Can the Simplest Two-Field Model of Open Inflation Survive?

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We investigate the quantum fluctuations of an inflaton field in a two-field model of one-bubble open inflation. One of the inflatons is assumed to be responsible for the first false vacuum stage of inflation and the other is assumed to be responsible for the second slow-roll stage of inflation. The mass of the second inflaton is assumed to be negligible throughout the whole era of inflation. We find the super-curvature fluctuations are enhanced by the factor given by the ratio of the Hubble constants at false vacuum and true vacuum. This gives a strong constraint on a class of open inflation models. In particular, this implies the simplest two-field model proposed by Linde and Mezhilumian is in trouble.

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There are growing observational evidences that our universe has negative spatial curvature [1]. Accordingly an inflationary universe scenario which predicts an open but homogeneous and isotropic universe has been proposed and studied [2–5]. We call it open inflation or one-bubble inflation.

This scenario is composed of two inflationary phases separated by the bubble nucleation due to quantum tunneling. As a result of the first inflation, the universe becomes sufficiently homogeneous and isotropic. Then a vacuum bubble is formed by quantum tunneling. This process is called the false vacuum decay. It is dominated by the so-called $O(4)$-symmetric bounce [6,7]. Reflecting this symmetry, the region inside the bubble can be regarded as an open Friedmann-Robertson-Walker universe. Subsequently the second slow-roll inflation occurs inside the bubble and solves the entropy problem.

Several models for inflaton potentials were proposed in this context [2,4,5]. Among them, the one that requires the least tuning of the model parameters is the simplest two-field model proposed by Linde and Mezhilumian [5]. In this model, we consider two inflaton fields, $\sigma$ and $\phi$, which have no interaction between them. The first inflaton field $\sigma$ has a tilted double well potential and is supposed to be in the false vacuum initially. The vacuum energy of $\sigma$ dominates the universe and is responsible for the first stage of inflation. The second inflaton field $\phi$ has a simple quadratic potential $\frac{1}{2}m^2\phi^2$ with the mass $m$ being assumed to be small compared with the expansion rate of the universe $H(t)$, just as in the chaotic inflation scenario. After the bubble nucleation occurs, the potential energy of $\sigma$ disappears. Then the potential energy of $\phi$ dominates and the slow-roll inflation occurs in the bubble. It is important to note that $\dot{\phi}$ at the first stage of inflation is much smaller than $\dot{\phi}$ at the second stage of inflation in this scenario, to ensure the $O(4)$ ($O(3,1)$) -symmetry of the background. This means that the Hubble constant at the first stage of inflation is much bigger than that at the second stage of inflation.

At the classical level, this model works well. However, it is still an open question if it is consistent with the observed spectrum of density fluctuations of our universe. In this letter, we estimate the amplitude of curvature perturbations predicted by a class of two-field open inflation models which includes the simplest model mentioned above.

First we state several working hypotheses. In order to predict the spectrum of perturbations of the universe, we need to specify their dominant origin. Here we assume that 1) the perturbations originate from the quantum fluctuations of the second inflaton field $\phi$, which we denote by $\phi(x)$, 2) the mass of $\phi$ is always negligible compared with the Hubble parameter $H(t)$ as in the simplest two-field model, and 3) $\phi$ has no strong coupling with other fields. Thus we approximate the fluctuation field $\phi$ as a noninteracting massless scalar field. Since the first stage of inflation is supposed to last long enough, we also assume that 4) the quantum state of $\phi$ is in the Euclidean (Bunch-Davies) vacuum before the false vacuum decay. Furthermore we assume that 5) the bubble wall between true and false vacua can be described by using the thin wall approximation. However, we will see in the end that relaxing the last assumption does not affect the essential part of the result. Under these assumptions, we can use the formulae given in our previous paper [8](Paper I) to calculate the amplitude of curvature perturbations on the comoving hypersurface, $R_c$. 


