in fluids is a rich and fascinating phenomenon which continues to elude a fundamental understanding.^[3] On the one hand, we know the underlying equations

$$\frac{\partial v^a(\mathbf{x}, t)}{\partial t} = -\frac{1}{\rho} \frac{\partial P}{\partial x^a} - v^b \frac{\partial}{\partial x^b} v^a + \nu \nabla^2 v^a \quad (1.1)$$

and

$$\nabla \cdot \mathbf{v}(\mathbf{x}, t) = 0 \quad (1.2)$$

(when the fluid is incompressible and homentropic). On the other hand, we are unable to account for the great complexity of turbulent flows. In a turbulent fluid the velocity $\mathbf{v}(\mathbf{x}, t)$ and pressure $P(\mathbf{x}, t)$ are subject to wild and erratic fluctuations which do not seem to be in direct response to the forces driving the flow.

One line of research, the dynamical systems approach,^[4] attempts to describe turbulence as deterministic chaos. When Reynolds number becomes high enough the nonlinearity in (1.1) causes solutions to become unstable against small perturbations. At the same time, dissipation acts to limit the effective number of degrees of freedom for large times. The result is a strange attractor^[5] where the motion is confined to a finite region in phase space while within that region trajectories differing infinitesimally in initial conditions diverge from one another exponentially. Various scenarios have been proposed for the transition from laminar to turbulent flow as Reynolds number increases.^{[5][6][7][8]} These models are usually based on simplified systems of evolution equations. Turbulence is expected to be a generic feature of such systems. However, we would like to have a more precise picture of how turbulence in real fluids arises from the Navier-Stokes equation. Thus far such a theory is lacking.

Another line of research is the statistical theory of turbulence.^{[1][2][9]} This approach treats the velocity field (after pressure has been eliminated) as a random variable and aims to calculate correlation functions. An infinite tower of equations relates these correlation functions. Various assumptions concerning the probability distribution are introduced in an effort to solve these relations. Recently^[10] the machinery of conformal field theory has been brought to bear on this problem for the case of turbulence in two dimensions. An essential ingredient in the statistical approach is the introduction of a random stirring force^[9] which simulates the production of a stationary probability distribution with desired properties such as homogeneity and isotropy^[11]. The turbulent probability distributions are believed to have

universal properties independent of the stirring forces. We would like to understand how the turbulent probability distributions arise from the dynamics of the Navier-Stokes equation under the action of driving forces which are not necessarily random.

This paper is the beginning of an attempt to build a bridge between the dynamical systems and statistical approaches to turbulence. The basis of our approach is a path integral for deterministic systems. Path integrals^[12] have proved to be remarkably fruitful in applications to quantum mechanics, quantum field theory and statistical mechanics. Recently a path integral for classical mechanics has been developed.^[13] This path integral is constructed directly from the deterministic equations of motion. At the same time, it propagates distributions of points in phase space and therefore lends itself to a statistical description of dynamical systems. The classical path integral contains in addition to the bosonic variables, fermionic ones which give information about deviations among classical trajectories. Correlations among the fermionic variables are related to the Lyapunov exponents.^[14] The classical path integral has symmetries which uncover the symplectic geometry of phase space^[13] and ergodic properties of Hamiltonian systems^[15].

In this paper a classical path integral is developed for fluids which obey Equations (1.1) and (1.2). Other functional integrals for turbulence have appeared in the literature.^{[16][17]} What is unique about the classical path integral is first that it is deterministic. No noise term is introduced in the form of a random stirring force. Second, the classical path integral incorporates fermionic variables which give additional information about the dynamics.

This paper is organized as follows. In Section 2. we review the construction of the classical path integral for finite dimensional dynamical systems and the symmetries which it has when the system is Hamiltonian. We also extend previous work by giving the path integral expression for the propagator of the Hopf characteristic functional which is relevant for turbulence theory. In Section 3. we construct the classical path integral for fluids using the velocity field as the fundamental variable and explore the symmetries connected with the space of velocities and its Hamiltonian structure. In Section 4. we investigate the space-time symmetries of the classical path integral including Kolmogorov scaling and the transformations of the Galilean group. Section 5. contains some concluding remarks and indicates directions for future research. In the Appendix we construct the classical path integral in the vorticity representation.

2. The Classical Path Integral for Finite Dimensional Systems

In previous work^[13] a path integral was developed specifically for classical Hamiltonian systems. The same construction, however, can be applied to any deterministic system

$$\dot{\phi}^a(t) = V^a(\phi(t)) \quad (2.1)$$

where ϕ^a ($a = 1, \dots, N$) are the dynamical variables and $V^a(\phi)$ are the components of the vector field which determines the flow on phase space .

A dynamical system may exhibit chaotic behavior which makes it impractical to track individual trajectories. Instead, we are interested in ensembles of systems described by the density distribution $\varrho(\phi)$. The classical path integral gives an expression for the kernel $K(\phi, t|\phi_0, t_0)$ which propagates the distribution according to

$$\varrho(\phi, t) = \int K(\phi, t|\phi_0, t_0) \varrho(\phi_0, t_0) d^N \phi_0 \quad (2.2)$$

For a deterministic system the kernel is just given by a delta-function

$$K(\phi, t|\phi_0, t_0) = \delta^N(\phi - \phi_{cl}(t; t_0, \phi_0)) \quad (2.3)$$

where $\phi_{cl}(t; t_0, \phi_0)$ is the unique solution of Eq.(2.1) satisfying the initial condition $\phi(t_0) = \phi_0$. Slicing the interval $[t_0, t]$ into $M + 1$ equal pieces, we get

$$\delta^N(\phi - \phi_{cl}(t; t_0, \phi_0)) = \int \delta^N(\phi - \phi_M) \prod_{j=1}^M \delta^N(\phi_j - \phi_{cl}(t_j; t_{j-1}, \phi_{j-1})) d^N \phi_j \quad (2.4)$$

In the continuum limit this goes over into

$$K(\phi, t|\phi_0, t_0) = \int \mathcal{D}' \phi \tilde{\delta}[\phi - \phi_{cl}] \quad (2.5)$$

where the initial and final points are held fixed in the integration. The functional delta-function forces the system to lie on the classical trajectory. Using a functional generalization of the formula

$$\delta^N(\phi - \phi_{root}) = \delta^N(F(\phi)) \left| \det \left[\frac{\partial F^a}{\partial \phi^b} \right] \right| \quad (2.6)$$

where $F(\phi) = (F^1(\phi), \dots, F^N(\phi))$ and $F(\phi) = 0$ is assumed to have a unique solution ϕ_{root} , we obtain

$$\tilde{\delta}[\phi - \phi_{cl}] = \tilde{\delta}[\dot{\phi} - V(\phi)] \left| \det \left[\partial_t \delta_b^a - \frac{\partial V^a}{\partial \phi^b} \right] \right| \quad (2.7)$$

We can now Fourier transform the delta-function on the right hand side of (2.7), using an auxiliary field Λ_a , and exponentiate the determinant via anticommuting variables $C^a, \bar{C}_b$

to arrive at the classical path integral for the kernal

$$K(\phi, t | \phi_0, t_0) = \int_{\phi_0}^{\phi} \mathcal{D}'\phi \mathcal{D}\Lambda \mathcal{D}C \mathcal{D}\bar{C} \exp[i\tilde{S}] \quad (2.8)$$

where $\tilde{S} = \int_{t_0}^t dt' \tilde{\mathcal{L}}$ is the classical path integral (CPI) action with the Lagrangian

$$\tilde{\mathcal{L}} = \Lambda_a \left[\dot{\phi}^a - V^a(\phi) \right] + i\bar{C}_a \left[\partial_t \delta_b^a - \frac{\partial V^a}{\partial \phi^b} \right] C^b \quad (2.9)$$

Since $\tilde{\mathcal{L}}$ contains only first derivatives in time, we can immediately read off the associated CPI Hamiltonian

$$\tilde{\mathcal{H}} = \Lambda_a V^a + i\bar{C}_a \frac{\partial V^a}{\partial \phi^b} C^b \quad (2.10)$$

Notice that $\tilde{\mathcal{H}}$ has been constructed directly from the equations of motion. It exists even for systems which are not Hamiltonian.

The arena for the classical path integral is a superspace spanned by the bosonic variables (ϕ, Λ) and the fermionic variables $(C, \bar{C})$. In (2.8) the Grassmann variables C^a are saturated at the initial point through the insertion of the delta-function $\delta^N(C_0)$.^[18] The integration over C and $\bar{C}$ then yields the determinant in (2.7). Actually, it turns out^[13] that this determinant is equal to one for Hamiltonian systems. Therefore, if we are only interested in the propagation of scalar density distributions $\varrho(\phi)$ in Hamiltonian systems, we can omit the Grassmann variables. However, we shall find that the Grassmann variables yield important physical information about dynamical systems.

