Fermionic Matrix Models

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Abstract

We review a class of matrix models whose degrees of freedom are matrices with anticommuting elements. We discuss the properties of the adjoint fermion one-, two- and gauge invariant $D$-dimensional matrix models at large-$N$ and compare them with their bosonic counterparts which are the more familiar Hermitian matrix models. We derive and solve the complete sets of loop equations for the correlators of these models and use these equations to examine critical behaviour. The topological large-$N$ expansions are also constructed and their relation to other aspects of modern string theory such as integrable hierarchies is discussed. We use these connections to discuss the applications of these matrix models to string theory and induced gauge theories. We argue that as such the fermionic matrix models may provide a novel generalization of the discretized random surface representation of quantum gravity in which the genus sum alternates and the sums over genera for correlators have better convergence properties than their Hermitian counterparts. We discuss the use of adjoint fermions instead of adjoint scalars to study induced gauge theories. We also discuss two classes of dimensionally reduced models, a fermionic vector model and a supersymmetric matrix model, and discuss their applications to the branched polymer phase of string theories in target space dimensions $D > 1$ and also to the meander problem.
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1 Introduction: From Hermitian to Fermionic Random Matrix Models

In this Paper we shall review the theory and applications of random matrix models whose matrices have anticommuting elements. Typically, these models possess a $U(N)$ symmetry and the matrices transform under the adjoint representation of the symmetry group. They can be analyzed, and often solved, in the limit of large $N$ where they have an interesting and non-trivial structure. Uses of these models ranges from attempts to formulate dynamical random surface theories with alternating genus sums to the solution of combinatorial problems such as the meander problem which we shall review, to lattice gauge theories and the formulation of models of induced gauge theory.

Fermionic matrix models have several features which distinguish them from the more familiar bosonic Hermitian matrix models. They are not eigenvalue models in the sense that matrices with anticommuting elements cannot be diagonalized by unitary transformations. They can nevertheless often be analyzed by methods similar to those used for Hermitian matrix models, such as the method of loop equations. Furthermore, the integration over anticommuting numbers is straightforward and is often more convergent than ordinary integrals over commuting variables. In perturbation theory, this improved convergence is seen as the result of an alternating perturbation series.

To help motivate the study of fermionic matrix models, we first present a brief overview of the techniques which are used to study the large $N$ limit of the standard Hermitian matrix models.

1.1 Hermitian Matrix Models, String Theory and Induced Gauge Theories

For completeness, and so that we can later compare techniques for solving Hermitian and fermionic matrix models, we begin this Section with a brief review of some of the technical aspects of solving random Hermitian matrix models in the large-$N$ limit. We also discuss some of their uses, particularly in lower dimensional quantum gravity, string theory and lattice gauge theories. A more detailed discussion of the material can also be found in many other reviews that will be cited as we proceed. For a particularly lucid discussion, see [38].

1.1.1 One-matrix Models

A Hermitian one-matrix model is defined as a statistical theory of Hermitian matrices with partition function

\[ Z = \int d\phi \ e^{-N \ tr \ V(\phi)} \] (1.1)

where $V(\lambda)$ is a smooth potential and

\[ d\phi \equiv \prod_{k=1}^{N} d\phi_{kk} \ \prod_{i<j} d \Re \phi_{ij} \ d \Im \phi_{ij} \] (1.2)
is the integration measure on the space of $N \times N$ Hermitian matrices ($d\phi_{ij}$ is always understood as an ordinary Riemann-Lebesgue measure), i.e. the Haar measure on the Lie algebra $\text{Herm}(N)$. This model is invariant under the adjoint action of the unitary group $U(N)$,

$$\phi \rightarrow U\phi U^\dagger, \quad U \in U(N)$$ (1.3)

which restricts the observables to those which are invariant functions of $\phi$. The free energy $\log Z$ can be expanded in a power series in the parameter $\frac{1}{N^2}$. The leading term is of order $N^2$ and occurs in the infinite-$N$ limit. Each order in this expansion can be represented as an infinite series of “fat-graphs” with the topological property that all graphs of a given order can be drawn without crossing lines of the graphs on a two-dimensional surface of particular genus - the graphs with genus $g$ contribute the term of order $N^{2g-2}$. The leading term is the sum of genus zero (or planar) graphs, i.e. those which can be drawn on a plane or the surface of a sphere.

In this and several more elaborate Hermitian matrix models, the planar and higher genus graphs can be summed explicitly. These Hermitian matrix models can be thought of as examples of a $D=0$ quantum field theory where the topological large-$N$ expansion which was originally proposed by 'tHooft for quantum chromodynamics (QCD) [129] is explicitly solvable [20, 25]. In QCD and in Yang-Mills theory, this expansion is intractable, except in $(1+1)$ dimensions where Yang-Mills theory and QCD with fundamental representation matter are solvable.

The main applications of Hermitian matrix models are as non-perturbative approaches to low-dimensional string theory where the large-$N$ expansion of the matrix model coincides with the genus expansion [22, 53, 65] of the string partition function. For instance, for a polynomial potential of the form

$$V(\phi) = \frac{1}{2} \phi^2 + \frac{g}{K} \sum_{i=1}^{K} \phi_i$$ (1.4)

the perturbative expansion of (1.1) in the coupling $\bar{g}$ in terms of fat-graphs coincides with the formal sum over discretizations by regular $K$-gons of 2-dimensional compact Riemann surfaces. The fattening of lines in the Feynman graphs represents the 2 indices that a matrix has. In this interpretation, the Gaussian $\phi^2$ term in (1.4) represents the free boson term (with unit mass $m^2 = 1$) for the non-kinematical quantum field theory (1.1) which gives the propagator of the (fat) Feynman graphs,

$$(2\pi)^{-N^2/2} \int d\phi \phi_{ij}\phi_{kl} e^{-N \text{ tr } \phi^2/2} = \frac{1}{N} \delta_{il} \delta_{kj}$$ (1.5)

The $\phi^K$ interaction term produces $K$-valence vertices.

Diagrammatically, the partition function (1.1) is the sum over all possible connected and disconnected Feynman diagrams constructed by linking the propagators (1.5) and $K$-point vertices together in an orientation-preserving way. This is just the usual Wick expansion in quantum field theory. The Wick contractions between matrices, as in (1.5), assign a product of 2 delta-functions, one for the inner 2 indices of a matrix pair and one for the outer ones. The connected diagrams, divided by the appropriate symmetry factors for topologically equivalent graphs, are obtained by expanding the free energy $\log Z$. This gives the fat-graph expansion [38]

$$\log Z = \sum_{\mathcal{F}} \left( -\bar{g} \right)^N N^{v-e+l} \frac{\left| G(\mathcal{F}) \right|}{\left| G(\mathcal{F}) \right|}$$ (1.6)
where the sum is over all fat-graphs $\mathcal{F}$ with $v$ vertices, $e$ propagators and $l$ index loops, and $|G(\mathcal{F})|$ is the order of the symmetry group of $\mathcal{F}$ (the group of permutations of the vertices of $\mathcal{F}$). The connected diagrams generate a 2-dimensional lattice whose dual lattice is a discretization of a Riemann surface by regular polygons. This dual lattice of regular polygons is constructed by associating polygon faces with $K$-point vertices, sides with propagators, and polygon vertices with closed loops. The number $v - e + l$ appearing in (1.6) is then the Euler characteristic of the Riemann surface. A smooth Riemann surface is well approximated by its polygonization when the number of polygons is large and the area of each polygon is infinitesimal. General polynomial potentials would allow a variety of polygons in the polygonization of the Riemann surface. If the area of each polygon is taken as one, the statistical sum in (1.6) is a statistical sum over connected Riemann surfaces where each term in the sum is dual to a Feynman graph.

Note that the sign in front of the factor of $N$ in (1.1) is taken to be negative to ensure the convergence of the Gaussian statistical model ($\bar{g} = 0$ in (1.4)) and hence all Feynman graphs. However, the sum over Riemann surfaces in (1.6) has positive weights only when $\bar{g} < 0$. For these values of $\bar{g}$, the integration over Hermitian matrices in (1.1) diverges. It turns out that one can make sense of it only when $N$ is infinite.