The classical description of the spacetime is given by the $O(3,1)$-symmetric bubble. In what follows, we denote this classical background field configuration by $\phi_B$. In the thin-wall approximation, the whole spacetime is given by the junction of two different de Sitter spaces with different Hubble constants $H_L$ and $H_R$. We take the region with the Hubble constant $H_L$ to be the false vacuum. Hence $H_L > H_R$. We divide the spacetime into six regions as shown in Fig. 1. The coordinates in the regions $R$, $L$, $C_R$ and $C_L$, are given by

$$
\begin{align*}
&ds_J^2 = H_J^{-2} \left[ -dt_J^2 + \sinh^2 t_J \left( dr_J^2 + \sinh^2 r_J d\Omega^2 \right) \right], \\
&ds_{C,J}^2 = H_J^{-2} \left[ dt_{C,J}^2 + \cos^2 t_{C,J} \left( -dr_{C,J}^2 + \cosh^2 r_{C,J} d\Omega^2 \right) \right],
\end{align*}
$$

where $J = R$ or $L$. The relations among the coordinate systems are uniquely determined by the analyticity which was discussed in Paper I. Here we write the resulting relations,

$$
\begin{align*}
t_R &= it_{C,R} - \frac{\pi i}{2}, & r_R &= r_C + \frac{\pi i}{2}, \\
t_L &= -it_{C,L} - \frac{\pi i}{2}, & r_L &= r_C + \frac{\pi i}{2}.
\end{align*}
$$

For later convenience, we introduce the coordinate $T$ which covers both regions $C_R$ and $C_L$. Using $T$, the metric there is written as

$$
\begin{align*}
ds_C^2 &= dT^2 + a^2(T) \left( -dr_{C,J}^2 + \cosh^2 r_{C,J} d\Omega^2 \right),
\end{align*}
$$

where

$$
dT = \frac{dt_{C,J}}{H_J}, \quad a(T) = \frac{\cos t_{C,J}}{H_J},
$$

in the region $C_J$. The junction condition of the metric requires the continuity of the scale factor $a(T)$ on the wall,

$$
\left. \frac{\cos t_{C,L}}{H_L} \right|_{\text{wall}} = \left. \frac{\cos t_{C,R}}{H_R} \right|_{\text{wall}}.
$$

In order to calculate $R_c$, we need to solve the field equation for $\varphi$. First we consider the region $C$. By using the harmonics,

$$
Y_{p\ell m} = \frac{\Gamma(ip + \ell + 1)}{\Gamma(ip + 1)} \frac{p}{\sinh r_J} P_{ip-1/2}^{-\ell-1/2}(\cosh r_J) Y_{\ell m}(\Omega),
$$

and its analytic extension to the region $C$, we write the solution in the decomposed form as

$$
u_{p\ell m}(x) = u_p(T) Y_{p\ell m}(r_C, \Omega).
$$

Then the field equation in region $C$ becomes

$$
\left[ \frac{1}{a^3(T)} \frac{\partial}{\partial T} a^3(T) \frac{\partial}{\partial T} + \frac{p^2 + 1}{a^2(T)} \right] u_p(T) = 0.
$$

The field equations in the other regions are obtained by the analytic continuation.

As stressed in our previous paper [9], the perturbation modes are divided into two classes. One has usual continuous spectrum with $p^2 > 0$ and the other has discrete spectrum with $p^2 < 0$. We call the latter super-curvature modes since their characteristic wavelengths are greater than the spatial curvature scale.