If we fix C at the initial and final endpoints of the classical path integral we obtain the kernal

$$K(\phi, C, t | \phi_0, C_0, t_0) = \int_{\phi_0, C_0}^{\phi, C} \mathcal{D}'\phi \mathcal{D}\Lambda \mathcal{D}'C \mathcal{D}\bar{C} \exp[i\tilde{S}] \quad (2.11)$$

which propagates functions $\tilde{\varrho}(\phi, C)$ on the superspace spanned by ϕ and C according to

$$\tilde{\varrho}(\phi, C, t) = \int d^N \phi_0 d^N C_0 K(\phi, C, t | \phi_0, C_0, t_0) \tilde{\varrho}(\phi_0, C_0, t_0) \quad (2.12)$$

Using standard techniques^[12] we can derive from the classical path integral the following

equal-time graded commutators:[★]

$$\begin{aligned}\langle [\phi^a, \Lambda_b] \rangle &= \langle \phi^a \Lambda_b - \Lambda_b \phi^a \rangle = i \delta_b^a \\ \langle [\bar{C}_a, C^b] \rangle &= \langle \bar{C}_a C^b + C^b \bar{C}_a \rangle = \delta_b^a\end{aligned}\tag{2.13}$$

All other graded commutators vanish. In particular, the ϕ^a 's commute among themselves for all values of the indices a and b , showing that we are doing classical mechanics and not quantum mechanics. The operator algebra (2.13) can be realized by differential operators

$$\lambda_a = -i \frac{\partial}{\partial \phi^a}, \quad \bar{C}_a = \frac{\partial}{\partial C^a}\tag{2.14}$$

and multiplicative operators ϕ^a and C^a acting on functions $\tilde{\varrho}(\phi, C, t)$. An arbitrary function $\tilde{\varrho}(\phi, C, t)$ can be expanded in products of C 's

$$\tilde{\varrho}(\phi, C, t) = \sum_{p=0}^N \frac{1}{p!} \varrho_{a_1 \dots a_p}^{(p)} C^{a_1} \dots C^{a_p}\tag{2.15}$$

These are the physical states of the theory. They can be used to calculate expectation values of observables $O(\phi, \Lambda, C, \bar{C})$ as follows:

$$\langle O(\phi, \Lambda, C, \bar{C}) \rangle = \int d^N \phi d^N C O(\phi, \Lambda, C, \bar{C}) \tilde{\varrho}(\phi, C, t)\tag{2.16}$$

The physical states propagating according to (2.12) obey the Schroedinger-like equation

$$\partial_t \tilde{\varrho}(\phi, C, t) = -i \tilde{\mathcal{H}} \tilde{\varrho}(\phi, C, t)\tag{2.17}$$

where

$$\tilde{\mathcal{H}} = -i V^a \frac{\partial}{\partial \phi^a} - i C^b \frac{\partial V^a}{\partial \phi^b} \frac{\partial}{\partial C^a}\tag{2.18}$$

is the CPI Hamiltonian operator obtained by substituting (2.14) into (2.10) and ordering the differential operators to the right. Each term in the sum (2.15) then behaves like a

★ We indicate with the square brackets the graded commutators^[13] which are symmetric if both entries are Grassmann-odd and antisymmetric otherwise. The angular brackets $\langle \rangle$ indicate that the enclosed quantity is calculated within the path integral as explained in textbooks on path integrals^[12]. Henceforth, we shall omit the angular brackets from all expressions involving graded commutators.

p-form which is Lie transported along the vector field V

$$\begin{aligned}
\partial_t \tilde{\varrho}^{(p)}(\phi, C, t) &= -i \tilde{\mathcal{H}} \tilde{\varrho}^{(p)}(\phi, C, t) \\
&= -l_V \tilde{\varrho}^{(p)}(\phi, C, t) \\
&= -\frac{1}{p!} (V^a \partial_a \varrho_{a_1 \dots a_p}^{(p)} + p(\partial_{a_1} V^a) \varrho_{aa_2 \dots a_p}^{(p)}) C^{a_1} \dots C^{a_p}
\end{aligned} \tag{2.19}$$

In previous publications^{[13][14][15][19]} we have referred to C and $\bar{C}$ as ghosts and anti-ghosts, respectively. However, in order to avoid the impression that they are somehow unphysical, we shall call them forms and anti-forms. We can gain further insight into the physical role of the forms by going over to a Heisenberg picture in which the states are fixed and the variables $(\phi(t), \Lambda(t), C(t), \bar{C}(t))$ evolve according to the Euler-Lagrange equations[†]

$$\dot{\phi}^a = V^a \tag{2.20}$$

$$\dot{\Lambda}_a = -\frac{\partial V^b}{\partial \phi^a} \Lambda_b - i \bar{C}_e \frac{\partial^2 V^e}{\partial \phi^a \partial \phi^f} C^f \tag{2.21}$$

$$\dot{C}^a = \frac{\partial V^a}{\partial \phi^b} C^b \tag{2.22}$$

$$\dot{\bar{C}}_a = -\bar{C}_b \frac{\partial V^b}{\partial \phi^a} \tag{2.23}$$

We see that in the Heisenberg picture the forms behave as Jacobi fields. Equation (2.22) is just Jacobi's equation for the first order deviation $\delta\phi^a(t)$ from the classical trajectory $\phi(t)$. E. Gozzi and M. Reuter^[14] have shown that certain expectation values involving forms and anti-forms are related to the Lyapunov exponents which give information about how quickly neighboring trajectories diverge from one another. Consider the operator

$$\mathcal{O}_b^a(T, \phi_0) = C^a(T) \bar{C}_b(0) \delta^N(\phi(0) - \phi_0) \tag{2.24}$$

with vacuum expectation value

$$\langle \mathcal{O}_b^a(T, \phi_0) \rangle = \int \mathcal{D}\phi \mathcal{D}\Lambda \mathcal{D}C \mathcal{D}\bar{C} \mathcal{O}_b^a(T, \phi_0) \delta^N(C(-\infty)) \exp\left[i \int_{-\infty}^{\infty} dt \tilde{L}\right] \tag{2.25}$$

The trace of this expectation value is related to the one dimensional Lyapunov exponent

† These can also be derived as Heisenberg equations of motion using (2.13) and (2.10).

$\lambda^{(1)}(\phi_0)$ by[‡]

$$\lambda^{(1)}(\phi_0) = \lim_{T \rightarrow \infty} \frac{1}{T} \ln(\langle \mathcal{O}_a^a(T, \phi_0) \rangle) \quad (2.26)$$

Higher dimensional Lyapunov exponents, which measure the rate of growth of volumes in phase space, are related to expectation values of operators of the form

$$\mathcal{O}_f(T; \phi_0) = C^{a_1}(T) \cdots C^{a_f}(T) \bar{C}_{a_f}(0) \cdots \bar{C}_{a_1}(0) \delta^N(\phi(0) - \phi_0) \quad (2.27)$$

An important object in the theory of turbulence is the Hopf characteristic functional^[2]. The finite dimensional analog is the characteristic function of the distribution $\varrho(\phi)$

$$\begin{aligned} \Phi(\theta, t) &= \langle \exp[i\theta_a \phi^a(t)] \rangle \\ &= \int d^N \phi_0 \varrho(\phi_0) \exp[i\theta_a \phi_{cl}^a(t; t_0, \phi_0)] \end{aligned} \quad (2.28)$$

where $\theta = (\theta_1, \dots, \theta_N)$ is a variable dual to ϕ . Equal-time correlation functions can be derived from the characteristic function by functional differentiation[§].

$$\langle \phi^{a_1}(t) \cdots \phi^{a_r}(t) \rangle = (-i)^r \frac{\partial^r}{\partial \theta_{a_1} \cdots \partial \theta_{a_r}} \Phi[\theta, t] \quad (2.29)$$

The characteristic function evolves according to the Hopf equation^[20]

$$\frac{\partial \Phi[\theta, t]}{\partial t} = i\theta_a V^a \left(-i \frac{\partial}{\partial \theta}\right) \Phi[\theta, t] \quad (2.30)$$

Solutions of the Hopf equation are propagated by the turbulence kernel^[16] $K(\theta, \theta_0; t, t_0)$ according to

$$\Phi(\theta, t) = \int d^N \theta_0 K(\theta, \theta_0; t, t_0) \Phi(\theta_0, t_0) \quad (2.31)$$

Let us develop an expression for the turbulence kernel in terms of the classical path integral^[16]. We start by returning to the definition of the characteristic function

$$\begin{aligned} \Phi(\theta, t) &= \int d^N \phi_0 \varrho(\phi_0) \exp[i\theta_a \phi_{cl}^a(t; t_0, \phi_0)] \\ &= \int d^N \phi_0 d^N \phi \delta^N(\phi - \phi_{cl}(t; t_0, \phi_0)) \varrho(\phi_0) \exp[i\theta_a \phi^a] \\ &= \int d^N \phi_0 d^N \phi K(\phi, t | \phi_0, t_0) \varrho(\phi_0) \exp[i\theta_a \phi^a] \end{aligned} \quad (2.32)$$

[‡] ϕ_0 is usually taken to lie on a periodic orbit.

[§] In the finite dimensional case functional differentiation is just partial differentiation.