The corresponding discretized surface model (1.6) is associated with the (Euclidean) statistical ensemble of random surfaces with partition function

$$Z_{\text{str}} = \sum_{h=0}^{\infty} \int Dg \ e^{-\Lambda A(\Sigma^h;g) + G^{-1}\chi(h)}$$

(1.7)

where the action is the Einstein-Hilbert action for pure gravity with cosmological term. Here the functional integration is over all metrics $g_{\mu\nu}(x)$ on the 2-surface $\Sigma^h$ with $h$ ‘handles’, $\Lambda$ is the cosmological constant (or string tension) which multiplies the area $A(\Sigma^h;g) = \int_{\Sigma^h} \sqrt{g}$ of $\Sigma^h$, and $G$ is the gravitational constant which multiplies the topological scalar curvature term

$$\chi(h) = \int_{\Sigma^h} \sqrt{g} R(g)/4\pi = 2 - 2h$$

(1.8)

with $\chi(h)$ the Euler characteristic of $\Sigma^h$. This statistical model can be regarded as a $D = 0$ dimensional string theory, i.e. a pure theory of surfaces with no coupling to additional matter degrees of freedom on the string world-sheet or the propagation of strings in a non-existent embedding space.

The fat-graph expansion (1.6) of the Hermitian matrix model (1.1) represents a discretized version of the continuum quantum gravity model (1.7). This latticized model is the dynamically triangulated random surface theory [7, 33, 73, 76]

$$Z = \sum_{h=0}^{\infty} e^{G^{-1}\chi(h)} \sum_{T_h} e^{-\Lambda \mathcal{N}(T_h)}$$

(1.9)

where the second sum in (1.9) is over all possible triangulations $T_h$ at fixed genus $h$ and $\mathcal{N}(T_h)$ is the dynamical variable which counts the number of triangles in $T_h$ (so that $\Lambda$ plays the role here of a chemical potential). The dual lattice formed by the fat graphs of the matrix model represents the triangulation of the surface. The propagator (1.5) produces a factor of $\mathcal{N}^{2-2h}$ in the free energy (1.6) associated with a graph of genus $h$ [129]. The coupling constants of (1.9) are therefore related to those in (1.1),(1.4) by [129]

$$N = e^{G^{-1}} \quad , \quad \bar{g} = - e^{-\Lambda}$$

(1.10)
It is also possible to couple matter in (1.9) (represented by more complicated potentials in (1.1)) by including matter fields $X_i$ at each vertex $i$ of the dual triangulation lattice $\mathcal{F}$ (of fixed coordination number), adding appropriate interaction terms between nearest neighbor vertices to the action in (1.9) and summing over all of the $X_i$. For instance, to couple the Ising model to 2-dimensional quantum gravity we can replace the weight in (1.9) by the Ising spin partition function

$$Z_I(\beta; T_h) = \sum_{S_i = \pm 1} e^{\beta \sum_{(i,j) \in \epsilon} S_i S_j}$$

(1.11)

The large-$N$ limit of the Hermitian matrix model exhibits phase transitions which correspond to the continuum limits of the discretized random surface theories [22, 53, 38, 65].

Normally, Hermitian 1-matrix models are treated as statistical theories of the eigenvalues of the Hermitian matrices [25] and their partition function and correlators are computable in the large-$N$ limit using techniques such as loop equations [14, 36, 74] or orthogonal polynomials [20]. This is accomplished by diagonalizing the Hermitian matrices in (1.1). The measure can be written as [38]

$$d\phi = [dU] \prod_{i=1}^{N} d\lambda_i \Delta^2(\lambda)$$

(1.12)

where $[dU]$ is the Haar measure for integration over the unitary transformations (1.3) which diagonalize $\phi$, $\lambda_i$ are the eigenvalues of $\phi$ and the Vandermonde determinant,

$$\Delta(\lambda) = \det_{i,j} [\lambda_i^{j-1}] = \prod_{i<j} (\lambda_i - \lambda_j) = \exp \sum_{i<j} \log (\lambda_i - \lambda_j)$$

(1.13)

is the Jacobian. The partition function (up to an irrelevant overall numerical factor) is

$$Z = \int [dU] \int_{-\infty}^{+\infty} \prod_{i=1}^{N} d\lambda_i \Delta^2(\lambda) e^{-N \sum_{j=1}^{N} V(\lambda_j)}$$

(1.14)

The Haar measure for integration on the group $U(N)$ of unitary matrices

$$[dU] = \frac{\prod_{k=1}^{N} k!}{(2\pi)^{N(N+1)/2}} \prod_{i,j} dU_{ij} \delta \left( \sum_{k=1}^{N} U_{ik} U_{jk}^\dagger - \delta_{ij} \right)$$

(1.15)

is normalized ($\int [dU] = 1$) and has the symmetries

$$\int [dU] F(U^\dagger) = \int [dU] F(U) , \quad \int [dU] F(VU) = \int [dU] F(U) = \int [dU] F(UV)$$

(1.16)

for any $V \in U(N)$. These imply that some of its lower order moments are

$$\int [dU] U_{ij} = 0 , \quad \int [dU] U_{ij} U_{kl}^\dagger = \delta_{il} \delta_{jk}$$

(1.17)

$$\int [dU] U_{ij} U_{mn} U_{kl}^\dagger U_{pq}^\dagger = \frac{1}{N^2 - 1} \left( \delta_{il} \delta_{jk} \delta_{mp} \delta_{np} + \delta_{iq} \delta_{jp} \delta_{ml} \delta_{nk} \right)$$

$$+ \frac{1}{N(N^2 - 1)} \left( \delta_{jk} \delta_{lm} \delta_{np} \delta_{qi} + \delta_{jp} \delta_{nm} \delta_{nk} \delta_{li} \right)$$

(1.18)
If the group were $SU(N)$ rather than $U(N)$, there is an additional identity which follows from invariance of the determinant,

$$\int [dU] \, U_{i_1j_1} U_{i_2j_2} \cdots U_{i_Nj_N} = c \cdot \epsilon_{i_1i_2\cdots i_N} \epsilon_{j_1j_2\cdots j_N}$$

(1.19)

where $c$ is a constant. Similar expressions can be obtained for the higher-order correlators of the Haar measure.

Since the integration over unitary matrices in (1.14) decouples, the effective statistical theory of eigenvalues involves an action of order $N^2$ describing $N$ degrees of freedom. This makes the model (formally) an exactly-solvable one in the $N \to \infty$ limit. In particular, in this limit it can be evaluated by the saddle-point approximation. The Vandermonde determinant in (1.14) acts as a hard-core repulsive term in the action which prevents its minimum from being simply the minimum of the potential. The stationary condition for the effective action in (1.14) leads to the saddle-point equation

$$V'(\lambda) = \frac{1}{N} \sum_{j < i} \frac{1}{\lambda_i - \lambda_j}, \quad i = 1, \ldots, N$$

(1.20)

and the lowest-order contribution to (1.1) (i.e. the number of planar fat-graphs), which dominates for $N \to \infty$, is obtained by substituting into (1.14) the saddle-point value determined by (1.20). The common way to study the large-$N$ limit is to introduce the normalized spectral density of eigenvalues

$$\rho(\lambda) = \frac{1}{N} \sum_{i=1}^N \delta(\lambda - \lambda_i), \quad \lambda \in \mathbb{R}$$

(1.21)

which for finite-$N$ is supported at the discrete points $\lambda_1, \ldots, \lambda_N$. We can then represent all summations above as integrations over $\lambda \in \mathbb{R}$ using (1.21). To treat the large-$N$ limit, we order the eigenvalues so that $\lambda_1 < \lambda_2 < \ldots < \lambda_N$ and introduce a non-decreasing differentiable function $\lambda(x)$ of $x \in [0, 1]$ with $\lambda_i = N \lambda(i/N)$ for $i = 1, \ldots, N$. We then regard the continuous function $\rho(\lambda) \equiv \frac{d\lambda}{d\lambda(x)}$ as the formal large-$N$ limit of (1.21). The positive quantity $N \rho(\lambda) d\lambda$ is the number of eigenvalues in a range $d\lambda$ about $\lambda$, and it is supported on some finite intervals $[a_i, b_i] \subset \mathbb{R}$. The saddle-point equation (1.20) then becomes

$$V'(\lambda) = 2 \int d\alpha \, \frac{\rho(\alpha)}{\lambda - \alpha}$$

(1.22)

and the large-$N$ saddle-point free energy is

$$\lim_{N \to \infty} \frac{1}{N^2} \log Z = - \int d\alpha \, \rho(\alpha) V(\alpha) + \int \int d\alpha \, d\beta \, \rho(\alpha) \rho(\beta) \log |\alpha - \beta|$$

(1.23)

where $\rho(\alpha)$ is the solution of (1.22). Note that (1.22) now also follows from the functional variation of (1.23) with respect to the spectral density $\rho(\alpha)$.