We first consider the continuous modes. We introduce the fundamental mode functions which are most naturally defined in regions $R$ and $L$;

$$
u_p^{(R)} = \frac{H_R \cosh t_R - ip}{\sinh t_R \Gamma(2 - ip)} \left( \frac{\cosh t_R + 1}{\cosh t_R - 1} \right)^{ip/2}
$$

$$
= H_R e^{-\pi p/2} \frac{\sin t_{C,R} - ip}{\cos t_{C,R} \Gamma(2 - ip)} \left( \frac{1 + \sin t_{C,R}}{1 - \sin t_{C,R}} \right)^{ip/2},
$$

and

$$
\begin{align*}
\text{and}
\end{align*}
$$
\[ u^{(L)}_p = \frac{H_L \cosh t_L - ip}{\sinh t_L} \left( \frac{\cosh t_L + 1}{\cosh t_L - 1} \right)^{ip/2} \]
\[ = \frac{H_L e^{-ip/2} \sin t_{C,L} + ip}{-i \cos t_{C,L} \Gamma(2 - ip)} \left( \frac{1 + \sin t_{C,L}}{1 - \sin t_{C,L}} \right)^{-ip/2}. \]

In the second line of each equation, the functions are naturally extended to the region \( C_J \) by the analytic continuation. The extension of these functions to the other side of the wall is given by matching the mode functions at the wall. Since the first derivative of the scale factor, \( \dot{a}(T) \), where the dot means the derivative with respect to \( T \), is discontinuous but finite at the wall, the field equation requires the continuity of \( u_p \) and \( \dot{u}_p \). Putting
\[ u_{pℓm} := u^{(R)}_{pℓm} = \alpha_p u^{(L)}_{-pℓm} + \beta_p u^{(L)}_{pℓm}, \]
the junction condition determines \( \alpha_p \) and \( \beta_p \) as
\[ \alpha_p = ((u^{(L)}_p, u^{(R)}_p))/((u^{(L)}_p, u^{(L)}_{-p})), \]
\[ \beta_p = ((u^{(R)}_p, u^{(L)}_p))/((u^{(L)}_p, u^{(L)}_{-p})), \]

where
\[ ((u, v)) = \left. \frac{1}{a(T)} (uv - \dot{uv}) \right|_{wall}. \]

Then one can construct the orthonormalized mode functions \( v_{pℓm} \) (\( \sigma = \pm 1 \)) over the whole spacetime by forming linear combinations of \( u_{pℓm} \) and \( u_{-pℓm} \) (Paper I), which we write as \( v_{pℓm} = V_{pℓm}. \)

Next we turn to the discrete mode. By setting \( p = i\Lambda \), we write the fundamental mode function as
\[ u^{(J)}_{\Lambdaℓm} = u^{(J)}(\Lambda) Y_{ℓm}, \]

where
\[ Y_{\Lambdaℓm} = \frac{P_{\Lambda - 1/2}(\cosh r)}{\sqrt{\sinh r} Y_{ℓm}(\Lambda)}, \]

and
\[ u^{(R)}_{\Lambda} = \frac{H_R \cosh t_R + \Lambda}{\sinh t_R} \left( \frac{\cosh t_R + 1}{\cosh t_R - 1} \right)^{\Lambda/2} \]
\[ = \frac{H_R e^{-i\pi\Lambda/2} \sin t_{C,R} + \Lambda}{-i \cos t_{C,R} \Gamma(2 + \Lambda)} \left( \frac{1 + \sin t_{C,R}}{1 - \sin t_{C,R}} \right)^{-\Lambda/2}, \]
\[ u^{(L)}_{\Lambda} = \frac{H_L \cosh t_L + \Lambda}{\sinh t_L} \left( \frac{\cosh t_L + 1}{\cosh t_L - 1} \right)^{\Lambda/2} \]
\[ = \frac{H_L e^{-i\pi\Lambda/2} \sin t_{C,L} - \Lambda}{-i \cos t_{C,L} \Gamma(2 + \Lambda)} \left( \frac{1 + \sin t_{C,L}}{1 - \sin t_{C,L}} \right)^{\Lambda/2}. \]