Inserting (2.8) into (2.32) we obtain

$$\Phi(\theta, t) = \int \mathcal{D}\phi \mathcal{D}\Lambda \mathcal{D}C \mathcal{D}\bar{C} \exp[i\theta_a \phi^a] \exp[i\tilde{S}] \varrho(\phi_0) \quad (2.33)$$

But $\varrho(\phi_0)$ is just the inverse Fourier transform

$$\varrho(\phi_0) = \int \frac{d^N \theta_0}{(2\pi)^N} \Phi[\theta_0, t_0] \exp[-i\theta_{0a} \phi_0^a] \quad (2.34)$$

Substitution of (2.34) into (2.33) and comparison with (2.31) then yields the classical path integral expression for the turbulence kernel

$$K(\theta, \theta_0; t, t_0) = \int \mathcal{D}\phi \mathcal{D}\Lambda \mathcal{D}C \mathcal{D}\bar{C} \exp[i(\theta_a \phi^a - \theta_{0a} \phi_0^a)] \exp[i\tilde{S}] \quad (2.35)$$

When the Grassmann part of $\tilde{S}$ is omitted, (2.35) corresponds to the functional integral of Rosen^[16] for the turbulence kernel. In fact, integrating $\phi(t_0)$ and $\phi(t)$ out of (2.35) we get delta functions which allow us to identify Λ with θ at the limits, leading to

$$K[\theta, \theta_0; t, t_0] = \int_{\Lambda=\theta_0}^{\theta} \mathcal{D}\phi \mathcal{D}'\Lambda \exp\left[i \int_{t_0}^t dt' (\phi^a \dot{\Lambda}_a + V^a(\phi) \Lambda_a)\right] \quad (2.36)$$

Since we are treating the forms as dynamical variables which carry important information, let us generalize the characteristic function to

$$\begin{aligned} \tilde{\Phi}[\theta, \bar{\chi}, t] &= \int d^N \phi_0 d^N C_0 \tilde{\varrho}(\phi_0, C_0) \exp[i(\theta_a \phi_{cl}^a(t; t_0, \phi_0) + \bar{\chi}_a C_{cl}^a(t; t_0, \phi_0, C_0))] \\ &= \int d^N \phi_0 d^N C_0 d^N \phi d^N C \exp[i(\theta_a \phi^a + \bar{\chi}_a C^a)] K(\phi, C, t | \phi_0, C_0, t_0) \tilde{\varrho}(\phi_0, C_0) \end{aligned} \quad (2.37)$$

which we shall call the characteristic super-function, since it involves bosonic and fermionic variables. The characteristic super-function is propagated by the turbulence super-kernel

$$\begin{aligned} \tilde{K}[\theta, \theta_0; \bar{\chi}, \bar{\chi}_0; t, t_0] &= \int \mathcal{D}\phi \mathcal{D}\Lambda \mathcal{D}C \mathcal{D}\bar{C} \exp[i(\theta_a \phi^a - \theta_{0a} \phi_0^a + \bar{\chi}_a C^a - \bar{\chi}_{0a} C_0^a)] \exp[i\tilde{S}] \\ &= \int_{\Lambda=\theta_0, \bar{C}=\bar{\chi}_0}^{\theta, \bar{\chi}} \mathcal{D}\phi \mathcal{D}'\Lambda \mathcal{D}C \mathcal{D}'\bar{C} \exp\left[i \int_{t_0}^t dt' (\phi^a \dot{\Lambda}_a + i C^a \dot{\bar{C}}_a + \tilde{\mathcal{H}})\right] \end{aligned} \quad (2.38)$$

Equal-time correlation functions involving ϕ and C can be calculated from the characteristic super-function (2.37). In general we are interested in multi-time correlation functions among

all the variables $(\phi, \Lambda, C, \bar{C})$. These can be derived by functional differentiation from the generating functional

$$Z[J_\phi, J_\Lambda, J_C, J_{\bar{C}}] = \int \mathcal{D}\phi \mathcal{D}\Lambda \mathcal{D}C \mathcal{D}\bar{C} e^{i \int dt (\tilde{\mathcal{L}} + J_\phi \cdot \phi + J_\Lambda \cdot \Lambda + J_C \cdot C + J_{\bar{C}} \cdot \bar{C})} \tilde{\varrho}(\phi(-\infty), C(-\infty)) \quad (2.39)$$

where dot indicates here contraction over the phase space index a .

Let us now specialize to Hamiltonian systems and consider the symmetries of the classical path integral. Let $N = 2n$ be the dimension of phase space. In a Hamiltonian system there is a function $H(\phi)$ and a set of Poisson brackets $\omega^{ab} = \{\phi^a, \phi^b\}$ such that the vector field V in Eq. (2.1) is given by

$$V^a = h^a \equiv \omega^{ab} \frac{\partial H}{\partial \phi^b} \quad (2.40)$$

If there is a nondegenerate two-form with components ω_{ab} where $\omega^{ab}\omega_{bc} = \delta_c^a$, then the phase space is a symplectic manifold.

In Ref. [13] we found that for Hamiltonian systems $\tilde{S}$ is invariant under the transformations generated by the $ISp(2)$ charges given in canonical coordinates by

$$\begin{aligned} Q &= iC^a \Lambda_a, \\ \bar{Q} &= i\bar{C}_a \omega^{ab} \Lambda_b \\ Q_f &= C^a \bar{C}_a \\ K &= \frac{1}{2} \omega_{ab} C^a C^b, \\ \bar{K} &= \frac{1}{2} \omega^{ab} \bar{C}_a \bar{C}_b \end{aligned} \quad (2.41)$$

These $ISp(2)$ generators can be interpreted in terms of the symplectic geometry of phase space. If we identify the forms C^a with differential forms $d\phi^a$ on phase space and let the generators act through the graded commutator, we find that the Q is the *exterior derivative* on phase-space.[★] The anti-BRST charge $\bar{Q}$ is the symplectic dual to the exterior derivative, lowering the form number by one. K is the symplectic 2-form while $\bar{K}$ is its dual, and Q_f is the operator for form number.

★ Motivated by analogy with the exterior derivative of orbit space of gauge theory, we call Q the BRST charge^[21].

The generators (2.41) are not the only charges which commute with the CPI Hamiltonian $\tilde{\mathcal{H}}$. For every constant of the motion $I(\phi)$ we can construct the conserved charges

$$\begin{aligned} N_I &= [Q, I] \\ \bar{N}_I &= [\bar{Q}, I] \\ \tilde{I} &= i[\bar{Q}, [Q, I]] \end{aligned} \tag{2.42}$$

The third charge $\tilde{I}$ generates the Lie derivative along the Hamiltonian vector field corresponding to $I(\phi)$. In particular, let us apply this construction to the Hamiltonian $H(\phi)$ itself. Consider the following two combinations:^{[22]†}

$$\begin{aligned} Q_H &= Q - \beta N_H = C^a \left(i\Lambda_a - \beta \frac{\partial H}{\partial \phi^a} \right) \\ \bar{Q}_H &= \bar{Q} + \beta \bar{N}_H = \bar{C}_a \omega^{ab} \left(\Lambda_b + \beta \frac{\partial H}{\partial \phi^b} \right) \end{aligned} \tag{2.43}$$

It is easy to verify^[15] that

$$[Q_H, Q_H] = [\bar{Q}_H, \bar{Q}_H] = 0 \tag{2.44}$$

and

$$[Q_H, \bar{Q}_H] = 2i\beta\tilde{\mathcal{H}} \tag{2.45}$$

Equation (2.45) shows that the anticommutator of Q_H and $\bar{Q}_H$ is the CPI Hamiltonian, and thus these operators are genuine *supersymmetry* generators while the Q and $\bar{Q}$ of Eq. (2.41) are BRST and anti-BRST operators, they are nilpotent and anticommute with each other.

In Ref. [15] it was shown that this supersymmetry is connected with ergodic properties of Hamiltonian systems. Let us consider what states are annihilated by these supercharges. If we restrict our attention to observables O with form number zero, we need only consider states $\tilde{\varrho}(\phi, C)$ with form number $2n$ to obtain a nonvanishing result from the Grassmann integration in (2.16). Then

$$Q_H \tilde{\varrho} = C^a \left(\partial_a - \beta \frac{\partial H}{\partial \phi^a} \right) \tilde{\varrho}(\phi, C) = 0 \tag{2.46}$$

† β is an arbitrary complex parameter.

automatically and

$$\bar{Q}_H \tilde{\varrho}(\phi, C) = \frac{\partial}{\partial C^a} \omega^{ab} (\partial_a + \beta \frac{\partial H}{\partial \phi^b}) \tilde{\varrho}(\phi, C) = 0 \quad (2.47)$$

is satisfied^[15] only if

$$\tilde{\varrho}(\phi, C) = k e^{-\beta H(\phi)} C^1 \dots C^{2n} \quad (2.48)$$

where k is a constant. This result tells us that the supersymmetric invariant state has to be a Gibbs state where β can be identified with the inverse temperature.^{[23][24]} This state is ergodic since it depends only on the Hamiltonian. Now, when the supersymmetry is unbroken, the only ground state $\tilde{\varrho}_0$ with $\tilde{\mathcal{H}}\tilde{\varrho}_0 = 0$ is the super symmetric invariant state (2.48). Therefore, if the supersymmetry is unbroken, then the system is ergodic.[‡]

The symmetry generators (2.41) and (2.43) were derived for Hamiltonian systems. A fluid, however, is not a Hamiltonian system because of dissipation. It is therefore of interest to investigate to what extent these symmetries are preserved by a general dynamical system (2.1). The charges $\bar{Q}$, K and $\bar{K}$ are constructed from ω_{ab} and ω^{ab} which exist only when the phase space is a symplectic manifold, or, at least a Poisson manifold, in which case ω^{ab} exists. That leaves the charges Q and Q_f . The form-number charge Q_f will always be conserved. It is easy to verify that

$$[Q_f, \tilde{\mathcal{H}}] = 0 \quad (2.49)$$

for any $\tilde{\mathcal{H}}$ given by (2.10). For the BRST charge we find

$$[Q, \tilde{\mathcal{H}}] = i C^b \frac{\partial}{\partial \phi^b} \frac{\partial V^a}{\partial \phi^a} \quad (2.50)$$

The BRST charge is conserved if and only if the divergence of the flow V is constant on phase space

$$\frac{\partial V^a}{\partial \phi^a} = \text{constant} \quad (2.51)$$

This will turn out to be the case for fluids. For Hamiltonian systems $V^a = h^a$ the stronger condition $\frac{\partial h^a}{\partial \phi^a} = 0$ is satisfied, as a glance at (2.40) reveals. To construct the other $ISp(2)$ charges we would need to find antisymmetric tensor fields ω^{ab} and ω_{ab} which are Lie transported by V . For the supersymmetry we would need to construct a pair of Grassmann-odd charges for which the anticommutator yields (2.10). In this paper we shall be less ambitious.