To solve (1.22) for the eigenvalue distribution $\rho(\alpha)$, we introduce its Hilbert transform

$$\omega(z) \equiv \left\langle \frac{1}{N} \frac{1}{z - \phi} \right\rangle = \frac{1}{N} \sum_{i=1}^N \frac{1}{z - \lambda_i} = \int d\alpha \, \frac{\rho(\alpha)}{z - \alpha}, \quad z \in \mathbb{C}$$

(1.24)

and note that the saddle-point equation (1.22) is the real part of the discontinuity of the function $\omega(z)$ across the support of $\rho$, i.e.

$$\omega(\lambda \pm i0) = V'(\lambda)/2 \mp i\pi \rho(\lambda), \quad \lambda \in \text{supp } \rho$$

(1.25)
In (1.24), \( \langle \cdot \rangle \) denotes the normalized average with respect to the matrix ensemble (1.1),

\[
\langle Q(\phi) \rangle \equiv \frac{1}{Z} \int d\phi \, e^{-N \, \text{tr} \, V(\phi)} Q(\phi)
\]  

(1.26)

and the function (1.24) is the generating function, via its \( \frac{1}{z} \)-expansion, for the correlators \( \langle \frac{1}{z} \phi^k \rangle \) of the Hermitian matrix model. For \( s \in \mathbb{R}^+ \), the quantity \( \frac{1}{N} \phi^s \) can be thought of as an operator which creates a loop of length \( s \) on the surface [74], and the potential \( V(\phi) \) is a source for the inverse Laplace transform of (1.24),

\[
\frac{\text{tr}}{N} V(\phi) = \int_{0-i\infty}^{0+i\infty} \frac{dz}{2\pi i} \, V(z) \omega(z)
\]  

(1.27)

This method of determining the solutions to the saddle-point equation by solving an equation via the analytic properties of a function such as (1.24) is known as the Riemann-Hilbert method. Alternatively, one can derive explicitly an equation for \( \omega(z) \) by multiplying (1.20) by \( \frac{1}{N} \frac{1}{z - \lambda_i} \) and summing over \( i = 1, \ldots, N \). This leads to

\[
\omega(z)^2 + \omega'(z)/N - V'(z) \omega(z) = \int d\alpha \, \rho(\alpha) \frac{V'(z) - V'(\alpha)}{\alpha - z}
\]  

(1.28)

where the right-hand side of (1.28) is a polynomial of degree \( K - 2 \) when the potential \( V(\lambda) \) is a polynomial of degree \( K \).

The saddle-point technique relies heavily on the facts that (a) the matrix model has a finite large-\( N \) limit, or equivalently a finite contribution from planar (genus 0) diagrams to the free energy, and (b) the matrices \( \phi \) are diagonalizable. Another way to arrive at the equation (1.28) is to exploit the invariance of the partition function (1.1) under arbitrary changes \( \phi \rightarrow f(\phi) \) of the Hermitian matrix variables. The resulting equations represent the full set of Schwinger-Dyson equations of motion for the non-kinematical field theory (1.1). They are obtained by performing a shift of the matrix variables by an infinitesimal loop of length \( k \in \mathbb{Z}^+ \), i.e. the change of variables \( \phi \rightarrow \phi + \varepsilon \phi^k \) in the matrix integral (1.1), where \( \varepsilon \) is an infinitesimal real number. This leads to the identity

\[
\sum_{\ell=0}^{k-1} \left\langle \frac{\text{tr}}{N} \phi^\ell \frac{\text{tr}}{N} \phi^{k-\ell-1} \right\rangle = \left\langle \frac{\text{tr}}{N} \phi^k V'(\phi) \right\rangle
\]  

(1.29)

Multiplying (1.29) by \( \frac{1}{z^{k+1}} \) and summing over all \( k \in \mathbb{Z}^+ \) then leads to the loop equation (1.28), which represents the full set of equations of motion of the model because (1.24) contains a complete set of operators. In fact, this equation could also have been obtained from the shift

\[
\phi \rightarrow \phi + \frac{\varepsilon}{z - \phi}
\]  

(1.30)

under which the invariance of (1.1) leads to the matrix identity

\[
0 = \int d\phi \, \frac{\partial}{\partial \phi_{ij}} \left\{ \left( \frac{1}{z - \phi} \right)_{kl} e^{-N \, \text{tr} \, V(\phi)} \right\}
\]  

(1.31)

After expanding the terms in (1.31) into averages and summing over all \( i = k \) and \( j = \ell \), we arrive at the loop equation (1.28). However, if the potential \( V \) in (1.31) is not bounded from below, then the integration by parts on the right-hand side of (1.31) gives infinite contributions.
from boundary terms at infinity and is not formally zero. This derivation of the loop equations is appealing because it does not explicitly involve the eigenvalue distribution of the model.

As an example, consider the simplest case of the Gaussian model where \( V(\phi) = m^2 \phi^2 / 2 \). The loop equation (1.28) at \( N = \infty \) then leads to a quadratic equation for \( \omega(z) \) whose solution is

\[
\omega(z) = \frac{1}{2} \left( m^2 z - \sqrt{m^4 z^2 - 4m^2} \right)
\]

where the branch of the square root is chosen so that \( \omega(z) \sim 1/z \) at \( |z| \to \infty \) which follows from the definition (1.24) and the normalization condition \( \int d\lambda \rho(\lambda) = 1 \). Computing the discontinuities of (1.32) across its branch cuts leads to the well-known Wigner semi-circle distribution for the eigenvalues of \( \phi \),

\[
\rho(\lambda) = \frac{m}{\pi} \sqrt{1 - \frac{m^2 \lambda^2}{4}}
\]

which has support on the interval \(( -\frac{2}{m}, \frac{2}{m} ) \).

Generally, the Wilson loop average \( \langle \frac{1}{N} \text{tr} e^{i\phi} \rangle \) yields a weighted superposition of loops of different length on the surface and it has the same behaviour in the continuum as the correlator \( \langle \frac{1}{N} \phi^3 \rangle \). The resolvent function \( \omega(z) \) is the inverse Laplace transform of the Wilson loop (see (1.24))

\[
W(s) \equiv \langle \frac{1}{N} \text{tr} e^{i\phi} \rangle = - \oint_{\mathcal{C}} \frac{dz}{2\pi i} e^{iz} \omega(z)
\]

where the contour \( \mathcal{C} \) encircles the singularities of \( \omega(z) \) with counterclockwise orientation. The loop equation (1.28) at \( N = \infty \) can then also be written as [74]

\[
V'(\partial_s) W(s) = \frac{1}{N} \int_0^s dt \ W(t) W(s - t)
\]

where the right-hand side of (1.35) can be interpreted as the operation of gluing 2 boundary loops together [74]. (1.35) is very similar to a simplified (zero-dimensional) version of the Makeenko-Migdal equations for the Wilson loop functionals \( W[C] \) in multi-colour QCD [107].