Note that the regularity condition at \( t_J = 0 \) determines the form of \( u^{(J)}_{\Lambda} \) as given above, and further requires \( \Lambda > 0 \). Then the continuity condition of the mode function gives
\[ \frac{1 - \Lambda^2}{\sin t_{C,R} + \Lambda} = \frac{1 - \Lambda^2}{\sin t_{C,L} - \Lambda}, \]

which determines the eigenvalue \( \Lambda \). One easily sees the only positive root is \( \Lambda = 1 \). The corresponding mode also exists in the case of the pure de Sitter background [9]. Hence we call it the de Sitter super-curvature mode. In the present case, the explicit form of the mode function becomes
\[ u_{\Lambda} = u^{(R)}_{\Lambda} = \frac{H_R}{2}, \]
To determine the amplitude of the curvature perturbation due to this mode, we need to calculate the normalization factor $N_\Lambda$, which is defined by

$$N_\Lambda := \int dT a(T) |u_\Lambda(T)|^2$$

$$= \frac{1}{H^2} \int_{t_{\text{wall}}}^{\pi/2} dt C_R \cos t C_R |u_\Lambda|^2$$

$$+ \frac{1}{H^2} \int_{-\pi/2}^{t_{\text{wall}}} dt C_L \cos t C_L |u_\Lambda|^2.$$  

(19)

In the present case, $N_\Lambda$ is evaluated to be

$$N_\Lambda = 2 \left( \frac{H_R}{H_L} \right)^2 \left( \frac{1 + s_L}{1 + s_R} \right) \left( 1 + \frac{\Delta s}{2} \right).$$  

(20)

The normalized mode function is then given by

$$V_\Lambda = N_\Lambda^{-1/2} u_\Lambda.$$

Now let us evaluate the curvature perturbation on the comoving hypersurface. We expand $R_c$ by modes,

$$R_c = \sum_{\ell,m} \int_0^\infty dp \mathcal{R}_p Y_{\ell m}(r, \Omega) + \sum_{n,\ell,m} \mathcal{R}_{n,\ell m} \tilde{Y}_{n,\ell m}(r, \Omega).$$  

(21)

We first consider the contribution from the continuous modes. The power spectrum of the curvature perturbation $R_p$ is given by

$$|R_p|^2 = \left( \frac{H_R}{\phi_B} \right)^2 \lim_{t_B \to \infty} \sum_{\sigma = \pm 1} |V_{p\sigma}|^2,$$  

(22)

which is expressed as

$$|R_p|^2 = |R_p|_{BD}^2 (1 - Y),$$  

(23)

where

$$|R_p|_{BD}^2 = \frac{H_R^2}{(\phi_B)^2} \frac{\coth \pi p}{2p(1 + p^2)}.$$  

(24)

is the spectrum for the Bunch-Davies vacuum, and $Y$ is expressed in terms of $\alpha_p$ and $\beta_p$ as

$$Y = \frac{\Gamma(2 - i\tilde{p})}{\Gamma(2 + i\tilde{p})} \frac{e^{-\tilde{p} \beta_p}}{2 \cosh \tilde{p} \alpha_p} + \text{c.c.}.$$  

(25)

From the Eq. (12), after a straightforward calculation, we obtain

$$\alpha_p = e^{-\pi p} \frac{\Gamma(2 + i\tilde{p})}{\Gamma(2 - i\tilde{p})} \frac{(1 + s_R)(1 - s_L)}{(1 - s_R)(1 + s_L)} \frac{i\Delta s}{2p},$$

$$\beta_p = - \frac{(1 + s_R)(1 + s_L)}{(1 - s_R)(1 - s_L)} \frac{i\Delta s}{2p},$$  

(26)

where $s_R := \sin t_{C,R}|_{\text{wall}}, s_L := \sin t_{C,L}|_{\text{wall}}$ and $\Delta s := s_R - s_L$. Note that $(1 - s_R^2)/H_R^2 = (1 - s_L^2)/H_L^2$. Thus we finally obtain

$$1 - Y = 1 - \frac{(\Delta s)^2 \cos \tilde{p} + 2 \Delta s \sin \tilde{p}}{\cosh \pi p (4\tilde{p}^2 + (\Delta s)^2)},$$  

(27)

where

$$\tilde{p} = p \ln \frac{1 + s_R}{1 - s_R}.$$  

(28)
When $p$ is large, the second term in the right hand side is exponentially suppressed due to the factor, $1/\cosh \pi p$. On the other hand, when $p$ is small, it goes to 1. Thus it is expected that $1 - Y$ does not become much larger than unity. Therefore we conclude that the fluctuations due to the continuous modes are never enhanced much in comparison with those in the Bunch-Davies vacuum. That is, we have $\phi \sim H_R$ inside the bubble although $\phi \sim H_L$ outside the bubble, which is in accordance with an intuitive argument given by Linde and Mezhlumian [5].