‡ See Ref. [15] for more details in this argument.

Instead, we shall construct $ISp(2)$ charges and supercharges which generate exact symmetries for the classical path integral of the incompressible fluid when the viscosity is set equal to zero and then check which ones are preserved by the dissipation.

3. The Classical Path Integral for Incompressible Fluids

Let us consider the fundamental fluid equations (1.1) and (1.2). We can use the incompressibility condition (1.2) to eliminate the pressure term in (1.1) thus obtaining^[1] an evolution equation for the velocity field alone

$$\frac{\partial v^a(\mathbf{x}, t)}{\partial t} = -\frac{\partial}{\partial x^b} D^{ac}(\nabla)(v^b v^c) + \nu \nabla^2 v^a \quad (3.1)$$

where

$$D^{ab}(\nabla) = \delta^{ab} - \partial^a \Delta^{-1} \partial^b \quad (3.2)$$

is the operator which projects every vector field onto its solenoidal, or divergenceless part, Δ^{-1} being the inverse of the Laplacian given in position space by

$$\Delta^{-1} f(\mathbf{x}) = \int d^3 \mathbf{y} \left(-\frac{1}{4\pi |\mathbf{x} - \mathbf{y}|} \right) f(\mathbf{y})$$

The pressure can be obtained from the velocity according to

$$P(\mathbf{x}, t) = \Delta^{-1} \left(-\rho \frac{\partial v^a}{\partial x^b} \frac{\partial v^b}{\partial x^a} \right) \quad (3.3)$$

Therefore at any instant in time t the state of an incompressible homentropic fluid of constant density is specified by the velocity field $\mathbf{v}(\mathbf{x}, t)$.

The configuration space of an incompressible fluid is the group $\mathcal{D}_{vol}$ of volume-preserving diffeomorphisms on the region occupied by the fluid.^[25] The space of divergence-free vector fields (velocity fields) is the Lie algebra $\mathcal{G}_{vol}$ of $\mathcal{D}_{vol}$. The classical path integral will propagate points in this space. The dynamics of an incompressible fluid can be separated into a conservative evolution given by the Euler equation

$$\frac{\partial v^a(\mathbf{x}, t)}{\partial t} = -\frac{\partial}{\partial x^b} D^{ac}(\nabla)(v^b v^c) \quad (3.4)$$

plus dissipation coming from the term $\nu \nabla^2 v^a$ in the Navier-Stokes equation (3.1). The Euler equation can be formulated as a Hamiltonian system^{[26][27]} where the Hamiltonian is

just the kinetic energy[§]

$$H[\mathbf{v}] = \frac{1}{2} \int d^3\mathbf{x} \, \mathbf{v}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x}) \quad (3.5)$$

The canonical coordinates for this Hamiltonian formulation are the Clebsch variables^[28] $\lambda(\mathbf{x}, t)$ and $\mu(\mathbf{x}, t)$ which are related to the velocity of an incompressible fluid by

$$v^a(\mathbf{x}, t) = D^{ab}(\nabla)(\lambda(\mathbf{x}, t) \frac{\partial}{\partial x^b} \mu(\mathbf{x}, t)) \quad (3.6)$$

In terms of the Clebsch variables the Poisson bracket is given by the simple expression

$$\{F[\lambda, \mu], G[\lambda, \mu]\} = \int d^3\mathbf{x} \left[\frac{\delta F}{\delta \lambda(\mathbf{x})} \frac{\delta G}{\delta \mu(\mathbf{x})} - \frac{\delta F}{\delta \mu(\mathbf{x})} \frac{\delta G}{\delta \lambda(\mathbf{x})} \right] \quad (3.7)$$

Substituting (3.6) into (3.5) and using (3.7), one can obtain Hamilton's equations of motion for λ and μ .

In spite of the simple form of the Poisson bracket (3.7), there are several reasons why the Clebsch variables are not convenient for the formulation of the classical path integral for fluids. First, these variables become multivalued for topologically nontrivial flows^[29], which have nonvanishing helicity

$$I = \int d^3\mathbf{x} \, (\nabla \times \mathbf{v}) \cdot \mathbf{v} \quad (3.8)$$

Second, the Clebsch variables do not have a simple physical interpretation. We want to be able use the classical path integral to calculate correlation functions among physical quantities such as velocity or vorticity. Third, when dissipation is brought back into the picture (3.6) must be substituted into the full Navier-Stokes equation (3.1). Then it is no longer possible to disentangle equations for $\frac{\partial \mu}{\partial t}$ and $\frac{\partial \lambda}{\partial t}$ from each other.

For these reasons we have chosen to use the noncanonical variables $\mathbf{v}$. Using (3.6) to express (3.7) in terms of velocities, we obtain the Poisson bracket^[27]

$$\{F[\mathbf{v}], G[\mathbf{v}]\} = (\nabla \times \mathbf{v}) \cdot \left(\frac{\delta F}{\delta \mathbf{v}} - \nabla \Delta^{-1} \nabla \cdot \frac{\delta F}{\delta \mathbf{v}} \right) \times \left(\frac{\delta G}{\delta \mathbf{v}} - \nabla \Delta^{-1} \nabla \cdot \frac{\delta G}{\delta \mathbf{v}} \right) \quad (3.9)$$

with an implicit integration over $\mathbf{x}$. It is often convenient to Fourier transform to wave-vector space where the gradient acts through multiplication by $\mathbf{k}$ and the inverse Laplacian acts

§ The (constant) density has been set equal to one.

simply on each Fourier component according to

$$\Delta^{-1}f(\mathbf{k}) = -\frac{1}{|\mathbf{k}|^2}f(\mathbf{k})$$

In wave vector space the Poisson bracket is given by

$$\begin{aligned} \{F[\mathbf{v}], G[\mathbf{v}]\} &= \int d^3\mathbf{k}_1 d^3\mathbf{k}_2 (2\pi)^{-3} i(\mathbf{k}_1 + \mathbf{k}_2) \times \mathbf{v}(\mathbf{k}_1 + \mathbf{k}_2) \cdot \\ &\quad \left(\frac{\delta F}{\delta \mathbf{v}(\mathbf{k}_1)} - \frac{\mathbf{k}_1}{|\mathbf{k}_1|^2} \mathbf{k}_1 \cdot \frac{\delta F}{\delta \mathbf{v}(\mathbf{k}_1)} \right) \times \left(\frac{\delta G}{\delta \mathbf{v}(\mathbf{k}_2)} - \frac{\mathbf{k}_2}{|\mathbf{k}_2|^2} \mathbf{k}_2 \cdot \frac{\delta G}{\delta \mathbf{v}(\mathbf{k}_2)} \right) \end{aligned} \quad (3.10)$$

From this we derive the tensor

$$\begin{aligned} \omega^{ab}(\mathbf{k}_1, \mathbf{k}_2) &\equiv \{v^a(\mathbf{k}_1), v^b(\mathbf{k}_2)\} \\ &= \frac{i}{(2\pi)^3} [D^{ac}(\mathbf{k}_1) k_2^c D^{be}(\mathbf{k}_2) v^e(\mathbf{k}_1 + \mathbf{k}_2) - D^{ac}(\mathbf{k}_1) k_1^e D^{be}(\mathbf{k}_2) v^c(\mathbf{k}_1 + \mathbf{k}_2)] \end{aligned} \quad (3.11)$$

for the Poisson structure, where^[1]

$$D^{ab}(\mathbf{k}) = \delta^{ab} - \frac{k^a k^b}{|\mathbf{k}|^2} \quad (3.12)$$

is the Fourier transform of the operator $D^{ab}(\nabla)$ given by (3.2). Using the Hamiltonian in wave vector space

$$H[\mathbf{v}] = \frac{1}{2} \int d^3\mathbf{k} (2\pi)^3 \mathbf{v}(\mathbf{k}) \cdot \mathbf{v}(-\mathbf{k}) \quad (3.13)$$

we can derive the Hamiltonian vector field

$$\begin{aligned} h^a(\mathbf{k}) &= \int d^3\mathbf{k}' \omega^{ab}(\mathbf{k}, \mathbf{k}') \frac{\delta H}{\delta v^b(\mathbf{k}')} \\ &= -i \int d^3\mathbf{k}' k^b D^{ac}(\mathbf{k}) v^b(\mathbf{k}') v^c(\mathbf{k} - \mathbf{k}') \\ &= \int d^3\mathbf{k}' M^{abc}(\mathbf{k}) v^b(\mathbf{k}') v^c(\mathbf{k} - \mathbf{k}') \end{aligned} \quad (3.14)$$

where^[1]

$$M^{abc}(\mathbf{k}) = -\frac{i}{2} [k^c D^{ab}(\mathbf{k}) + k^b D^{ac}(\mathbf{k})] \quad (3.15)$$

is the Fourier transform of

$$M^{abc}(\nabla) = -\frac{1}{2} [\partial^c D^{ab}(\nabla) + \partial^b D^{ac}(\nabla)] \quad (3.16)$$

