Hawking radiation and masses in generalized dilaton theories

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Abstract

A generalized dilaton action is considered of which the standard dilaton black hole and spherically reduced gravity are particular cases. The Arnowitt-Deser-Misner (ADM) and the Bondi-Sachs (BS) mass are calculated. Special attention is paid to both the asymptotic conditions for the metric as well as for the reference space-time. For the latter one we suggest a modified expression thereby obtaining a new definition of energy. Depending on the parameters of the model the Hawking radiation behaves like a positive or negative power of the mass.

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1 Introduction

Over the last few years 1+1 dimensional dilaton theories have been studied extensively in their string inspired (CGHS) version [1] as well as in more general forms [2]. One of the main motivations to study such models arises from the hope that within the simplified setting of 2D models one can gain insight into physical properties of 4D gravity. They actually allow for example black hole solutions and Hawking radiation and are more amenable to quantum treatments than their 4D counterparts. The two most frequently considered theories, the string inspired CGHS as well as spherically reduced gravity (SRG) differ drastically in some of their physical properties. For example differences were observed with respect to the completeness of null geodesics for those two models [3]. These differences directly lead one to investigate physical properties of a generalized model of which the two prominent examples are simply particular cases.

Important classical quantities are the energy at spatial and null infinity, the so called Arnowitt-Deser-Misner (ADM) and the Bondi-Sachs (BS) mass, respectively (provided the global structure of the space-time is the same as for SRG). For general models, however, there exist in general no flat space time solutions [3] and therefore a proper reference space-time has to be chosen. A second important point is to realize that these “masses” depend on the asymptotic behavior of the metric and are therefore only defined with respect to a particular observer. Our approach is based on the second order formalism. Significant insight can also be gained from a first order formalism [4],[5].

An important feature in semiclassical considerations is the behavior of Hawking radiation. In the CGHS model it is just proportional to the cosmological constant whereas the dependence in SRG is inverse to its mass, which implies an accelerated evaporation towards the end of its lifetime. As we will show a generalized theory will exhibit Hawking radiation which is proportional to the black hole mass in terms of positive or negative powers of the black hole mass, depending on the parameters of the model.

In section 2 we repeat some relevant results of [3] for the metric and the global structure of the classical solution of a generalized dilaton Lagrangian which, for a certain range of parameters posesses a singularity structure coinciding with the one of the Schwarzschild black hole. The ADM and BS mass are calculated in section 3. Furthermore we will present a definition for energy, differing from the one proposed by Hawking [6], by taking a
modified reference space-time. The path integral measure and the problem of interpretation of various energy definitions will be discussed in section 4 before we finally demonstrate the dynamical formation of black holes in the framework of conformal gauge, with special emphasis on the necessary boundary conditions.

2 Classical Solution

Among the numerous different generalizations of the CGHS model [1] we consider the action

\[ L = \int d^2x \sqrt{-g} e^{-2\phi} (R + 4a(\nabla \phi)^2 + Be^{2(1-a-b)\phi}). \]

This form of the Lagrangian covers e.g. the CGHS model [1] for \( a = 1, b = 0 \), spherically reduced gravity [7] \( a = \frac{1}{2}, b = -\frac{1}{2} \), the Jackiw-Teitelboim model [8] \( a = 0, b = 1 \). Lemos and Sa [9] give the global solutions for \( b = 1 - a \) and all values of \( a \), Mignemi [10] considers \( a = 1 \) and all values of \( b \). The models of [11] correspond to \( b = 0, a \leq 1 \).

The classical solutions of (1) were already found in [3] but since the solutions for the metric will be used extensively in the following sections they shall be repeated here. Letting \( \phi \) represent one of the coordinates the line element reads

\[ (ds)^2 = g(\phi) \left( 2d\phi^2 + l(\phi)dv^2 \right), \]

with

\[ g(\phi) = e^{-2(1-a)\phi} \]

\[ l(\phi) = \begin{cases} \frac{e^{2\phi}}{8} \left( C - \frac{2B}{b+1} e^{-2(b+1)\phi} \right), & b \neq -1; \\ \left( \tilde{C} + 4B\phi \right), & b = -1 \end{cases} \]

The transformation

\[ a \neq 1: \quad u = \frac{e^{-2(1-a)\phi}}{2(a-1)}, \]

\[ a = 1: \quad u = \phi, \]

brings the metric into Eddington-Finkelstein form (EF) [12]

\[ (ds)^2 = 2dvdu + l(u)dv^2 \]
with

\[ m(9) \neq -1: \quad a \neq 1: \quad l(u) = B_1 |u|^{a/b - 1} - B_2 |u|^{1 - a/b}, \]

\[ m(10) \quad a = 1: \quad l(u) = \frac{1}{8} e^{2u} \left( C - \frac{2B}{b + 1} e^{-2(b+1)u} \right), \]

\[ m(b) \neq -1: \quad a \neq 1: \quad l(u) = \frac{1}{8} |2(a - 1)u|^{a/b - 1} \left( C + \frac{2B}{a - 1} \ln |2(a - 1)u| \right), \]

\[ m(12) \quad a = 1: \quad l(u) = \frac{1}{8} e^{2u} \left( \tilde{C} + 4Bu \right), \]

where the constants are given by

\[ m(13) \quad B_1 = C \left( 2 |a - 1| \right)^{a/b - 1}; \quad B_2 = \frac{B}{4(b + 1) (2 |a - 1|)^{1 - a/b}}. \]