In the large-\( N \) limit the branch cut singularities of the solution \( \omega(z) \) to the quadratic equation (1.28) determine the spectral density \( \rho(\lambda) \). There will be regions in general in coupling constant space where the analytic features of the continuous function \( \rho \) change (e.g. it becomes negative and acquires multiple branch cuts) [25, 38]. For a discretization potential (1.4), the area, or number of \( K \)-point vertices \( v \), appears in the free energy \( F \) as the power \( g^v \), and so the average area of each diagram is \( \langle v \rangle \sim \partial_g \log F \). These singular points of the free energy therefore describe critical behaviours in the random matrix models at which the areas of the individual discretization polygons become infinite [88, 38]. As \( g \) then approaches its critical value \( g_c \), the areas of the individual polygons may be rescaled to zero giving a continuum surface with finite area. This is precisely what is needed to define continuum surfaces, in that one tunes the coupling \( g \) to the point where the perturbation series for (1.1) diverges and the integral becomes dominated by ’tHooft diagrams with infinite numbers of vertices. A phase transition for \( \Lambda \to \Lambda_c \) at \( D = 0 \) is indeed possible – it is a third order Gross-Witten type phase transition [68] and it occurs for \( N \to \infty \) when the number of degrees of freedom becomes infinite. The continuum limit is therefore reached as \( N \to \infty \) and \( \Lambda \to \Lambda_c \) where the discrete partition function (1.9) approaches the physical continuum one (1.7). The singularities in the coupling constants in this case occur at points where some of the zeroes
of the singular parts of \( \omega(z) \) (i.e. the zeroes of \( \rho(\alpha) \)) coalesce with an endpoint of its branch cuts [38]. These divergences arise because of the entropy factor (number of graphs) in (1.9), whereas for \( \Lambda \) large enough (\( \bar{g} \) small) the sum over triangulations \( T_h \) in (1.9) converges.

One prediction of the continuum Liouville field theory approach to non-critical string theory and quantum gravity [122, 83, 33, 45] (see [38, 112] for reviews) is the occurrence of a critical string exponent \( \gamma_{\text{str}} \), which is defined in terms of the area dependence of the free energy for surfaces of fixed large area \( A \) as

\[
\log Z_{\text{str}}(A) \sim A^{(\gamma_{\text{str}} - 2) \chi(h)/2 - 1}
\]

If we couple 2-dimensional gravity to conformal field theory, in particular the unitary discrete series which are labelled by an integer \( m \geq 2 \) (see [61] for a review) and the central charge

\[
D = 1 - \frac{6}{m(m+1)}
\]

then the continuum Liouville theory prediction for the critical exponent \( \gamma_{\text{str}} \) is

\[
\gamma_{\text{str}} = \frac{1}{12} \left( D - 1 - \sqrt{(D-1)(D-25)} \right) = -\frac{1}{m}
\]

The constant \( \gamma_{\text{str}} \) is universal and it represents the nature of the geometry. If we think of the gravity-coupled conformal field theory as a string theory where the conformal matter fields are identified as the string embedding functions and the 2-dimensional space-time is thought of as the string world-sheet [122], then the central charge (1.37) represents the (fractal) dimension of the embedding space. The case \( m = 2 \) corresponds to \( D = 0 \), i.e. \( \gamma_{\text{str}} = -\frac{1}{2} \) for pure gravity. The case \( m = 3 \) corresponds to \( D = \frac{1}{2} \), i.e. \( \gamma_{\text{str}} = -\frac{1}{3} \) for a half-boson or fermion, or equivalently for the conformal field theory of the 2-dimensional critical Ising model coupled to gravity (for which the continuum limit is also associated with the magnetization transition of the ordinary Ising model). As (1.38) ceases to make sense for \( D > 1 \), there is a “conformal barrier” in the model at \( D = 1 \) (which corresponds to the conformal field theory of a single free boson).

These features of string theory are all reproduced by the Hermitian one-matrix model with polynomial potentials which therefore serve as tools for extracting non-perturbative information about string theories, i.e. the one-matrix models can be explicitly solved non-perturbatively at each order of the \( \frac{1}{N} \)-expansion. For instance, for \( K = 3 \) in (1.4), we can consider the “double-scaling limit” [22, 53, 65, 89] where we take the limits \( N \to \infty \) and \( \bar{g} \to \bar{g}_c \) in a correlated fashion so that the renormalized string coupling \( N(\bar{g} - \bar{g}_c)^{2-\gamma_{\text{str}}}/2 \) remains finite. Notice that the area dependence in (1.36) is realized in terms of the coupling \( \bar{g} \) in the model partition function (c.f. (1.10)), except that in the latter case the overall exponent is down by 1 power (c.f. (1.34)) because of the Laplace transformation from the area dependence. The limit \( N = \infty \) by itself corresponds to planar diagrams or genus 0 (the spherical approximation), while the higher-genus terms \( (h \geq 1, \chi(h) \leq 0) \) in the \( \frac{1}{N} \)-expansion are suppressed by \( \frac{1}{N^h} \) for \( N \to \infty \) but are enhanced as \( \bar{g} \to \bar{g}_c \). The double scaling limit of Hermitian 1-matrix models therefore results in a coherent contribution from all genus surfaces and allows the construction of the genus expansion of 2-dimensional quantum gravity [22, 53, 65, 38]. In terms of the renormalized cosmological constant

\[
x \equiv \Lambda_R = (1 - \bar{g}/\bar{g}_c) N^{4/5}
\]
it can be shown that the leading singular part, as $\bar{g} \to \bar{g}_e$, of the specific heat (or string susceptibility)

$$\chi(x) \equiv -\frac{\partial^2 \log Z}{\partial x^2}$$

obeys the transcendental Painlevé I equation [22, 53, 65, 38]

$$x = \chi^2(x) - \frac{1}{3} \chi''(x)$$

which is also known as the string equation [19, 67].

The characteristic property of this second order differential equation is that its only removable singularities in the complex plane are double poles which have residue 2 and correspond to double zeroes of the free energy $\log Z$. Its solutions are determined by 2 boundary conditions. The solutions which have an asymptotic expansion for $x$ large (the topological expansion) that begins with the leading spherical result $\chi(x) \sim \sqrt{x}$ are determined as

$$\chi(x) = \sqrt{x} \left( 1 - \sum_{k=1}^{\infty} \chi_k \ x^{-\gamma_k/2} \right)$$

where $\chi_k > 0$ for large-$k$ grow asymptotically as $(2k)!$. The solution for $\chi(x)$ (or the topological expansion of the free energy) is therefore not Borel summable and thus does not define a unique function [38]. Thus the Hermitian matrix models reproduce the well-known fact that quantum gravity in 2-dimensions is ill-defined as a statistical theory of random surfaces because the topological genus expansion is not Borel summable (and therefore has terrible convergence properties). This is reflected directly in the Hermitian matrix integrals (1.1) in which the divergence of the large-$N$ expansion is a consequence of the fact that the integration over Hermitian matrices diverges in the region of interest. The coefficients of the genus expansion (1.42) can be explicitly determined by the Bessis method of orthogonal polynomials for the eigenvalue model (1.14) [20] or alternatively by examining the $1/N$-expansions of the loop equations [14]. In this case, the string exponent $\gamma_{\text{str}} = -\frac{1}{2}$ appears for the discretization potentials (1.4), while the values $\gamma_{\text{str}} = -\frac{1}{m}$ are associated with more complicated polynomial potentials for which similar Painlevé expansions can be constructed [38].