In analogy to the finite dimensional case, let us introduce the auxilliary functions Λ , forms C and anti-forms $\bar{C}$, which each have three space indices and can depend on either $\mathbf{x}$

or $\mathbf{k}$. We use the following conventions for the Fourier transformation between $\mathbf{x}$ -space and $\mathbf{k}$ -space:

$$\begin{aligned}
v^a(\mathbf{x}) &= \int d^3\mathbf{k} v^a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \\
v^a(\mathbf{k}) &= \int \frac{d^3\mathbf{x}}{(2\pi)^3} v^a(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \\
C^a(\mathbf{x}) &= \int d^3\mathbf{k} C^a(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \\
C^a(\mathbf{k}) &= \int \frac{d^3\mathbf{x}}{(2\pi)^3} C^a(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \\
\Lambda_a(\mathbf{x}) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \Lambda_a(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \\
\Lambda_a(\mathbf{k}) &= \int d^3\mathbf{x} \Lambda_a(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} \\
\bar{C}_a(\mathbf{x}) &= \int \frac{d^3\mathbf{k}}{(2\pi)^3} \bar{C}_a(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} \\
\bar{C}_a(\mathbf{k}) &= \int d^3\mathbf{x} \bar{C}_a(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}}
\end{aligned} \tag{3.17}$$

In analogy to (2.13), these variables satisfy the graded commutators

$$\begin{aligned}
[\Lambda_a(\mathbf{x}), v^b(\mathbf{x}')] &= -i\delta_a^b \delta^3(\mathbf{x} - \mathbf{x}') \\
[\Lambda_a(\mathbf{k}), v^b(\mathbf{k}')] &= -i\delta_a^b \delta^3(\mathbf{k} - \mathbf{k}') \\
[\bar{C}_a(\mathbf{x}), C^b(\mathbf{x}')] &= \delta_a^b \delta^3(\mathbf{x} - \mathbf{x}') \\
[\bar{C}_a(\mathbf{k}), C^b(\mathbf{k}')] &= \delta_a^b \delta^3(\mathbf{k} - \mathbf{k}')
\end{aligned} \tag{3.18}$$

The CPI Hamiltonian $\tilde{\mathcal{H}}_0$ for the conservative part of the evolution is given in analogy to (2.10) by

$$\begin{aligned}
\tilde{\mathcal{H}}_0 &= \int d^3\mathbf{k} \Lambda_a(\mathbf{k}) h^a(\mathbf{k}) \\
&\quad + i \int d^3\mathbf{k} d^3\mathbf{k}' \bar{C}_a(\mathbf{k}) \frac{\delta h^a(\mathbf{k})}{\delta v^b(\mathbf{k}')} C^b(\mathbf{k}') \\
&= \int d^3\mathbf{k} d^3\mathbf{k}' [-i\Lambda_a(\mathbf{k}) k^b D^{ac}(\mathbf{k}) v^b(\mathbf{k}') v^c(\mathbf{k} - \mathbf{k}') \\
&\quad + \bar{C}_a(\mathbf{k}) [k^c D^{ab}(\mathbf{k}) + k^b D^{ac}(\mathbf{k})] v^c(\mathbf{k} - \mathbf{k}') C^b(\mathbf{k}')] \\
&= \int d^3\mathbf{k} d^3\mathbf{k}' M^{abc}(\mathbf{k}) [\Lambda_a(\mathbf{k}) v^b(\mathbf{k}') v^c(\mathbf{k} - \mathbf{k}') \\
&\quad + 2i\bar{C}_a(\mathbf{k}) v^c(\mathbf{k} - \mathbf{k}') C^b(\mathbf{k}')]
\end{aligned} \tag{3.19}$$

In position space $\tilde{\mathcal{H}}_0$ is given by

$$\begin{aligned} \tilde{\mathcal{H}}_0 = \int d^3\mathbf{x} [\Lambda_a(\mathbf{x})M^{abc}(\nabla)(v^b(\mathbf{x})v^c(\mathbf{x})) \\ + 2i\bar{C}_a(\mathbf{x})M^{abc}(\nabla)(C^b(\mathbf{x})v^c(\mathbf{x}))] \end{aligned} \quad (3.20)$$

The dissipative part of the evolution is generated by

$$\tilde{\mathcal{H}}_d = \nu \int d^3\mathbf{x} (\Lambda_a(\mathbf{x})\nabla^2 v^a(\mathbf{x}) + i\bar{C}_a(\mathbf{x})\nabla^2 C^a(\mathbf{x})) \quad (3.21)$$

in position space or

$$\tilde{\mathcal{H}}_d = -\nu \int d^3\mathbf{k} (\Lambda_a(\mathbf{k})|\mathbf{k}|^2 v^a(\mathbf{k}) + i\bar{C}_a(\mathbf{k})|\mathbf{k}|^2 C^a(\mathbf{k})) \quad (3.22)$$

in wave vector space. The total CPI Hamiltonian is then

$$\tilde{\mathcal{H}} = \tilde{\mathcal{H}}_0 + \tilde{\mathcal{H}}_d \quad (3.23)$$

It generates the evolution according to (3.1). Including the appropriate kinetic terms we obtain the CPI Lagrangian for incompressible fluids in the velocity representation

$$\tilde{\mathcal{L}} = \mathbf{\Lambda} \cdot \dot{\mathbf{v}} + i\bar{\mathbf{C}} \cdot \dot{\mathbf{C}} - \tilde{\mathcal{H}} \quad (3.24)$$

where the dot here indicates integration over $\mathbf{x}$ or $\mathbf{k}$ as well as contraction over the spacial index a .

In addition, the velocity field must satisfy the subsidiary condition $\nabla \cdot \mathbf{v} = 0$. We can account for this condition according to the prescription of Section 2. by adding the terms

$$\int d^3\mathbf{x} [\eta(\mathbf{x}, t)\nabla \cdot \mathbf{v}(\mathbf{x}, t) + i\bar{\xi}(\mathbf{x}, t)\nabla \cdot \mathbf{C}(\mathbf{x}, t)]$$

to the CPI Lagrangian. Since η and $\bar{\xi}$ do not propagate, we integrate them out of the path integral leaving

$$\tilde{\delta}[\nabla \cdot \mathbf{v}]\tilde{\delta}[\nabla \cdot \mathbf{C}]$$

in the measure.

We may wish to add an external force to the action to counteract the effects of dissipation. Since we want the path integral to contain information about how small variations in driving force induce variations in flow^[30], we would introduce the Grassmann variable $\mathbf{C}_r(\mathbf{k}, t)$ along with the force $\mathbf{f}(\mathbf{k}, t)$ so that the terms to be added to the CPI Langrangian are

$$\tilde{\mathcal{L}}_f = \mathbf{\Lambda} \cdot \mathbf{f} + \bar{\mathbf{C}} \cdot \mathbf{C}_r \quad (3.25)$$

Now, in the full Langrangian $\tilde{\mathcal{L}} + \tilde{\mathcal{L}}_f$ $\mathbf{\Lambda}$ multiplies the forced Navier-Stokes equation while $\bar{\mathbf{C}}$ multiplies Jacobi's equation for the first order variation in velocity. We emphasize that in our approach the external force need not be random. This is a deterministic path integral. Ultimately, we hope to understand turbulence as deterministic chaos arising from the Navier-Stokes equation.

The physical states $\tilde{q}[\mathbf{v}, \mathbf{C}, t]$ are now functionals on the superspace containing velocity fields and their first order variations, the forms. They are propagated by the kernel

$$K_{(f, C_f)}[\mathbf{v}, \mathbf{C}, t | \mathbf{v}_0, \mathbf{C}_0, t_0] = \int_{\mathbf{v}_0, \mathbf{C}_0}^{\mathbf{v}, \mathbf{C}} \mathcal{D}[\mathbf{v}] \mathcal{D}[\mathbf{\Lambda}] \mathcal{D}[\mathbf{C}] \mathcal{D}[\bar{\mathbf{C}}] \tilde{\delta}[\nabla \cdot \mathbf{v}] \tilde{\delta}[\nabla \cdot \mathbf{C}] \exp[i \int_{t_0}^t dt' (\tilde{\mathcal{L}} + \tilde{\mathcal{L}}_f)] \quad (3.26)$$

Similarly, we can construct the Hopf functional, turbulence kernel and the generating functional as in Section 2. Only the solenoidal parts of the fields $\mathbf{\Lambda}$ and $\bar{\mathbf{C}}$ contribute to the path integral, since they are both contracted with divergenceless vectors in every term of $\tilde{\mathcal{L}} + \tilde{\mathcal{L}}_f$.^{*} Every vector field, for example $\mathbf{\Lambda}$, can be decomposed into a solenoidal part $\mathbf{\Lambda}_{sol}$ plus the gradient of a scalar function ∇L . Therefore each term in the CPI action involving $\mathbf{\Lambda}$ has the form

$$\begin{aligned} \int d^3\mathbf{x} (\mathbf{\Lambda}_{sol} + \nabla L) \cdot \mathbf{w} &= \int d^3\mathbf{x} (\mathbf{\Lambda}_{sol} \cdot \mathbf{w} - L \nabla \cdot \mathbf{w}) \\ &= \int d^3\mathbf{x} \mathbf{\Lambda}_{sol} \cdot \mathbf{w} \end{aligned}$$

where $\mathbf{w}$ is a divergenceless vector field constructed from $\mathbf{v}$.[†]

Now, let us consider the $ISp(2)$ and supersymmetry charges which are exact symmetries

^{*} The vector fields $\mathbf{f}$ and $\mathbf{C}_r$ are also chosen to be divergenceless.