The scalar curvature of the metric (2) has the form

\[ b \neq -1 \quad R = \frac{1}{2} e^{2(2-a)\phi} \left( aC - \frac{2Bb}{b + 1} (b + 1 - a) e^{-2(1+b)\phi} \right) \]

\[ m(14) \quad b = -1 \quad R = \frac{1}{2} e^{2(2-a)\phi} (aC + 2(a + 1)B + 4aB\phi) \]

or equivalently for (9)

\[ m(15) - 1, \quad a \neq 1: \quad R = B'_1 u^{2-a} + B'_2 u^{a+b-1} \]

\[ m(16) - 1, \quad a = 1: \quad R = \frac{C}{2} e^{2u} - \frac{Bb^2}{b + 1} e^{-2bu} \]

\[ m(17) - 1, \quad a \neq 1: \quad R = |2(a - 1)u|^{2-a/b - 1} \left( \frac{aC}{2} + B \frac{a^2 - 1 + a \ln |2(a - 1)u|}{a - 1} \right) \]

\[ m(18) - 1, \quad a = 1: \quad R = \frac{1}{2} e^{2u} (\tilde{C} + 4B + 4Bu). \]

Finally, a third form of the metric, the generalized Schwarzschild metric,

\[ m(19) \quad (ds)^2 = l(u) dt^2 - \frac{1}{l(u)} du^2, \]

is obtained by means of the transformation

\[ m(20) \quad dv = dt - \frac{du}{l(u)}. \]
Since below we will repeatedly make use of the special cases of asymptotically Minkowski, Rindler and de Sitter space-times we list them for completeness. After a suitable rescaling (see eq. (34) below) they are

**Asymptotically Minkowski space:**  
\( (b = a - 1) \)

\[
(ds)^2 = (M U^{a-1} - 1) dt^2 - (M U^{a-1} - 1)^{-1} dU^2
\]

**Asymptotically Rindler space:**  
\( (b = 0) \)

\[
(ds)^2 = (M U^{a-1} - B_1^2 U) dt^2 - (M U^{a-1} - B_1^2 U)^{-1} dU^2
\]

**Asymptotically de Sitter space:**  
\( (b = 1 - a) \)

\[
(ds)^2 = (M U^{a-1} - B_2 U^2) dt^2 - (M U^{a-1} - B_2 U^2)^{-1} dU^2
\]

with \( M = B_1 B_2^{2(a-1)} \). As can be seen from (15) and (21) for \( 0 < a < 2 \) these metrics go to the unit metric or to de Sitter metric at the asymptotic flat or constant curvature regions. For \( a \notin (0,2) \) we cannot construct static metrics with proper asymptotic behavior. Throughout this paper asymptotically Minkowski, Rindler or de Sitter solution will imply \( 0 < a < 2 \).

The global structure is completely determined by \( l(u) \) or equivalently by \( l(\phi) \), which is nothing else but the norm of the Killing vector whose zeros correspond to the horizons. The corresponding Penrose diagrams for all possible values of the parameters of the action (1) were presented in [3], where it was shown that genuine Schwarzschild like behavior is restricted to the region \( b \leq 0 \) and \( a < 1 \).

## 3 ADM and BS Mass

### 3.1 ADM Mass

The aim of this section is to calculate the ADM mass for our model. More precisely, we will obtain the value of a generalized definition for the energy at spatial infinity, for space-times whose lapse does not approach 1 asymptotically. The standard approach to obtain this quantity is to split off the dynamical part \( h_{\mu\nu} \) and \( \rho \) from the metric \( g_{\mu\nu} = h_{\mu\nu} + \eta_{\mu\nu} \) and the dilaton field \( \phi = \phi_{\text{vac}} + \psi \), respectively and then calculate the energy defined as the
value of the standard Hamiltonian taken on the shell of zero constraints [13] [14]. Instead of going through these lengthy arguments we shall first review some recent very elegant methods that arise from the canonical formalism of gravity [7] [15] and will later on apply them to solutions of our model. We review what kind of slicing we have to work with and will then outline the main steps on how to obtain the expression for the total energy.

Consider a one-dimensional spacelike slice $\Sigma$ drawn in our spacetime (see Fig.1). Assume that $\Sigma$ has a boundary point $B = \partial \Sigma$. Let the unit, outward-pointing, spacelike normal of the point $B$ as embedded in $\Sigma$ be $n^\mu$. If $\Sigma$ is a surface of constant $t$ with metric $\Lambda^2 dr^2$ then the space-time metric near $\Sigma$ can be written in ADM form [16]

$$ds^2 = \Lambda^2 dr^2 - N^2 (dt + \Lambda^t dr)^2.$$  

(24)

Figure 1: Spacetime foliation: $\Lambda^t$ denoting the radial shift and $\Lambda$ the radial lapse

To obtain the expression for the total energy one has to cast the action into hamiltonian form. Boundary terms have to be added to the action to ensure that its associated variational principle fixes the induced metric and the dilaton on the boundary. As shown in [7], the form of a suitable Hamiltonian with boundary terms at $B$ is the following:

$$H = \int_\Sigma d\nu (N\mathcal{H} + N^r \mathcal{H}_r) + N(E_{ql} + N^r \Lambda P_\Lambda)|_B,$$  