### 1.1.2 Multi-matrix Models

Although the simplest Hermitian one-matrix models above describe pure 2-dimensional quantum gravity, the simplest $\mathbb{Z}_2$-symmetric multi-matrix models describe gravity interacting with matter in a $D \leq 1$ dimensional embedding space [23, 24, 29, 32, 50, 38, 62, 66]. The typical multi-matrix models are defined by the partition functions

$$Z_n = \int \prod_{\ell=1}^{n} d\phi_\ell \ e^{-N \ tr \ S[\phi]}$$

where $\phi_\ell$ are $N \times N$ Hermitian matrices and the action is

$$S[\phi] = \sum_{\ell=1}^{n} V_\ell(\phi_\ell) - \sum_{\ell=1}^{n-1} \phi_\ell \phi_{\ell+1}$$

11
The generalization of the one-matrix eigenvalue distribution follows from the Itzykson-Zuber formula [71]

\[ I[\phi_m, \phi_1] \equiv \int [dU] \ e^{\lambda_m U \phi_1 U^\dagger} = \frac{\det_{i,j} \left[ e^{\lambda_i^{(m)} \lambda_j^{(1)}} \right]}{\Delta(\lambda^{(m)}) \Delta(\lambda^{(1)})} \]  

(1.45)

where \( \lambda_i^{(\ell)} \) are the eigenvalues of the Hermitian matrix \( \phi_\ell \). This formula enables one to write (1.43) as the multi-eigenvalue model

\[ Z_n = \int_{-\infty}^{+\infty} \prod_{\ell=1}^{n} \prod_{i=1}^{N} d\lambda_i^{(\ell)} \ \Delta(\lambda^{(1)}) \Delta(\lambda^{(n)}) e^{-N \sum_{j=1}^{N} S[\lambda_j]} \]  

(1.46)

The diagrammatic expansion of these multi-matrix models generates a sum over discretized surfaces, where the different matrices \( \phi_\ell \) represent \( n \) different matter states that can exist at the vertices. The cross-terms between the \( \phi_\ell \)'s in (1.44) links the graphs generated by each \( \phi_\ell \) together. The quantity (1.43) thereby admits an interpretation as the partition function of 2-dimensional gravity coupled to matter, or of string theory in a \( D \leq 1 \) dimensional embedding space. In particular, by taking \( n \to \infty \) (a matrix chain) one can represent a \( D = 1 \) model (i.e., a single free boson) coupled to gravity. When \( V_\ell = V_{n+1-\ell} \), the matrix problem has a \( \mathbb{Z}_2 \)-symmetry corresponding to the mapping \( \phi_\ell \to \phi_{n+1-\ell} \) of Hermitian matrices, which manifests itself as an Ising-like reflection symmetry in the discretized random surface model. In these cases the statistical theory can be solved for using techniques such as loop equations [1, 59, 104, 128, 95], orthogonal polynomials or canonical commutation relations [32, 38]. The continuum limits of the discretized random surface models in the case of the \( \mathbb{Z}_2 \)-symmetric 2-matrix model (\( n = 2 \) in (1.43)) represent the \((p,q)\)-minimal models of conformal field theory [32, 38, 61, 112], with the relatively prime integers \( p \) and \( q \) associated with the scaling behaviours generated by the matrices \( \phi_1 \) and \( \phi_2 \). The central charge of these models is

\[ D = 1 - \frac{6(p - q)^2}{pq} \]  

(1.47)

and the associated string susceptibility exponent is

\[ \gamma_{\text{str}} = -\frac{2|p - q|}{p + q - |p - q|} \]  

(1.48)

The unitary discrete series which is generated by the Hermitian one-matrix models is then recovered for \( (p,q) = (m+1,m) \), and the \( \mathbb{Z}_2 \) symmetry of the two-matrix model corresponds to the well-known \( p-q \) duality of conformal field theory. The large-\( N \) phase transitions in these cases can also be studied using saddle point methods analogous to those in Hermitian one-matrix models [38].

1.1.3 The Kazakov-Migdal Model

Unitary matrix models play a role in 2-dimensional QCD [21, 68], mean-field computations in lattice gauge theory [54] and various other approaches to higher-dimensional continuum gauge theories [113]. Recently, a unitary matrix model for induced QCD has been proposed
by Kazakov and Migdal [75]. Their model is the bosonic lattice field theory

\[ Z_{KM} = \int \prod_{x \in \mathcal{L}^D} d\phi(x) \prod_{\langle x, y \rangle \in \mathcal{L}^D} [dU(x, y)] \times \exp \left\{ -N \mathrm{tr} \left( \sum_{x \in \mathcal{L}^D} V(\phi(x)) - \sum_{\langle x, y \rangle \in \mathcal{L}^D} \phi(x)U(x, y)\phi(y)U^\dagger(x, y) \right) \right\} \]  

(1.49)

where \( \mathcal{L}^D \) is a \( D \)-dimensional oriented hypercubic lattice, \( \phi(x) \) is a scalar field which, for each site \( x \in \mathcal{L}^D \), is an \( N \times N \) Hermitian matrix and \( U(x, y) \) is a gauge field which, for each link \( \langle x, y \rangle \in \mathcal{L}^D \) connecting nearest neighbour sites \( x \) and \( y \), is an \( N \times N \) unitary matrix. This model is invariant under the gauge transformation

\[ \phi(x) \rightarrow \Xi(x)\phi(x)\Xi^\dagger(x) \quad , \quad U(x, y) \rightarrow \Xi(x)U(x, y)\Xi^\dagger(y) \]  

(1.50)

where \( \Xi(x) \) is an arbitrary \( U(N) \)-valued function of \( x \in \mathcal{L}^D \). The second term in the action in (1.49) is the usual gauge invariant kinetic term for a scalar field in the adjoint representation of the colour gauge group [54, 133]. The absence of the usual Wilson kinetic term for the gauge field makes this model exactly solvable in the large-\( N \) limit. The model (1.49) is the natural \( D > 1 \) dimensional extension of the Hermitian one-matrix model (1.1). It reduces to the standard \( D \leq 1 \) matrix chains discussed above if the lattice \( \mathcal{L}^D \) is just a 1-dimensional sequence of points in which case the gauge field can be absorbed by a unitary transformation (1.50) of \( \phi(x) \).

For the Gaussian potential \( V(\phi) = m^2\phi^2/2 \), it is straightforward to calculate the Gaussian integrals over the Hermitian matrix fields to rewrite (1.49) as the lattice gauge theory

\[ Z_{KM} = \int \prod_{\langle x, y \rangle \in \mathcal{L}^D} [dU(x, y)] e^{-S_{\text{ind}}[U(x, y)]} \]  

(1.51)

where the induced action is given by the large mass expansion [75]

\[ S_{\text{ind}}[U] = -\frac{1}{2} \sum_{\Gamma \in \mathcal{L}^D} \left| \mathrm{tr} U(\Gamma) \right|^2 \frac{1}{l(\Gamma)m^2|\Gamma|} \]  

(1.52)

The sum in (1.52) is over all closed loops \( \Gamma \) with \( l(\Gamma) \) links and \( U(\Gamma) \) denotes the path-ordered product of the gauge fields along \( \Gamma \) with counterclockwise orientation. Alternatively, we can treat (1.49) using standard matrix model techniques. As usual, because of the gauge invariance (1.50) the integral (1.49) depends only on the eigenvalues \( \phi_i(x) \) of the Hermitian matrices \( \phi(x) \). Using the Itzykson-Zuber formula (1.45) it can be written as the eigenvalue model

\[ Z_{KM} = \int_{-\infty}^{+\infty} \prod_{x \in \mathcal{L}^D} \prod_{i=1}^{N} d\phi_i(x) \prod_{x \in \mathcal{L}^D} \Delta^2(\phi(x)) \exp \left( -N \sum_{x \in \mathcal{L}^D} \sum_{i=1}^{N} V(\phi_i(x)) \right) \times \prod_{\langle x, y \rangle \in \mathcal{L}^D} \det \left[ e^{N\phi_i(x)\phi_j(y)} \Delta(\phi(x))\Delta(\phi(y)) \right] \]  

(1.53)
When \( N \) is large the integral (1.53) is dominated by the saddle-point of the effective eigenvalue action. The saddle-point equation is

\[
\frac{2D}{N} \frac{\partial}{\partial \phi_i} \log I[\phi, \chi] \bigg|_{\chi=\phi} = V'(\phi_i) - \frac{1}{N} \sum_{j < i} \frac{1}{\phi_i - \phi_j}, \quad i = 1, \ldots, N
\]  

(1.54)

for each site \( x \in \mathcal{L}^D \). The solutions \( \Phi_i(x) \) to (1.54) are called the master fields of the theory and the value of the integral (1.53) when \( N \to \infty \) is equal to the integrand evaluated when \( \phi_i(x) \) is set equal to the master field \( \Phi_i(x) \). The mean-field approximation to the partition function consists of assuming that the master field \( \Phi_i(x) \) is frozen at some constant value \( \Phi_0 \) at each site \( x \) of the lattice \( \mathcal{L}^D \).