[†] We assume throughout that the fields vanish on the boundaries or at infinity.

of the CPI action when $\tilde{\mathcal{H}}_d$ is omitted. The BRST charge is

$$\begin{aligned} Q &= i \int d^3\mathbf{x} \Lambda_a(\mathbf{x}) C^a(\mathbf{x}) \\ &= i \int d^3\mathbf{k} \Lambda_a(\mathbf{k}) C^a(\mathbf{k}) \end{aligned} \quad (3.27)$$

It acts as an exterior derivative on the space $\mathcal{G}_{vol}$ of solenoidal velocity fields. An explicit calculation reveals that Q is conserved not only by the conservative, but also by the dissipative part of the evolution

$$[Q, \tilde{\mathcal{H}}] = 0 \quad (3.28)$$

This comes about because $\tilde{\mathcal{H}}_d$ is linear in $\mathbf{v}$ and therefore the dissipative part of the flow on $\mathcal{G}_{vol}$ satisfies (2.51). The BRST charge generates an exact symmetry of the classical path integral. The form number charge

$$Q_f = \mathbf{C} \cdot \bar{\mathbf{C}} \quad (3.29)$$

also generates an exact symmetry. Physically this just means that the form number of any state $\tilde{\varrho}[\mathbf{v}, \mathbf{C}]$ is conserved.

The other $ISp(2)$ charges are constructed from $\omega^{ab}(\mathbf{k}, \mathbf{k}')$, given in (3.11), and its inverse $\omega_{ab}(\mathbf{k}, \mathbf{k}')$ (if it exists). The dual to the symplectic form is given explicitly by

$$\begin{aligned} \bar{K} &= \frac{1}{2} \int d^3\mathbf{k} d^3\mathbf{k}' \omega^{ab}(\mathbf{k}, \mathbf{k}') \bar{C}_a(\mathbf{k}) \bar{C}_b(\mathbf{k}') \\ &= \frac{i}{2} \int d^3\mathbf{k} d^3\mathbf{k}' (2\pi)^{-3} [D^{ac}(\mathbf{k}_1) k_2^c D^{be}(\mathbf{k}_2) v^e(\mathbf{k}_1 + \mathbf{k}_2) - D^{ac}(\mathbf{k}_1) k_1^e D^{be}(\mathbf{k}_2) v^c(\mathbf{k}_1 + \mathbf{k}_2)] \\ &\quad \bar{C}_a(\mathbf{k}) \bar{C}_b(\mathbf{k}') \\ &= - \int d^3\mathbf{x} \mathbf{v} \cdot (\bar{\mathbf{C}} \cdot \nabla \bar{\mathbf{C}}) \end{aligned} \quad (3.30)$$

Some care must be taken in the construction of the anti-BRST charge since we are working in noncanonical coordinates. In Ref. [19] it was shown that when ω is not constant on phase space, we must modify Eq. (2.41) for $\bar{Q}$ by adding of the term $-\frac{1}{2} \frac{\partial \omega^{ab}}{\partial \phi^e} C^e \bar{C}_a \bar{C}_b$, in order to obtain a conserved charge which can still be interpreted as the symplectic dual to the exterior derivative. Applying the infinite dimensional generalization of this construction

$$\begin{aligned} \bar{Q} &= i \int d^3\mathbf{k} d^3\mathbf{k}' \omega^{ab}(\mathbf{k}, \mathbf{k}') \bar{C}_a(\mathbf{k}) \Lambda_b(\mathbf{k}') \\ &\quad - \frac{1}{2} \int d^3\mathbf{k} d^3\mathbf{k}' d^3\mathbf{k}'' \frac{\delta \omega^{ab}(\mathbf{k}, \mathbf{k}')}{\delta v^c(\mathbf{k}'')} C^c(\mathbf{k}'') \bar{C}_a(\mathbf{k}) \bar{C}_b(\mathbf{k}') \end{aligned} \quad (3.31)$$

we obtain

$$\begin{aligned} \bar{Q} = \int d^3\mathbf{k} d^3\mathbf{k}' (2\pi)^{-3} [D^{ac}(\mathbf{k}) k^e D^{be}(\mathbf{k}') - D^{ae}(\mathbf{k}) k'^e D^{bc}(\mathbf{k}')] \\ [v^c(\mathbf{k} + \mathbf{k}') \bar{C}_a(\mathbf{k}) \Lambda_b(\mathbf{k}') + \frac{i}{2} C^c(\mathbf{k} + \mathbf{k}') \bar{C}_a(\mathbf{k}) \bar{C}_b(\mathbf{k}')] \end{aligned} \quad (3.32)$$

Taking $\mathbf{\Lambda}$ and $\bar{\mathbf{C}}$, as well as $\mathbf{v}$ and $\mathbf{C}$, to be solenoidal vector fields, and Fourier transforming, we obtain the simpler expression

$$\bar{Q} = \int d^3\mathbf{x} (i\mathbf{v} \cdot (\mathbf{\Lambda}, \bar{\mathbf{C}}) + \mathbf{C} \cdot (\bar{\mathbf{C}} \cdot \nabla \bar{\mathbf{C}})) \quad (3.33)$$

where

$$(\mathbf{\Lambda}, \bar{\mathbf{C}}) = \mathbf{\Lambda} \cdot \nabla \bar{\mathbf{C}} - \bar{\mathbf{C}} \cdot \nabla \mathbf{\Lambda}$$

is the Lie bracket of vector fields.

$\bar{K}$ and $\bar{Q}$ both commute with $\tilde{\mathcal{H}}_0$; they are exact symmetries of Euler's equation. However, they are not conserved by the dissipative part of the evolution

$$\begin{aligned} [\bar{K}, \tilde{\mathcal{H}}_d] &= i\nu \int d^3\mathbf{x} (\nabla_1^2 - \nabla_2^2 - \nabla_3^2) \mathbf{v} \cdot (\bar{\mathbf{C}} \cdot \nabla \bar{\mathbf{C}}) \\ [\bar{Q}, \tilde{\mathcal{H}}_d] &= i\nu \int d^3\mathbf{x} (\nabla_1^2 - \nabla_2^2 - \nabla_3^2) (i\mathbf{v} \cdot (\mathbf{\Lambda}, \bar{\mathbf{C}}) + \mathbf{C} \cdot (\bar{\mathbf{C}} \cdot \nabla \bar{\mathbf{C}})) \end{aligned} \quad (3.34)$$

where ∇_1^2 acts on the first slot, ∇_2^2 acts on the second slot and ∇_3^2 acts on the third slot for each term to the right. The Poisson structure is not Lie transported by the dissipative part of the evolution.

The supersymmetry charges are constructed from the Hamiltonian (3.13).

$$\begin{aligned} Q_H &= Q - \beta[Q, H] \\ &= \int d^3\mathbf{x} \mathbf{C}(\mathbf{x}) \cdot (i\mathbf{\Lambda}(\mathbf{x}) - \beta\mathbf{v}(\mathbf{x})) \\ \bar{Q}_H &= \bar{Q} + \beta[\bar{Q}, H] \\ &= \int d^3\mathbf{x} (\mathbf{v} \cdot (i\mathbf{\Lambda} + \beta\mathbf{v}, \bar{\mathbf{C}}) + \mathbf{C} \cdot (\bar{\mathbf{C}} \cdot \nabla \bar{\mathbf{C}})) \end{aligned} \quad (3.35)$$

They satisfy

$$[Q_H, \bar{Q}_H] = 2i\beta\tilde{\mathcal{H}}_0 \quad (3.36)$$

They each commute with $\tilde{\mathcal{H}}_0$ but they are not conserved by the dissipative part of the

evolution

$$[Q_H, \tilde{\mathcal{H}}_d] = -2i\beta\nu \int d^3\mathbf{x} \mathbf{C}(\mathbf{x}) \cdot \nabla^2 \mathbf{v}(\mathbf{x}) \quad (3.37)$$

$$[\bar{Q}_H, \tilde{\mathcal{H}}_d] = i\nu \int d^3\mathbf{x} (\nabla_1^2 - \nabla_2^2 - \nabla_3^2) (\mathbf{v} \cdot (i\boldsymbol{\Lambda}, \bar{\mathbf{C}}) + \mathbf{C} \cdot (\bar{\mathbf{C}} \cdot \nabla \bar{\mathbf{C}})) \\ \beta(\nabla_1^2 + \nabla_2^2 - \nabla_3^2) \mathbf{v} \cdot (\mathbf{v}, \mathbf{C}) \quad (3.38)$$

The supersymmetry is broken as a result of dissipation. In Section 2. we saw that the supersymmetric state is the Gibbs state. The probability distribution for turbulence is not Gibbsian and a central problem is to understand how non-Gibbsian distributions arise.^{[10][31][32]} An interesting topic for future research is the relationship between the explicit breaking of supersymmetry in the classical path integral and the appearance of non-Gibbsian stationary states.