(25)

where $P_\Lambda$ is the ADM momentum conjugate to $\Lambda$ and $\mathcal{H}$ and $\mathcal{H}_r$ are the Hamiltonian and the momentum constraint respectively. Since the expression
for the Hamiltonian (25) diverges in general, a reference Hamiltonian \( H_0 \) has to be subtracted to obtain the physical Hamiltonian. \( E \), which defines the quasilocal energy is given by

\[
E_{ql} = e^{-2\phi}(n[\phi] - n[\phi]^0)
\]

where the second term indicates that the value is referenced to a background. We shall only consider the case when \( N^r = 0 \) at \( B \), i.e the on-shell value of Hamiltonian will be associated only with time translations (\( H \) does not generate displacements normal to the boundary). Defining the total energy as the value of the physical hamiltonian on the boundary we finally get

\[
E \equiv H|_B = NE_{ql} = 4Ne^{-2\phi}(n[\phi] - n[\phi]^0)|_B
\]

Only for \( N = 1 \) this coincides with the ADM mass, whereas the additional factor \( N \) in (27) gives the proper definition for the total energy for space-times whose lapse doesn’t go to one asymptotically [17, 6]. However, as we shall see at the end of this section, the form of (27) is not unique.

To apply this result to our action (1) we take the line element

\[
(ds)^2 = g(\phi)l(\phi)dt^2 - g(\phi)l(\phi)d\phi^2
\]

with the same \( g(\phi) \) and \( l(\phi) \) as in (3) and (4), respectively. This form of the metric is obtained from (2) by a transformation analogous to (20). For \( g = 1 \) \( (a = 1) \) this is just the metric in generalized Schwarzschild coordinates. We see that \( \phi \) now corresponds to the radial coordinate of the ADM-metric (24), \( \phi \equiv r \).

Since the radial shift is zero the unit normal \( n \) to the boundary \( B \) is therefore simply

\[
n = \frac{1}{\Lambda} \left( \frac{\partial}{\partial \phi} \right).
\]

Using

\[
\Lambda^2 = -\frac{g(\phi)}{l(\phi)} \quad N^2 = -g(\phi)l(\phi)
\]

we obtain

\[
E_{ADM} = 4e^{-2\phi} \left( l(\phi) - \sqrt{l(\phi)l^0(\phi)} \right)|_{\phi \to \infty}
\]
where \( l^0 \) denotes the function in the metric (2) of a reference space-time. Where possible we will use flat space-time as our reference. It should be noted that this expression is independent of the function \( g(\phi) \) and therefore we have no \( a \)-dependence on \( E \) for the action (1). Calculating this quantity at spacelike infinity corresponds to the ADM mass. Inserting (4) into (31) with \( C = 0 \) for \( l^0 \) the diverging terms cancel each other whereas the next order terms give a finite contribution. We obtain

\[
\begin{align*}
\text{m(32)} & \quad E = \frac{C}{4} + O(e^{2(b+1)\phi}) \quad \text{for} \quad (b+1)\phi \to -\infty \\
\text{m(33)} & \quad E = \frac{C}{2} + O(e^{-(b+1)\phi}) \quad \text{for} \quad (b+1)\phi \to +\infty.
\end{align*}
\]

This coincides with the expectation that the parameter \( C \) is proportional to the mass. Notice that for these expressions the reference space-time \((C = 0)\) itself contains already a curvature singularity [3] except for the special cases considered below. For spherically reduced gravity \((a = -b = \frac{1}{2})\) the flat space-time region is located at \( \phi \to -\infty \). Therefore, by using (6) it is straightforwardly verified that the next order term of (32) becomes \( O(u^{-1}) \) which coincides with the usual ADM expression for the Schwarzschild black hole.

However, the results (32) and (33) still retain some arbitrariness. To show this consider the transformation of (8)

\[
\text{m(34)} \quad dV = K^{\frac{1}{2}}dv \quad dU = K^{-\frac{1}{2}}du.
\]

This preserves the form of the metric

\[
\text{m(35)} \quad ds^2 = 2dUdV + L(U(u))dV^2 \quad L(U(u)) = \frac{l(u)}{K}
\]

and at the same time we are able to obtain dimensions of length for \( U \) and \( V \). Simultaneously \( E_{ADM} \) is changed. We see that an unambiguous definition of \( E_{ADM} \) includes two important ingredients. The first one is the reference point for energy which is specified by the choice of ground state solution. We define the ground state configurations by \( C = 0 \). For the important particular cases of asymptotically Minkowski, Rindler and de Sitter models this choice gives zero or constant curvature ground state solutions. The second ingredient is the asymptotic condition for the metric. It corresponds to the choice of an observer who measures energy and Hawking radiation at the asymptotic
region. This ambiguity in the choice of time for the asymptotic observer is expressed in (35). In the spirit of these remarks, the value (32) and (33) of the ADM mass should be specified as "the ADM mass with respect to \( C = 0 \) solution measured by an asymptotic observer with time and length scales defined by metric (2)". This value is unambiguously defined for all values of \( a \) and \( b \). For the important particular cases mentioned above other definitions are more relevant:

- **Asymptotically Minkowski and Rindler spaces:**

As mentioned above, eq. (34) can be used to obtain dimensions of length for \( U \) and \( V \). This happens to be the case if \( K \) has dimensions of energy squared. Therefore a combination of \( B_1 \) and \( B_2 \), e.g. \( B_1^m B_2^{1-m} \), or equivalently of \( C \) and \( B \), is a natural choice since all of these quantities have dimension of energy squared. However, because our vacuum is defined as \( C = 0 \), i.e. \( B_1 = 0 \), only \( B_2 \) should contribute to \( K \). If we require in addition that for the minkowskian case our metric approaches asymptotically the unit one \( \eta_{\mu\nu} = \text{diag}(-1,1) \), we have to use

\[
K = B_2
\]

which in turn gives

\[
L(U) = B_1 B_2^{\frac{2-a}{2(a-1)}} U^{\frac{a}{a-1}} - B_2^{\frac{a-1}{2(a-1)}} U^{1 - \frac{b}{a-1}}.
\]

With

\[
n = \frac{1}{\Lambda} \frac{\partial}{\partial U} = \frac{\sqrt{K}}{\Lambda g(\phi)} \frac{\partial}{\partial \phi}
\]

\[
N^2 = \Lambda^{-2} = -L(U)
\]

we obtain from (27)

\[
E_{ADM} = \frac{C}{2} \left( \frac{b+1}{B} \right)^{\frac{3}{2}} (2(a-1))^{\frac{a-1}{2(a-1)}}.
\]

For the special case of asymptotically Minkowski and Rindler space-times this gives

\[
E_{ADM} = \frac{C}{2} \left( \frac{a}{B} \right)^{\frac{3}{2}} \quad \text{and} \quad E_{ADM} = \frac{C}{2} (2B(a-1))^{-\frac{1}{2}},
\]
respectively.

- **Asymptotically de Sitter spaces:**
  A rescaling of (8) as in (34) does not change the asymptotic behavior of (19), which is given by
  \[ (ds)^2 = -B_2 u^2 dt^2 + (B_2 u^2)^{-1} du^2, \]
  but the coefficient of the asymptotically vanishing term is changed as can be seen from (37). Therefore we get
  \[ E_{ADM} = \frac{C}{4(a-1)} \left( \frac{2-a}{B} \right)^2. \]

- **Modified Definition of Mass:**
  Finally we want to present a modification to Hawkings definitions for energy. Eq.(27) was obtained by subtracting the value of the reference space-time \((NE_{ql})^0|_B\) from the Hamiltonian (25). Following the line of argument in ref.[6] \(N\) was pulled out of the subtraction term by setting \(N|_B = N^0|_B\). However, this condition is not easily implemented because \(N\) diverges in general at the boundary. Here we suggest the more natural choice of leaving the lapse in the reference term which then still fulfills the foregoing condition. As a consequence this modifies (27) and (31) to
  \[ E = 4e^{-2\phi} \left( N n[\phi] - N^0 n[\phi]^0 \right)|_B = 4e^{-2\phi} \left( l(\phi) - l^0(\phi) \right)|_B \]
  and by inserting \(l(\phi)\) we obtain
  \[ E = \frac{C}{2}, \]
  which now does not only hold for \(|\phi| \to \infty\) but on the whole space-time. This value depends again in the same manner as above on the asymptotic condition for the metric. Note that for asymptotically Minkowski, Rindler or de Sitter solutions the two expressions (33) and (45) for \(E\) agree. On the other hand, for arbitrary asymptotic behavior the two values of \(E\) correspond to two different definitions of the asymptotic observer in reference space-time. In a recent paper [18] Mann also considered a modified reference space-time for the energy to obtain the so called quasilocal mass, which also diverges from Hawking’s proposal. Note that the same rescaling as in (34) could certainly be applied to (44) thereby giving additional coefficients of \(B\) and the parameters. A similar result was also obtained in [19] by applying a Regge-Teitelboim argument.
3.2 Bondi-Sachs Mass

In the first part of this chapter we obtained the value of the total energy which corresponds to the ADM mass at spatial infinity. Here we will show that $E_{\text{ADM}}$ on $\mathcal{I}^+$, the so called BS mass $E_{\text{BS}}$, equals $E_{\text{ADM}}$ and does not depend on the value of the retarded time coordinate $v$. However, the procedure to obtain this quantity is logically different. The main difference with respect to the calculation of the ADM mass above is that we do not take the limit $\phi \to \infty$ along a single slice but instead we will consider the limit of expression (27) along a particular null hypersurface $N$ associated with different slices along a line of constant retarded time [20] (see Fig. 2). For convenience we will work again with the EF-metric (8). First we define a future directed lightlike vector $k^\mu$ and pick another lightlike vector $h^\mu$ such that $h_\mu k^\mu = 1$. They are

\begin{align*}
 m(46) & \quad k^\mu \partial / \partial x^\mu = \sqrt{\frac{l}{2}} \partial / \partial u \\
 m(47) & \quad h^\mu \partial / \partial x^\mu = -\sqrt{\frac{l}{2}} \partial / \partial u + \sqrt{\frac{2}{l}} \partial / \partial v.
\end{align*}
Next, along the null hypersurface $N$ we let

\[ \tilde{u}^\mu \partial/\partial x^\mu := \frac{1}{\sqrt{2}} (k^\mu + h^\mu) \]

\[ = \frac{1}{\sqrt{l}} \partial/\partial v \]

define the timelike normal to the $\Sigma$ slices spanning the points of $N$. Similarly we define its spacelike normal

\[ \tilde{n}^\mu \partial/\partial x^\mu := \frac{1}{\sqrt{2}} (k^\mu - h^\mu) \]

\[ = \sqrt{l} \partial/\partial u - \frac{1}{\sqrt{l}} \partial/\partial v. \]