The saddle-point equation (1.54) can be written in terms of the one-link pair correlator for the gauge fields,

\[
\frac{1}{N} C_{ij} = \frac{\int [dU] |U_{ij}|^2 e^{N \text{ tr} \phi(x) U(x) U^\dagger(y)}}{\int [dU] e^{N \text{ tr} \phi(x) U(x) U^\dagger(y)}}
\]  

(1.55)

as

\[
\frac{1}{N} \sum_{k=1}^{N} C_{ik} \phi_k = \frac{1}{2D} \left( V'(\phi_i) - \frac{1}{N} \sum_{j < i} \frac{1}{\phi_i - \phi_j} \right)
\]  

(1.56)

In the large-\( N \) limit, as usual we replace the index of the eigenvalues of \( \phi \) with a continuous label and introduce the eigenvalue density \( \rho(\lambda) \) for the master field \( \lambda = \Phi \). Then the saddle-point equation (1.56) is

\[
\int d\alpha \rho(\alpha) C(\lambda, \alpha) \alpha = \frac{1}{2D} \left( V'(\lambda) - 2\int d\alpha \frac{\rho(\alpha)}{\lambda - \alpha} \right)
\]  

(1.57)

The singular non-linear integral equation (1.57) is rather complicated [111] and its explicit solution even for the simplest Gaussian potential \( V(\phi) = m^2 \phi^2 / 2 \) is non-trivial [64]. In this latter case, the quantities appearing in (1.57) can be solved for explicitly to yield the usual semi-circle eigenvalue distribution [47]

\[
F(\lambda) \equiv \int d\alpha \rho(\alpha) C(\lambda, \alpha) \alpha = \Pi \lambda = \frac{2\lambda}{\mu + \sqrt{\mu^2 + 4}}
\]  

(1.58)

\[
C(\alpha, \beta) = \frac{\Pi^{-1}}{\alpha^2 - (\Pi + \Pi^{-1}) \alpha \beta + \beta^2 + \mu}, \quad \rho(\alpha) = \frac{1}{\pi} \sqrt{\mu - \frac{\mu^2 \alpha^2}{4}}
\]

(1.59)

where

\[
m^2 = D \sqrt{\mu^2 + 4} - (D - 1) \mu, \quad \mu = \Pi^{-1} - \Pi
\]

An alternative way to solve for the eigenvalue distribution, which is equivalent to this large-\( N \) Riemann-Hilbert method, is again by the method of loop equations, which in this case provides as well the full set of one-link correlators of the gauge and scalar fields [47, 93] (see [95] for a review).

It was originally hoped that the Kazakov-Migdal model would be a description of QCD in the limit where the number \( N \) of colours is large [75]. It was thought that the model would have a second order order phase transition and that the critical behaviour should be represented by continuum QCD [75, 84, 108], the only known non-trivial 4-dimensional quantum field theory with non-abelian gauge symmetry. It was shown, however, that even in
the simplest Gaussian model, for which the phase transition was thought to occur at $m^2 = 2D$, there is no such critical behaviour [64, 79]. This problem, combined with other problems such as the “hidden” local $\mathbb{Z}_N$-symmetry which forces the Wilson loops to vanish [85], has led to the consensus that the Kazakov-Migdal model does not induce QCD. Nevertheless, the model (1.49) is interesting in its own right both as a higher-dimensional gauge theory which is exactly solvable at large-$N$ and as an interesting example of a matrix model in dimensions greater than 1 for which a solution in the large-$N$ limit may be attainable. In the former interpretation the model can be studied using the standard techniques of lattice gauge theory (see [54] for a review), while in the latter case one can employ the basic methods of matrix models discussed above.

Furthermore, it has been recently suggested that, as a matrix model, the partition function (1.49) may serve as some sort of statistical random surface model for strings in dimension $D > 1$ [85, 96, 97]. For instance, some remarkable self-consistent scaling solutions with non-trivial scaling indices have been found for a quartic potential $V$ in (1.49) at large-$N$ and for any $D$ [108, 111]. However, it is not clear what physical system these scaling solutions are associated with. The relation of the Kazakov-Migdal model to discretized random surfaces and strings has also been recently suggested in its equivalence at large-$N$ with the gauged Potts matrix model [96]. The latter model has a natural connection with triangulated random surface theories. The mechanism which governs the discretized random surface approach to string theory in dimensions $D > 1$, where the susceptibility formula (1.38) breaks down, has been a subject of much discussion over the years. The random surface models for $D \geq 1$ are perfectly well-defined and there is no obvious pathology at $D = 1$. It has been suggested [31] that the conformal barrier is due to some change in the geometry at $D = 1$. There is some evidence that this change is a transition to a tree-like or branched polymer phase (rather than a stringy phase) [5, 6, 9, 10, 16]. The polymer trees are thought of as connecting 2-dimensional baby universes together [97] so that the string constant is modified to [10, 56]

$$\gamma = \frac{\gamma_{\text{str}}}{\gamma_{\text{str}} - 1}$$

(1.60)

with $\gamma_{\text{str}} = -1/m < 0$ the critical exponent of each 2-dimensional surface in the $D$-dimensional embedding space.

In this picture, the case $\gamma_{\text{str}} = -1$, when there is no critical behaviour at all on the 2-dimensional surfaces, leads to an exponent $\gamma = 1/2$, which is the typical mean field value for pure branched polymers. The exponents $\gamma_{\text{str}} < 0$ are generically associated with a 2-dimensional, or stringy, random surface summation, while $\gamma > 0$ signifies a fragmentation of the surface or a “crumpling” of the strings [7, 8, 15]. The multicritical string exponents $\gamma_{\text{str}} = -1/m$ lead to $\gamma = 1/(m + 1)$ due to the polymerization. This differs from the mean field value $\gamma = 1/2$ due to the effects of the 2-dimensional gravitational dressing, just like the critical indices of the Ising model on a random surface differ from the usual mean field exponents of the Ising model on a fixed regular lattice. The polymer sum is the 1-dimensional reduction of the dynamically triangulated random surface model (1.9),

$$Z_{\text{poly}} = \sum_{h=0}^{\infty} \sum_{\mathcal{P}_h} \kappa^{2h-2} e^{-L_L(\mathcal{P}_h)} \int \prod_{i \in \mathcal{P}_h} d^D X_i \ e^{-\frac{1}{2} \sum_{\langle i,j \rangle \in \mathcal{P}_h} [X_i - X_j]^2}$$

(1.61)

where the sum is over all polymers $\mathcal{P}_h$ with $h$ loops, $L(\mathcal{P}_h)$ is the intrinsic length of the polymer chains, the constant $\kappa$ is related to the fugacity of the branching chains, and $X_i$ are matter degrees of freedom placed at the vertices $i$ of the polymer graphs. These models can
be described by vector field theories \([40, 41, 115, 116, 138]\), where the reduction from matrix
to vector degrees of freedom is equivalent to the reduction above from random surfaces to
randomly branched chains. The Kazakov-Migdal model yields an explicit realization of the
crumpled surfaces above \([96, 97]\), and the above picture of the branched polymer phase is expected
for any matrix model describing discretized random surfaces in target space dimensions
\(D > 1\).