As for the contribution from the de Sitter super-curvature mode, we apply the formula obtained in Paper I,

$$|\mathcal{R}_A|^2 = \left( \frac{H_R}{\phi_B} \right)^2 |V_A|^2. \quad (29)$$

Then

$$|\mathcal{R}_A|^2 = \frac{2}{N_A} |\mathcal{R}_A|_{BD}^2, \quad (30)$$

where

$$|\mathcal{R}_A|_{BD}^2 = \frac{1}{2} \left( \frac{H_R^2}{\phi_B^2} \right)^2, \quad (31)$$

is the amplitude in the Bunch-Davies vacuum limit. Thus we conclude

$$|\mathcal{R}_A|^2 = |\mathcal{R}_A|_{BD}^2 \times O \left( \frac{H_L^2}{H_R^2} \right). \quad (32)$$

This means that the suppression mechanism as discussed by Linde and Mezhlumian [5] does not work for the de Sitter super-curvature mode, contrary to the case for the continuous modes. This difference is caused by the fact that the normalization of the mode functions for the continuous modes is determined essentially by their behaviors at $t_{C,R} \to \pi/2$ and $t_{C,L} \to -\pi/2$ [9], while that for the super-curvature mode is determined by its behavior over the whole region $C$.

In the above we have used the thin-wall approximation. However the conclusion will not change even if the wall is thick. In order to understand this, we note that $u_\Lambda = \text{const.}$ is a solution of the field equation with $p = -i$ independent of the detail of $a(T)$. Hence we have $N_\Lambda = |u_\Lambda|^2 \int dT a(T)$. Since this integral is determined predominantly by the Hubble radius $H_L$ of the false vacuum region, one has $\int dT a(T) \sim H_L^{-2}$. Therefore $N_\Lambda = O(H_R^2/H_L^2)$ for $u_\Lambda = H_R/2$. Thus we conclude that Eq. (32) holds generally independent of the thin-wall approximation.

Now let us consider implications of our results. We have found that the amplitude of the perturbations due to the continuous modes are not much different from the case of the Bunch-Davies vacuum but that due to the de Sitter super-curvature mode is enhanced by a factor of $O(H_L/H_R)$. In one of our previous papers [3], we discussed the effect of the super-curvature mode on the power spectrum of the cosmic microwave background (CMB) anisotropies $C_\ell$. There we assumed the Bunch-Davies vacuum. We found the contribution from the super-curvature mode reaches more than 10% for models with $\Omega_0 < 0.2$. On the other hand, it gives only a negligible contribution to the CMB spectrum at $\ell > 10$. Since only the super-curvature mode are enhanced in models we have studied in this letter, the models will predict a large enhancement of the CMB spectrum at $\ell < 10$ if $H_L \gg H_R$, which will be in contradiction with observed spectrum [10]. Thus the simplest two field model proposed by Linde and Mezhlumian [5] is in trouble, though the other two-field models proposed by them are still viable.

In conclusion, in the context of one-bubble open inflation, we have investigated the quantum fluctuations of an inflaton field whose mass is negligible at both the first and second stages of inflation and which is responsible for the second stage of inflation. We have found the super-curvature fluctuations are enhanced by the factor given by the ratio of the Hubble constants at false vacuum and true vacuum. This gives a strong constraint on a class of open inflation models in which the Hubble constant outside a vacuum bubble is much greater than that inside the bubble.

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