Figure 3: Construction of the proper slicing

Notice that the choice of $k^\mu$ will ensure that the energy that we compute is associated with a proper rest frame such that the timelike normal behaves like $\tilde{u}^\mu \partial/\partial x^\mu = l^{-\frac{1}{2}} \partial/\partial v = N^{-1} \partial/\partial t$, which is obtained by using (20). Therefore, with $N = \sqrt{l(u)}$ we get

\[ N \left( \tilde{n}[\phi] - \tilde{n}[\phi]^0 \right) = \left( l(u) - \sqrt{l(u)l^0(u)} \right) \frac{\partial \phi}{\partial u} \]

and

\[ N\tilde{n}[\phi] - (N\tilde{n}[\phi])^0 = \left( l(u) - l^0(u) \right) \frac{\partial \phi}{\partial u}. \]

Using (52) in (27) and remembering that $g(\phi)d\phi = du$ and $l(u) = l(\phi)g(\phi)$ we finally obtain

\[ E_{BS} = 4e^{-2\phi} \left( l(\phi) - \sqrt{l(\phi)l^0(\phi)} \right) \]
which is exactly the same as expression (31) and therefore the ADM mass and the Bondi-Sachs mass turn out to have the same value. This can be readily understood by recalling that the difference between the BS and the ADM mass is the integral of a stress energy flux which vanishes at this purely classical level. Similarly (53) exactly reproduces the result of (44) thereby showing that also the modified definition of mass on \( I^+ \) agrees with the value at spatial infinity.

4 Hawking radiation of generalized Schwarzschild black holes

There are a number of ways of calculating the Hawking radiation [21]. One of them consists in comparing vacua before and after the formation of a black hole. In the case of generalized dilaton gravity this way is technically rather involved. We prefer a simpler approach based on an analysis of static black hole solutions.

Consider a generalized Schwarzschild black hole given by

\[
\begin{align*}
\text{m}(55) \quad ds^2 &= -L(U)d\tau^2 + L(U)^{-1}dU^2,
\end{align*}
\]

where \( L(U) \) has a fixed behavior at the asymptotic region \( I^+ \):

\[
\begin{align*}
\text{m}(56) \quad L(U) &\to L_0(U)
\end{align*}
\]

with \( L_0(U) \) corresponding to the ground state solution. At the horizon we have \( L(U_h) = 0 \). We can calculate the geometric Hawking temperature as the normal derivative of the norm of the Killing vector \( \partial/\partial \tau \) at the (nondegenerate) horizon [22]

\[
\begin{align*}
\text{m}(57) \quad T_H &= \left| \frac{1}{2} L'(U_h) \right|.
\end{align*}
\]

Let us introduce the coordinate \( z \) which is an analog of the Regge-Wheeler tortoise coordinate [23] \( r^* \) for the ordinary Schwarzschild black hole

\[
\begin{align*}
\text{m}(58) \quad dU &= dzL(U).
\end{align*}
\]

Then the metric takes the conformally trivial form with

\[
\begin{align*}
\text{m}(59) \quad ds^2 &= e^{2\rho}(-d\tau^2 + dz^2) \quad \rho = \frac{1}{2} \ln L.
\end{align*}
\]
In conformal coordinates the stress energy tensor looks like [21]

\[ T_{--} = -\frac{1}{12}((\partial_-\rho)^2 - \partial_-\partial_-\rho) + t_- = T_{--}[\rho(L)] + t_- \]

One can choose coordinates such that in the asymptotic region

\[ T_{--}[\rho(L_0)] = 0. \]

This choice ensures that there is no radiation in the ground state. It means that we measure Hawking radiation of a black hole without any contribution from background Unruh radiation.

The constant \( t_- \) is defined by the condition at the horizon

\[ T_{--}|_{\text{hor}} = 0 \]

in the spirit of [24]. The corresponding vacuum state is called the Unruh vacuum. In this state there is no energy flux at the black hole horizon. Of course, there cannot be such a thing as an observer at the horizon. However, as we shall demonstrate below, predictions of the theory with regard to measurements made at infinity are independent of the choice of coordinates at the horizon. In the case of four dimensional black holes this was shown in [25]. Taking into account equations (60), (61) and (62) one obtains a relation between \( T_{--}[\rho] \) at the horizon and the asymptotic value of \( T_{--} \):

\[ T_{--}|_{\text{asymp}} = -T_{--}[\rho]|_{\text{hor}} \]

The following simple identities are useful:

\[ \partial_+ = \frac{1}{2} \partial_z, \quad \partial_+\rho = \frac{1}{2}L', \quad \partial_+^2\rho = \frac{1}{2}L''L, \]

where prime denotes differentiation of \( L \) with respect to \( U \). By substituting (64) into (63) we obtain the Hawking flux

\[ T_{--}|_{\text{asymp}} = \frac{1}{48} \left( \frac{1}{2}L'(U) \right)^2 |_{\text{hor}}. \]

It is easy to demonstrate that our result is independent of a particular choice of conformal coordinates provided the behavior at the asymptotic region is fixed. Without destroying the conformal gauge one can change coordinates so that

\[ \rho \rightarrow \rho + h_+(x^+) + h_-(x^-) \]
with two arbitrary functions $h_{\pm}$. Since the behavior of $\rho$ at $I^+$ is fixed, $h_- = 0$. The transformation with arbitrary $h_+$ does not change $T_{--}$.