### 1.2 Adjoint Fermion Matrix Models

Although the scalar matrix models discussed above reproduce some nice features of non-critical string theory and 2-dimensional quantum gravity, and they provide non-perturbative approaches to these theories which serve as useful tools for string theory (beyond the realm of the usual perturbative approaches), they have several undesirable characteristics. For instance, as mentioned above, there is the problem that the observables of the Hermitian matrix models are not well-defined because the integrations over Hermitian matrices diverge in the region of interest for a description of discretized quantum gravity (reflecting the divergence of the genus sum in the random surface model). Furthermore, in Hermitian matrix models the stringy phase does not exist in dimension \(D > 1\) because of the conformal barrier, and even their \(D > 1\) generalizations have instability problems associated with the unbounded scalar actions \([95]\). It is therefore desirable to look for alternative matrix models which describe more general types of string theories and which have well-defined convergent observables. In particular, it would be interesting to find some alternative candidate to the Kazakov-Migdal model which has the same solvability features and might really induce QCD in the continuum limit.

One such possibility has been recently introduced by Makeenko and Zarembo \([101]\) who
considered a class of matrix models where the degrees of freedom are matrices whose elements
are anticommuting Grassmann numbers. The simplest example is the fermionic one-matrix model \([12, 101, 105, 126]\) which is defined by the partition function

\[
Z_1 = \int d\psi \ d\bar{\psi} \ e^{N^2 \text{tr } V(\psi)}
\]

where \(V\) is some potential and \(\psi\) and \(\bar{\psi}\) are independent complex Grassmann-valued \(N \times N\) matrices, i.e. matrices with anti-commuting nilpotent elements,

\[
\psi_{ij} \psi_{kl} = -\psi_{kl} \psi_{ij} \quad \psi_{ij} \bar{\psi}_{kl} = -\bar{\psi}_{kl} \psi_{ij} \quad \bar{\psi}_{ij} \bar{\psi}_{kl} = -\bar{\psi}_{kl} \bar{\psi}_{ij}
\]

\[
(\psi_{ij})^2 = (\bar{\psi}_{ij})^2 = 0
\]

The rules for left differentiation are given by the anticommutators

\[
\left\{ \frac{\partial}{\partial \psi_{ij}}, \psi_{kl} \right\} = \left\{ \frac{\partial}{\partial \bar{\psi}_{ij}}, \bar{\psi}_{kl} \right\} = \delta_{ik} \delta_{jl} \quad \left\{ \frac{\partial}{\partial \psi_{ij}}, \bar{\psi}_{kl} \right\} = \left\{ \frac{\partial}{\partial \bar{\psi}_{ij}}, \psi_{kl} \right\} = 0
\]

and the usual rules for complex conjugation of these Grassmann numbers are

\[
\bar{\psi}_{ij}^* = -\psi_{ji} \quad \psi_{ij}^* = -\bar{\psi}_{ji} \quad (\psi_{ij} \psi_{kl})^* = \psi_{kl}^* \bar{\psi}_{ij}^*
\]
The integration measure in (1.62) (the Haar measure on the Grassmann algebra \text{Grass}(N)),

\[ d\psi 
\psi \equiv \prod_{i,j} d\psi_{ij} d\tilde{\psi}_{ij} \]  

is defined using the usual Berezin rules for integrating Grassmann variables,

\[ \int d\psi_{ij} \psi_{ij} = 1 \quad , \quad \int d\psi_{ij} 1 = 0 \]  

which is equivalent to left differentiation. We normalize all traces here and in the following as

\[ \text{tr} \ A \equiv \frac{1}{N} \sum_{i=1}^{N} A_{ii} \]  

Matrix models of this kind were originally motivated by the studies of induced gauge theories using adjoint matter where the Yang-Mills interactions of gluons are induced by loops with heavy adjoint scalar fields [75, 95] as in the Kazakov-Migdal model or other kinds of matter such as heavy adjoint or fundamental representation fermions [80, 95, 101, 109, 110, 126]. In these latter types of adjoint theories one can think of the quark fields of the model as being represented by the fermion matrices in the adjoint representation of the gauge group \text{U}(N). This differs from previous approaches to lattice gauge theories which used fermion fields that transform in the fundamental representation of the gauge group [18, 70]. In particular, since there is no asymptotic freedom in the gauge coupling constant in the adjoint fermion model (if the number of fermion species is large enough), the kinetic term for the gauge field is not essential in this case (just like in quantum electrodynamics) [101].

To see the connection between the fermionic matrix model (1.62) and a statistical theory of discretized random surfaces as is described by the Hermitian matrix models, consider the simple Grassmann matrix field theory defined by

\[ Z_{1}^{(S)} = \int d\psi \ d\tilde{\psi} \ e^{N^2 \text{tr} \left( \tilde{\psi} \psi + g(\tilde{\psi} \psi)^{K} / K \right)} \]  

The perturbative expansion of (1.69) is the series

\[ Z_{1}^{(S)} = \sum_{k \geq 0} \frac{(N^2 g)^k}{K^k k!} \int d\psi \ d\tilde{\psi} \left[ \text{tr} \left( \tilde{\psi} \psi \right)^K \right]^k e^{N^2 \text{tr} \tilde{\psi} \psi} \]  

The perturbative solution of (1.69) therefore requires the evaluation of the normalized fermionic Gaussian moments

\[ \left\langle \left[ \text{tr} \left( \tilde{\psi} \psi \right)^K \right]^k \right\rangle \equiv \frac{\int d\psi \ d\tilde{\psi} \left[ \text{tr} \left( \tilde{\psi} \psi \right)^K \right]^k e^{N^2 \text{tr} \tilde{\psi} \psi}}{\int d\psi \ d\tilde{\psi} \ e^{N^2 \text{tr} \tilde{\psi} \psi}} \]  

These moments can be obtained from the generating functional

\[ Z_{1}(\eta, \bar{\eta}) = \frac{\int d\psi \ d\tilde{\psi} \ e^{N \text{tr} \left( \tilde{\psi} \psi + g(\tilde{\eta} \psi + \psi \eta) \right)}}{\int d\psi \ d\tilde{\psi} \ e^{N^2 \text{tr} \tilde{\psi} \psi}} , \]  

where \( \eta \) and \( \bar{\eta} \) are also independent \( N \times N \) Grassmann-valued matrices, through the identity

\[ \left\langle \tilde{\psi}_{i_{1}j_{1}} \psi_{k_{1}l_{1}} \cdots \tilde{\psi}_{i_{n}j_{n}} \psi_{k_{n}l_{n}} \right\rangle = \left. \frac{\partial}{\partial \eta_{j_{1}i_{1}}} \cdots \frac{\partial}{\partial \eta_{j_{n}i_{n}}} \frac{\partial}{\partial \tilde{\eta}_{l_{1}k_{1}}} \cdots \frac{\partial}{\partial \tilde{\eta}_{l_{n}k_{n}}} \right|_{\eta=\bar{\eta}=0} Z_{1}(\eta, \bar{\eta}) \]  

\[ = \left. \frac{\partial}{\partial \eta_{j_{1}i_{1}}} \cdots \frac{\partial}{\partial \eta_{j_{n}i_{n}}} \frac{\partial}{\partial \tilde{\eta}_{l_{1}k_{1}}} \cdots \frac{\partial}{\partial \tilde{\eta}_{l_{n}k_{n}}} e^{-\text{tr} \eta \bar{\eta}}} \right|_{\eta=\bar{\eta}=0} \]  

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In particular, the fermion propagator is

\[ \langle \bar{\psi}_{ij} \psi_{kl} \rangle = \frac{1}{N} \delta_{il} \delta_{kj} \]  

(1.74)

and it coincides with the scalar propagator (1.5). Thus the fermionic propagator has the same diagrammatic representation as in the Hermitian case. The difference in the two models lies in the vertices that each produces. The fat graphs in the fermionic case have 2K-point vertices. However, a 2K-point vertex is topologically equivalent to the contraction of two (K + 1)-point vertices along an external leg. Thus the dual lattice of the fat graph discretization of surfaces in the fermionic one-matrix model (1.69) with degree K polynomial potential \( V(\bar{\psi}\psi) = \bar{\psi}\psi + g(\bar{\psi}\psi)^K/K \) is the same as the triangulations of 2-surfaces produced in the Hermitian one-matrix model (1.1) with a degree \( K + 1 \) polynomial potential \( V(\phi) = \phi^2/2 + g\phi^{K+1}/(K + 1) \). This correspondence shows that the combinatorics of the surface discretizations in the fermionic case are modified because of a doubling of degrees of freedom at each vertex.