4. Kolmogorov Scaling and Galilean Invariance

The symmetries found in the previous section are connected with the space $\mathcal{G}_{vol}$ of velocity fields and its Hamiltonian structure. The Euler equation also possesses space and time symmetries. From the CPI Lagrangian (3.24) we can derive for every symmetry the Noether charge

$$Q_\alpha = -\tilde{\mathcal{H}} \frac{\partial}{\partial \epsilon^\alpha} \delta t + \int d^3\mathbf{x} \left[-(\Lambda_a \frac{\partial v^a}{\partial x^b} + i\bar{C}_a \frac{\partial C^a}{\partial x^b}) \frac{\partial}{\partial \epsilon^\alpha} \delta x^b + \Lambda_a \frac{\partial}{\partial \epsilon^\alpha} \delta v^a + i\bar{C}_a \frac{\partial}{\partial \epsilon^\alpha} \delta C^a \right] \quad (4.1)$$

(ϵ^α being the parameter of the transformation). In this section we construct the symmetry charges for Kolmogorov scaling and the generators of the Galilean group.

The Euler equation is symmetric under the rescaling of space and time.[★]

$$\begin{aligned} \mathbf{x} &\longrightarrow \mathbf{x}' = \lambda \mathbf{x} \\ t &\longrightarrow t' = \lambda^{1-h} t \\ \mathbf{v} &\longrightarrow \mathbf{v}' = \lambda^h \mathbf{v} \end{aligned} \quad (4.2)$$

Kolmogorov^[33] assumed that this symmetry holds statistically for the Navier-Stokes equation and used it to derive spectral properties of turbulence. The requirement that the energy flux

★ For a discussion of Kolmogorov scaling based on this symmetry see the second of Reference 3.

through a given length scale is scale invariant then leads to $h = \frac{1}{3}$ in Eq. (4.2). The Noether charge which generates the symmetry (4.2) in the classical path integral is

$$Q_K = -(1-h)t\tilde{\mathcal{H}} + \int d^3\mathbf{x} \left[-x^b(\Lambda_a \frac{\partial v^a}{\partial x^b} + i\bar{C}_a \frac{\partial C^a}{\partial x^b}) + h(\Lambda_a v^a + i\bar{C}_a C^a) \right] \quad (4.3)$$

This charge is conserved in the zero viscosity limit. In the presence of dissipation we have

$$\frac{d}{dt}Q_K = (h+1)\tilde{\mathcal{H}}_d \quad (4.4)$$

The CPI action is exactly symmetric under spacial translations

$$\delta x^a = \epsilon^a, \quad \delta \mathbf{v} = \delta \mathbf{C} = 0 \quad (4.5)$$

and rotations

$$\delta x^a = \varepsilon^{abc} x^b \epsilon^c, \quad \delta v^a = \varepsilon^{abc} v^b \epsilon^c, \quad \delta C^a = \varepsilon^{abc} C^b \epsilon^c \quad (4.6)$$

The generators for (4.5) and (4.6) are given, respectively, by

$$Q_{T^a} = - \int d^3\mathbf{x} \left(\Lambda_b \frac{\partial v^b}{\partial x^a} + i\bar{C}_b \frac{\partial C^b}{\partial x^a} \right) \quad (4.7)$$

and

$$Q_{R^j} = \int d^3\mathbf{x} \left[-(\Lambda_a \frac{\partial v^a}{\partial x^b} + i\bar{C}_a \frac{\partial C^a}{\partial x^b}) \varepsilon^{jbc} x^c + (\Lambda_b v^c + i\bar{C}_b C^c) \varepsilon^{jbc} \right] \quad (4.8)$$

The velocity field transforms inhomogeneously under Galilean boosts.

$$\mathbf{v}(\mathbf{x}, t) \longrightarrow \mathbf{v}'(\mathbf{x}', t') = \mathbf{v}(\mathbf{x} - \mathbf{u}t, t) + \mathbf{u}$$

Since shifting all velocities by a constant amount should not affect deviations among velocities, we assume that

$$\mathbf{C}(\mathbf{x}, t) \longrightarrow \mathbf{C}'(\mathbf{x}', t') = \mathbf{C}(\mathbf{x} - \mathbf{u}t, t)$$

Thus, taking

$$\begin{aligned} \delta \mathbf{x} &= -t\delta \mathbf{u} \\ \delta \mathbf{v} &= -t\delta \mathbf{u} \cdot \nabla \mathbf{v} + \delta \mathbf{u} \\ \delta \mathbf{C} &= -t\delta \mathbf{u} \cdot \nabla \mathbf{C} \end{aligned} \quad (4.9)$$

we find the generator for Galilean boosts

$$Q_{B^j} = \int d^3\mathbf{x} \Lambda_j(\mathbf{x}) \quad (4.10)$$

This is not conserved even in the absence of viscosity, as can be readily seen from Eq. (2.21).

In fact, eliminating surface terms we find

$$\frac{d}{dt}Q_{B^j} = \int d^3\mathbf{x} [\Lambda_a D^{ab}(\nabla) \frac{\partial v^b}{\partial x^j} - i\bar{C}_a D^{ab}(\nabla) \frac{\partial C^b}{\partial x^j}] \quad (4.11)$$

5. CONCLUSIONS

We have constructed a classical path integral for incompressible fluids based on the deterministic equation of motion (3.1). This formalism treats an incompressible fluid as a dynamical system on the space $\mathcal{G}_{vol}$ of velocity fields. The path integral also lends itself to a statistical description of turbulence by propagating ensembles of velocity fields on $\mathcal{G}_{vol}$. The probability distribution of fully developed turbulence would then be stationary under this propagation.

The next stage will involve the more difficult task of applying the classical path integral to the study of turbulence. A particularly important feature of the classical path integral is the presence of fermionic variables which encode information about deviations among trajectories on $\mathcal{G}_{vol}$. Expectation values of operators such as (2.24) and (2.27) can be used to study the stability of flows near the onset of turbulence.

The classical path integral possesses a set of symmetries associated with the geometry of $\mathcal{G}_{vol}$ and its Hamiltonian structure. An additional set of symmetries is connected with scaling and Galilean transformations of space and time. These symmetries can be used to derive Ward-Takahashi identities which yield relations among the correlation functions.

There are other symmetries which pose difficulties for this framework. They are associated with the Lagrangian invariants such as helicity (3.8) and the circulation

$$\Gamma_{\mathcal{C}}[\mathbf{v}] = \oint_{\mathcal{C}} \mathbf{v} \cdot d\mathbf{l} \quad (5.1)$$

Let us concentrate on the circulation. In the integral (5.1) the loop $\mathcal{C}$ follows the flow of the fluid. We can introduce the position variable $\mathbf{X}(t, \theta)$ for points on the loop to reexpress (5.1) as follows:

$$\Gamma_{\mathcal{C}} = \int_0^{2\pi} d\theta \int d^3\mathbf{x} \frac{\partial \mathbf{X}(t, \theta)}{\partial \theta} \cdot \mathbf{v}(\mathbf{x}, t) \delta^3(\mathbf{x} - \mathbf{X}(t, \theta)) \quad (5.2)$$

Taking into account the Eulerian evolution of the velocity field (3.4) and the fact that

$$\frac{\partial \mathbf{X}(t, \theta)}{\partial t} = \mathbf{v}(\mathbf{X}(t, \theta), t) \quad (5.3)$$

we find that the circulation is conserved. Now, we would like to construct conserved charges of the classical path integral according to the prescription (2.42). Naively, we could suppose that the functional derivative with respect to velocity acts only on the velocity. Then we would find that

$$N_\Gamma = \oint \mathbf{C} \cdot d\mathbf{l}$$

But this is not conserved. In fact, we find that

$$\frac{d}{dt} \oint \mathbf{C} \cdot d\mathbf{l} = \oint C^a \Omega^{ab} dl^b \quad (5.4)$$

with $\Omega^{ab} = \partial^a v^b - \partial^b v^a$, the components of the vorticity two-form. The problem is to find a sensible expression for the variational derivative with respect to velocity of loops which flow with the velocity.

This brings in another question of great importance. How can we use the classical path integral to propagate global flow structures such as eddies, vortex sheets and vortex cells^[32]? Such structures move under the action of diffeomorphisms in $\mathcal{D}_{vol}$. The classical path integral propagates velocity fields in $\mathcal{G}_{vol}$. Perhaps some additional variables^[34] are needed in the formulation of the classical path integral.

APPENDIX A

A. A. Migdal^[32] has argued that the fluctuations in the velocity field are too strong for $\mathbf{v}$ to be the useful variable in studies of turbulence with infinite Reynolds number. Moreover, the lack of invariance under Galilean boosts (4.9) results in infrared divergences in the two-point correlation function of velocities. The vorticity $\boldsymbol{\Omega}(\mathbf{x}, t) = \nabla \times \mathbf{v}(\mathbf{x}, t)$, on the other hand, is invariant under boosts

$$\boldsymbol{\Omega}(\mathbf{x}, t) \longrightarrow \boldsymbol{\Omega}'(\mathbf{x}', t') = \boldsymbol{\Omega}(\mathbf{x} - t\mathbf{u}, t)$$

Therefore, vorticity may be the more appropriate variable for the study of turbulence.