Before we start to apply (65) to some specific space-times we remark that as in the case of the ADM and Bondi mass the value will once again depend on our choice of the asymptotic behavior of the metric. We will use the same form of the metric as for the calculation of the ADM mass, equations (35) and (37), which give the metric (55) after a transformation like (20). For the most interesting space-times of our action (1) we will explicitly present the solutions.

- **Asymptotic Minkowski spacetimes:**
  As mentioned above this corresponds to $b = a - 1$. From (37) we get for $a \neq 1$

\[
L(U) = B_1B_2^{\frac{2-a}{a-1}} U^\frac{a}{a-1} - 1.
\]

The horizon determined by $L(U_h) = 0$ is located at

\[
U_h = B_1 \frac{1-a}{a} B_2 \frac{a-2}{2a}
\]

and therefore we get

\[
T_{--}|_{\text{asym}} = \frac{a^2}{384} C^{\frac{2(a-1)}{a}} \left( \frac{2B}{a} \right)^{\frac{2-a}{a}}.
\]

For the special case $a = 1$, the CGHS model, we have

\[
L(U) = e^{\sqrt{BU}} C \frac{e^{\sqrt{2B}}}{2B} - 1
\]

which results in

\[
T_{--}|_{\text{asym}} = \lambda^2 \frac{\lambda}{48},
\]

thereby confirming the well known result for the CGHS model, where we used the identification $B = 4\lambda^2$ such that (1) reproduces the CGHS action. For the case of

- **Asymptotic de-Sitter spacetimes**
  we have from (23)

\[
L(U) = B_1B_2^{\frac{2-a}{a-1}} U^\frac{a}{a-1} - B_2 U^2,
\]
and get for the value of Hawking radiation

\[ T_{-+}|_{\text{asymp}} = \frac{2 - a}{192(a - 1)} \left( B_1^{1-a} B_2^a \right)^{\frac{1}{2a}}. \]

Setting \( b = 0 \) we have

- **Asymptotic Rindler spacetime.** According to (37) we have

\[ L(U) = B_1 B_2^{\frac{2-a}{2(a-1)}} U^\frac{a}{a-1} - B_2 \frac{1}{U} \]

and the Hawking radiation reads

\[ T_{-+}|_{\text{asymp}} = \frac{B}{384 (a - 1)}. \]

Notice, that if we had not used the rescaling coefficient as in (37), but if instead we had set

\[ K = \frac{B}{4}, \]

we would have obtained

\[ T_{-+}|_{\text{asymp}} = \frac{\lambda^2}{48}, \]

which is just the result given in [11].

### 5 Path integral measure, asymptotic conditions and the problem of interpretation

It is well known (see e.g. [26]) that there is a unique ultralocal path integral measure for a scalar field \( f \) yielding a covariantly conserved stress energy tensor. This measure is defined by the equation

\[ \int Df \exp \left( i \int \sqrt{-g} d^4 x f^2 \right) = 1. \]

It is also known [27] that in four dimensional Einstein gravity one can define energy in a unique and self-consistent way if and only if in the asymptotic region the metric is approaching fast enough that of flat Minkowski space.
As a consequence, Hawking radiation and ADM mass depend on a choice of asymptotic conditions and path integral measure.

In dilaton gravity we have a new entity which is absent in Einstein gravity, the dilaton field. One is tempted to use a rescaled metric at some steps of the calculations. In the present paper such a rescaling is trivial everywhere. This means that we are using just the metric $g_{\mu\nu}$ which appears in the action (1) to define the path integral measure, stress energy tensor etc. Of course, this is just a matter of interpretation. One can claim that the rescaled metric $\Phi(\phi) g_{\mu\nu}$ describes the geometry of space-time and thus should be used in the definitions of the above mentioned quantities. However, the rescaled metric should be used everywhere. Otherwise, the quantum theory becomes inconsistent. For example, a part of diffeomorphism invariance can be lost. We believe that if one wishes to keep contact to four dimensional Einstein gravity, which is diffeomorphism invariant, one should retain general coordinate invariance\(^3\). In the context of dilaton gravity models a transition from $g_{\mu\nu}$ to $g^\Phi_{\mu\nu} = \Phi(\phi) g_{\mu\nu}$ means replacement of one particular dilaton interaction by another. Such a replacement should in general change the Hawking radiation because the definitions of the stress energy tensor and path integral measure and the asymptotic behavior of metric are not conformally invariant. At least, conformal invariance of the Hawking radiation was never proved. Such conformal equivalence was conjectured in a recent paper by Cadoni [29] who used conformal transformation in general dilaton theory to remove kinetic term $(\nabla \phi)^2$ of the dilaton field. He also used a $\phi$-dependent path integral measure and $\phi$-dependent stress energy tensor to define the Hawking radiation. A comparison shows that his results for the Hawking temperature differ by an $a$ and $b$ dependent factor from ours, which were obtained directly without use of conformal transformation. We conclude that conformally equivalent theories do really give different results for the Hawking radiation. This statement can be illustrated by an example from [30], where the CGHS black hole was transformed to the flat Rindler space-time. Due to the absence of black hole curvature singularity any radiation in the latter space should be considered as the Unruh radiation rather than the Hawking one.

We have seen that both ADM mass and Hawking radiation depend on the asymptotic behavior of the metric and on the subtraction procedure of refer-

\(^3\)Some new possibilities appear if one trades diffeomorphism invariance for Weyl invariance [28]. We shall not consider such theories here.
ence space-time contribution. This dependence is quite natural from a physical point of view. Since energy and energy flux are not coordinate invariant, in addition to a zero energy state one should also define two observers. One of them corresponds to a particular solution, and the other to the reference zero point configuration. For asymptotically Minkowski, Rindler or de Sitter solution there is a choice which is clearly preferable. With this choice our results for ADM mass and Hawking radiation are independent of coordinate transformations which vanish rapidly enough at asymptotic region. In the case of four dimensional Einstein gravity appropriate asymptotic conditions were formulated by Faddeev [27]. For the CGHS model with Polyakov–Liouville term the suitable asymptotic conditions were found recently [14].

6 Dynamical formation of black holes

6.1 General solution

Consider a scalar matter field $f$ minimally coupled to gravity. We add the following term to the action (1)

$$L_m = -\frac{1}{2} \int d^2x \sqrt{-g} g^{\mu\nu} \nabla_\mu f \nabla_\nu f.$$  \hspace{1cm} \text{(79)}

In the presence of this matter field the equations of motion take the form

$$g_{\mu\nu}(4-2a)(\nabla \phi)^2 - 2\nabla^2 \phi - \frac{B}{2} e^{2\phi(1-a-b)} + 4(a-1)\partial_\mu \phi \partial_\nu \phi$$

$$+ 2\nabla_\mu \nabla_\nu \phi + \frac{1}{2} e^{2\phi} T_{\mu\nu} = 0$$

$$R - 4a(\nabla \phi)^2 + 4a\nabla^2 \phi + B(a + b) e^{2(1-a-b)\phi} = 0$$

$$\nabla^2 f = 0$$  \hspace{1cm} \text{(80)}

and in conformal gauge these equations are

$$2\partial_\pm \partial_- \rho + 4a \partial_+ \phi \partial_- \phi - 4a \partial_+ \partial_- \phi + \frac{B}{4} (a + b) e^{2(\rho + (1-a-b)\phi)} = 0$$

$$-4\partial_+ \phi \partial_- \phi + 2\partial_+ \partial_- \phi - \frac{B}{4} e^{2(\rho + (1-a-b)\phi)} = 0$$

$$\partial_+ \partial_- f = 0$$  \hspace{1cm} \text{(82)}

$$4(a-1)\partial_+ \phi \partial_+ \phi + 2\partial_\pm \partial_\pm \phi - 4\partial_\pm \rho \partial_\pm \phi = \frac{1}{2} e^{2\phi} \partial_\pm f \partial_\pm f.$$  \hspace{1cm} \text{(84)}
The matter equation of motion (83) can be solved by setting $f = f(x^+)$. Then $\rho$ can be determined from the $(-,-)$ component of (84):

\[ \partial_- \rho = (a - 1) \partial_- \phi + \frac{1}{2} \partial_- \ln |\partial_- \phi| \]

which gives

\[ \rho = (a - 1) \phi + \frac{1}{2} \ln |\partial_- \phi| + \xi (x^+) \]

The arbitrary function $\xi (x^+)$ can be removed by using residual gauge freedom of the conformal gauge. In what follows we set $\xi = 0$. $\rho$ can be removed from the equations of motion which are reduced to the following three independent equations:

\[ \frac{1}{2} (\partial_+ \phi) \partial_+ [\ln |\partial_+ \phi| - \ln |\partial_- \phi|] = \frac{e^{2\phi}}{8} (f')^2 \]

\[ \partial_+ \partial_- (\phi - \frac{1}{2} \ln |\partial_- \phi|) = \frac{Bb}{8} |\partial_- \phi| e^{-2b\phi} \]

\[ -2 \partial_+ \partial_- \phi + 4 \partial_+ \phi \partial_- \phi + \frac{B}{4} |\partial_- \phi| e^{-2b\phi} = 0 \]

After some algebra the eq. (88) gives

\[ \partial_+ e^{-2\phi} \mp \frac{B}{8(b + 1)} e^{-2\phi(b+1)} = -2\eta(x^+) e^{-2\phi} + \lambda(x^+) \]

with two arbitrary functions $\eta$ and $\lambda$. From eq. (89) one obtains

\[ \partial_+ e^{-2\phi} = \frac{B}{8(b + 1)} e^{-2\phi(b+1)} = \kappa(x^+) \]

This means that $\eta = 0$, $\kappa = \lambda$ and

\[ \partial_+ \phi = \mp \frac{B}{16(b + 1)} e^{-2b\phi} - \frac{1}{2} \lambda(x^+) e^{2\phi} \]

Here $\pm$ is the sign of $\partial_- \phi$. From (87) we can express the arbitrary function $\lambda$:

\[ \partial_+ \lambda(x^+) = -\frac{1}{2} (f')^2 \]
The two equations (92) and (93) completely define \( \phi \) for a given matter distribution. Let us take the matter field in the form of shock wave

\[
m(94) f'(x^+)^2 = D\delta(x^+ - x_0^+)
\]