In this appendix we shall formulate the classical path integral in terms of the vorticity field. The solenoidal velocity field can always be recovered from the vorticity by means of the relation

$$\begin{aligned}\mathbf{v}(\mathbf{k}) &= \text{rot}^{-1} \boldsymbol{\Omega}(\mathbf{k}) \\ &= i \frac{\mathbf{k} \times \boldsymbol{\Omega}(\mathbf{k})}{|\mathbf{k}|^2} \\ \mathbf{v}(\mathbf{x}) &= \text{rot}^{-1} \boldsymbol{\Omega}(\mathbf{x}) \\ &= -\frac{1}{4\pi} \int d^3\mathbf{y} \frac{(\mathbf{x} - \mathbf{y}) \times \boldsymbol{\Omega}(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|^3}\end{aligned}\tag{A.1}$$

In terms of the Clebsch variables the vorticity is given by

$$\boldsymbol{\Omega} = \nabla \lambda \times \nabla \mu \tag{A.2}$$

Reexpressing (3.7) in terms of vorticity we then obtain^[27]

$$\begin{aligned}\{F[\boldsymbol{\Omega}], G[\boldsymbol{\Omega}]\} &= \int d^3\mathbf{x} \boldsymbol{\Omega}(\mathbf{x}) \cdot \left[\text{rot} \frac{\delta F}{\delta \boldsymbol{\Omega}}(\mathbf{x}) \times \text{rot} \frac{\delta G}{\delta \boldsymbol{\Omega}}(\mathbf{x}) \right] \\ &= - \int d^3\mathbf{k} d^3\mathbf{k}' (2\pi)^{-3} \boldsymbol{\Omega}(\mathbf{k} + \mathbf{k}') \cdot \left[\left(\mathbf{k} \times \frac{\delta F}{\delta \boldsymbol{\Omega}(\mathbf{k})} \right) \times \left(\mathbf{k}' \times \frac{\delta G}{\delta \boldsymbol{\Omega}(\mathbf{k}')} \right) \right]\end{aligned}\tag{A.3}$$

from which we read off

$$\omega^{ab}(\mathbf{k}, \mathbf{k}') = -\frac{1}{(2\pi)^3} \boldsymbol{\Omega}^c(\mathbf{k} + \mathbf{k}') \varepsilon^{cde} \varepsilon^{dfa} \varepsilon^{egb} k^f k'^g \tag{A.4}$$

Expressing the Hamiltonian in terms of vorticity

$$H[\boldsymbol{\Omega}] = -\frac{1}{2} \int d^3\mathbf{k} (2\pi)^3 \frac{\boldsymbol{\Omega}(\mathbf{k}) \cdot \boldsymbol{\Omega}(-\mathbf{k})}{|\mathbf{k}|^2} \tag{A.5}$$

we find as a result of

$$h^a(\mathbf{k}) = \int d^3\mathbf{k}' \omega^{ab}(\mathbf{k}, \mathbf{k}') \frac{\delta H[\boldsymbol{\Omega}]}{\delta \Omega^b(\mathbf{k})} \tag{A.6}$$

the Hamiltonian vector field on the space of vorticities

$$\mathbf{h}(\mathbf{k}) = i \int d^3\mathbf{k}' \mathbf{k} \times (\text{rot}^{-1} \boldsymbol{\Omega}(\mathbf{k}') \times \boldsymbol{\Omega}(\mathbf{k} - \mathbf{k}')) \tag{A.7}$$

Again we introduce the fields Λ , C and $\bar{C}$ which are tied to the vorticity and obey the commutation relations (3.27), vorticity taking the place of velocity. We construct the CPI

Hamiltonian for the conservative part of the evolution according to (3.19), obtaining

$$\begin{aligned}\tilde{\mathcal{H}}_0 = i \int d^3\mathbf{k} d^3\mathbf{k}' & [\mathbf{\Lambda}(\mathbf{k}) \cdot \mathbf{k} \times (\text{rot}^{-1}\mathbf{\Omega}(\mathbf{k}') \times \mathbf{\Omega}(\mathbf{k} - \mathbf{k}')) \\ & + i\bar{\mathbf{C}}(\mathbf{k}) \cdot \mathbf{k} \times (\text{rot}^{-1}\mathbf{C}(\mathbf{k}') \times \mathbf{\Omega}(\mathbf{k} - \mathbf{k}') + \text{rot}^{-1}\mathbf{\Omega}(\mathbf{k}') \times \mathbf{C}(\mathbf{k} - \mathbf{k}'))]\end{aligned}\quad (\text{A.8})$$

or

$$\begin{aligned}\tilde{\mathcal{H}}_0 = \int d^3\mathbf{x} & [\mathbf{\Lambda}(\mathbf{x}) \cdot \nabla \times (\text{rot}^{-1}\mathbf{\Omega}(\mathbf{x}) \times \mathbf{\Omega}(\mathbf{x})) \\ & + i\bar{\mathbf{C}}(\mathbf{x}) \cdot \nabla \times (\text{rot}^{-1}\mathbf{C}(\mathbf{x}) \times \mathbf{\Omega}(\mathbf{x}) + \text{rot}^{-1}\mathbf{\Omega}(\mathbf{x}) \times \mathbf{C}(\mathbf{x}))]\end{aligned}\quad (\text{A.9})$$

The dissipative part of the CPI Hamiltonian is

$$\begin{aligned}\tilde{\mathcal{H}}_d &= \nu \int d^3\mathbf{x} (\Lambda_a(\mathbf{x}) \nabla^2 \mathbf{\Omega}^a(\mathbf{x}) + i\bar{C}_a(\mathbf{x}) \nabla^2 C^a(\mathbf{x})) \\ &= -\nu \int d^3\mathbf{k} (\Lambda_a(\mathbf{k}) |\mathbf{k}|^2 \mathbf{\Omega}^a(\mathbf{k}) + i\bar{C}_a(\mathbf{k}) |\mathbf{k}|^2 C^a(\mathbf{k}))\end{aligned}\quad (\text{A.10})$$

The CPI Langrangian is

$$\tilde{\mathcal{L}} = \mathbf{\Lambda} \cdot \dot{\mathbf{\Omega}} + i\bar{\mathbf{C}} \cdot \dot{\mathbf{C}} - \tilde{\mathcal{H}}_0 - \tilde{\mathcal{H}}_d \quad (\text{A.11})$$

Since the vorticity is solenoidal the delta-functionals

$$\tilde{\delta}[\nabla \cdot \mathbf{\Omega}] \tilde{\delta}[\nabla \cdot \mathbf{C}]$$

need to be included again in the classical path integral measure.

The BRST and form number charges again generate exact symmetries. In the vorticity representation the dual to the symplectic form is given by

$$\begin{aligned}\bar{K} &= - \int d^3\mathbf{k} d^3\mathbf{k}' \mathbf{\Omega}(\mathbf{k} + \mathbf{k}') \cdot (\mathbf{k} \times \bar{\mathbf{C}}(\mathbf{k})) \times (\mathbf{k}' \times \bar{\mathbf{C}}(\mathbf{k}')) \\ &= \int d^3\mathbf{x} \mathbf{\Omega}(\mathbf{x}) \cdot (\nabla \times \bar{\mathbf{C}}(\mathbf{x})) \times (\nabla \times \bar{\mathbf{C}}(\mathbf{x}))\end{aligned}\quad (\text{A.12})$$

The anti-BRST charge is given in the vorticity representation by

$$\begin{aligned}\bar{Q} &= \int d^3\mathbf{k} d^3\mathbf{k}' (2\pi)^{-3} [-i\mathbf{\Omega}(\mathbf{k} + \mathbf{k}') \cdot (\mathbf{k} \times \bar{\mathbf{C}}(\mathbf{k})) \times (\mathbf{k}' \times \mathbf{\Lambda}(\mathbf{k}')) \\ &\quad + \frac{1}{2}\mathbf{C}(\mathbf{k} + \mathbf{k}') \cdot (\mathbf{k} \times \bar{\mathbf{C}}(\mathbf{k})) \times (\mathbf{k}' \times \bar{\mathbf{C}}(\mathbf{k}'))]\end{aligned}\quad (\text{A.13})$$

or

$$\begin{aligned}\bar{Q} = \int d^3\mathbf{x} [i\boldsymbol{\Omega}(\mathbf{x}) \cdot (\nabla \times \bar{\mathbf{C}}(\mathbf{x})) \times (\nabla \times \boldsymbol{\Lambda}(\mathbf{x})) \\ - \frac{1}{2}\mathbf{C}(\mathbf{x}) \cdot (\nabla \times \bar{\mathbf{C}}(\mathbf{x})) \times (\nabla \times \bar{\mathbf{C}}(\mathbf{x}))]\end{aligned}\quad (\text{A.14})$$

The supersymmetry charges, constructed as in Section 3., are given by

$$Q_H = \int d^3\mathbf{x} C^a(\mathbf{x})(i\Lambda_a(\mathbf{x}) - \beta \Delta^{-1}\Omega^a(\mathbf{x})) \quad (\text{A.15})$$

and

$$\begin{aligned}\bar{Q}_H = \int d^3\mathbf{x} [\boldsymbol{\Omega}(\mathbf{x}) \cdot (\nabla \times \bar{\mathbf{C}}(\mathbf{x})) \times (i\nabla \times \boldsymbol{\Lambda}(\mathbf{x}) - \beta \text{rot}^{-1}\boldsymbol{\Omega}(\mathbf{x})) \\ - \frac{1}{2}\mathbf{C}(\mathbf{x}) \cdot (\nabla \times \bar{\mathbf{C}}(\mathbf{x})) \times (\nabla \times \bar{\mathbf{C}}(\mathbf{x}))]\end{aligned}\quad (\text{A.16})$$

$\bar{K}$, $\bar{Q}$, Q_H and $\bar{Q}_H$ generate exact symmetries of the classical path integral without viscosity but they are not conserved in the presence of dissipation. Their commutators with $\tilde{\mathcal{H}}_d$ are analogous to (3.34), (3.37) and (3.38).

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