Before the shock wave starts we have the empty space solution:

\[
\phi = \frac{1}{2b} \ln \left| \frac{Bb}{4(b+1)} \right|, \quad \text{for} \quad \frac{B}{b+1} > 0
\]

\[
m(95) \phi = \frac{1}{2b} \ln \left| \frac{Bbx}{4(b+1)} \right|, \quad \text{for} \quad \frac{B}{b+1} < 0
\]

Here we consider only the case \( b \neq 0 \), since the dynamical formation of black holes for \( b = 0 \) was already considered in [11]. After the shock wave the black hole solution is formally given by the integral

\[
m(96) \int_{\phi(x,t)}^{\phi_{\text{bou}}(x^+)} \frac{de^{-2\phi}}{\pm \frac{B}{8(b+1)}e^{-2(b+1)\phi} - D} = x^+ + h(x^-)
\]

Now we should glue together the solutions (95) and (96) by using the arbitrary function \( h(x^-) \). For the sake of simplicity we put \( x_0^+ = 0 \). Note, that we can restrict ourselves to positive or negative values of coordinates \( x \) and \( t \) in (95) since in these regions the dilaton field changes from \(-\infty\) to \(+\infty\). On the line \( x^+ = 0 \) we obtain from (95)

\[
m(97) \phi_{\text{bou}}(x^-) = \frac{1}{2b} \ln \left| \frac{Bbx^-}{8(b+1)} \right|.
\]

Next one can find the function \( h(x^-) \) which ensures continuity of \( \phi \):

\[
m(98) h(x^-) = \int_{\phi_{\text{bou}}(x^-)}^{\phi(x,t)} \frac{de^{-2\phi}}{\pm \frac{B}{8(b+1)}e^{-2(b+1)\phi} - D}
\]

The lower limit in the integrals in (96) and (98) plays no role, however it should be the same and constant in both equations. The \( \pm \) sign in (96) can be fixed by calculating \( \partial_- \phi \).

### 6.2 Asymptotically Minkowski solutions

Consider now the case \( a = b + 1 \) corresponding to asymptotically Minkowski solutions in more detail. Let \( 0 < a < 2 \). One can easily demonstrate that
in the asymptotic regions before and after the formation of a black hole the metric of the solutions (96) and (95) behaves as

\[ e^{2\rho} \to |B/16a|. \]

Thus to obtain a black hole surrounded by Minkowski space with unit metric we need a proper rescaling of coordinates.

Let us modify the procedure of the previous subsection in order to obtain the rescaled solution. First of all, one should take the function \( \xi \) in (86) in the form

\[ \xi = \frac{1}{4} \ln \left| \frac{16a}{B} \right|. \]

With this choice equation (86) now reads

\[ \rho = (a - 1)\phi + \frac{1}{2} \ln \left| \frac{a}{B} \sqrt{\frac{a}{B} \partial \phi} \right|. \]

One can repeat all the steps of the previous subsection with this form of \( \rho \). As a result, one obtains a unit metric for \( x^+ < 0 \) and an asymptotically unit metric for \( x^+ > 0 \). Such a configuration is again produced by the shock wave (94), where the constant \( C \) is now related to the parameter \( D \) in a different manner,

\[ D = \frac{C}{4} \left( \frac{a}{B} \right)^{\frac{3}{2}}. \]

This constant is proportional to the energy of the shock wave measured by a Minkowski space observer. This confirms our previous result (41) for the ADM mass of black hole in asymptotically minkowskian space-time.

It is tempting to calculate the Hawking radiation by comparing normal ordering prescriptions and Green functions in two asymptotic regions, before and after formation of a black hole [31]. If such a method could be applied in the coordinate systems \((x^+, x^-)\) and \((x^+, h(x^-))\), the stress energy tensor of the Hawking radiation would be calculated by differentiating the function \( h(x^-) \). This appeared, however, not to be the case. The full ground state solution is covered by positive (or negative) values of \( x \) or \( t \). Hence the Green function differs from the standard expression valid for infinite range of all coordinates.
7 Conclusions

In the present paper we calculated the ADM and BS mass and Hawking radiation of the black hole solutions in generalized dilaton gravities described by the action (1). Special attention was paid to asymptotic conditions for the metric field. It turned out that both mass and stress energy tensor depend on these conditions. From a physical point of view this means that they are influenced by the choice of an asymptotic observer. For generic $a$ and $b$ this choice is not unique. It rather depends on the interpretation of particular two dimensional model and its relation to four dimensional gravity. For the selected values of $a$ and $b$ corresponding to asymptotically Minkowski, Rindler or de Sitter solutions there exists a preferred coordinate frame at infinity. This fact allows some physical conclusions. We can see that, depending on the parameters of the dilatonic action, the Hawking temperature can be related to the black hole mass with positive or negative power. In a fully quantized theory the parameters $a$ and $b$ should become running couplings being functions of a scale parameter which could be the black hole mass. Thus it could happen that a black hole which starts its evolution near the point $a = -b = \frac{1}{2}$, corresponding to spherically symmetric gravity, migrates in the course of evaporation to another region of the $(a, b)$ plane where the heat capacity becomes positive. With our present level of understanding the last statement is nothing more than an unsupported speculation, which should, however, be able to describe black hole evolution in some realistic quantum theory.

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