However, an important difference in the statistical theory generated by (1.69) now occurs. The Wick contraction rules given by (1.73) yield a product of 2 delta-functions (as in (1.74)) in each pairing \( \bar{\psi}\psi \), one for each contraction of the inner and outer indices of \( \bar{\psi} \) with \( \psi \). A contraction of the form \( \psi\bar{\psi} \) gives the same result but with the opposite sign. This leads to the well-known Feynman rule from perturbative quantum field theory that Feynman diagrams for fields which obey Fermi statistics have an extra factor of \((-1)^L\) compared to the bosonic case, where \( L \) is the number of closed fermion loops. Since for the fat graphs of the fermionic matrix model the number \( L \) is the area or number of vertices of the triangulated random surface \([38, 129]\), this indicates that the genus expansion of the free energy in the fermionic case may be an alternating series.

### 1.2.1 Penner Matrix Models

In the case of fermionic matrix models, since it is not possible to diagonalize matrices which have anticommuting elements using a unitary transformation, they are not natural eigenvalue models. It has been argued though that many of the analytical tools which are used to analyze Hermitian matrix models, such as the concept of “eigenvalue distribution” and a “master field”, are also useful in the fermionic case \([101]\). In fact, Makeenko and Zarembo showed that the adjoint fermion matrix model (1.62) has many of the features of the more familiar Hermitian one-matrix model and that its loop equations are identical to those for the Hermitian one-matrix model with generalized Penner potential \([12, 101]\)

\[ Z_P = \int d\phi \; e^{-\frac{N^2}{2} \text{tr} \left( V(\phi) - 2 \log \phi \right)} \]  

(1.75)

When \( N \) is infinite, the models (1.62) and (1.75) therefore have the same solution. However, beyond the leading order in the large-\( N \) expansion, the loop equations for the 2 models should be solved with different boundary conditions and the solution is different in the 2 cases.

The formal equivalence between the models defined by (1.62) and (1.75) at large-\( N \) can be seen by inserting the matrix-valued delta function

\[ 1 = \int d\phi \; \delta(\phi - \bar{\psi}\psi) \]  

(1.76)
where $\phi$ is a Hermitian matrix, to write (1.62) as

$$Z_1 = \int d\psi \, d\bar{\psi} \int d\phi \, e^{N^2 \, \tr V(\phi) \delta(\phi - \bar{\psi}\psi)}$$

and using the identity

$$\int \frac{d\lambda}{(2\pi)^N} \int d\psi \, d\bar{\psi} \, e^{i \tr \lambda(\phi - \bar{\psi}\psi)} = \int \frac{d\lambda}{(2\pi)^N} \det[-i(I \otimes \lambda)] e^{i \tr \lambda \phi}$$

where $I$ is the $N \times N$ identity matrix. In the infinite-$N$ limit, the integral over the Hermitian matrices $\lambda$ is evaluated on a saddle point. In that case, when $\phi$ has no zero eigenvalues, the coordinate transform $\lambda = \lambda^2 \phi^{-1}$ can be used to show

$$\int \frac{d\lambda}{(2\pi)^N} \det N(-i\lambda) e^{i \tr \lambda \phi} \propto \det^{-2N} \phi$$

so that

$$Z_1 \sim \int d\phi \, e^{N^2 \, \tr (V(\phi) - 2 \log})$$

However, the integral over Hermitian matrices in (1.75) is ill-defined for finite $N$ because of the logarithmic divergence at $\phi = 0$. This representation of the fermionic matrix model partition function in terms of a Hermitian model with effective action involving a hard-core logarithmic interaction can be thought of as the analog of the effective eigenvalue representation involving the Vandermonde determinant in Hermitian matrix models.

The Penner matrix models (1.75) were originally used to calculate the virtual Euler characteristics of the moduli spaces of compact Riemann surfaces [119]. This latter quantity is defined as follows [46]. The moduli space $\mathcal{M}_{h,n}$ of a Riemann surface $\Sigma^h_n$ of genus $h$ with $n$ distinguished punctures can be discretized by a simplicial decomposition. The simplices can be represented by dual fat graphs $\mathcal{F}$ such that the dimension $\dim \mathcal{F}$ of each simplex is the number of lines in the graph minus the number $n$ of punctures in the Riemann surface. Then the virtual Euler characteristic of $\mathcal{M}_{h,n}$ is

$$\chi_v(\mathcal{M}_{h,n}) = \sum_{\mathcal{F} \in \mathcal{M}_{h,n}} \frac{(-1)^{\dim \mathcal{F}}}{|G(\mathcal{F})|}$$

where $|G(\mathcal{F})|$ is the order of a stabilizer of the subgroup of the mapping class group of $\Sigma^h_n$ which fixes the topological class of the fat graph $\mathcal{F}$ (this is precisely the order of the symmetry group of the fat graph itself).

A Hermitian matrix model which reproduces the topological invariant (1.81) should therefore have very special features. Since arbitrary valence vertices appear in the simplicial discretization of $\mathcal{M}_{h,n}$, the matrix model should include arbitrary powers of interaction terms $\tr \phi^m$, $m \geq 3$. The weighting factor for the order of the symmetry group in (1.81) appears in the perturbative expansion of the free energy if we include a coupling factor of $1/m$ in front of the $m$-th interaction term, as is standard in perturbative quantum field theory. Finally, to produce the correct sign in (1.81) for each fat graph $\mathcal{F}$ the potential must associate a factor of $-1$ for each vertex [119, 46], which we recall was precisely the case for the fermionic diagrammatics. Thus the matrix model potential which reproduces these Feynman rules and whose free energy perturbative expansion therefore coincides with (1.81) is

$$V_P(\phi) = -\sum_{m=2}^{\infty} \frac{\phi^m}{m} = \phi + \log(1 - \phi)$$

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It has been shown that the Hermitian matrix model with Penner potential (1.82) has a continuum limit which is described by a logarithmic susceptibility behaviour at criticality [28, 46]. Such a critical behaviour is associated with a string constant $\gamma_{\text{str}} = 0$, or equivalently central charge $D = 1$, i.e. a free boson conformal field theory [61]. Thus the Penner matrix models are also deeply connected to $D = 1$ string theory. More precisely, they describe matter compactified on a self-dual radius circle interacting in $D = 1$ dimension with 2-dimensional quantum gravity [28, 46, 130]. By adding more complicated polynomial interactions as in (1.75), it is possible to localize multi-critical points with $\gamma_{\text{str}} = -1/m$ which are associated with the $D = 1$ string compactified to the critical radius of the Kosterlitz-Thouless phase transition [12, 28, 130, 96]. These models therefore capture various intriguing features of 2-dimensional quantum gravity and certain topological aspects of moduli space [135]. One important technical feature of Penner matrix models is that in the derivation of their loop equations, the field transformation (1.30) must be slightly modified because it produces a singular pole term $\langle \text{tr} (1/\phi) \rangle$ from the variation of $\log \phi$ in the action. This term is easily removed by the considering instead the field redefinition [12]

$$\phi \rightarrow \phi + \varepsilon \sum_{k=1}^{\infty} \frac{\phi^k}{z^{k+1}} = \phi \left(1 + \frac{\varepsilon}{z(\bar{z} - \phi)} \right)$$

However, the resulting loop equations lead to an algebraic equation of order $2K$ when the potential in (1.75) is a polynomial of degree $K$.

The connection between the Penner models and matrix models involving fermionic degrees of freedom were originally suggested in the work of Gilbert and Perry [60] (see also [2]) who considered a $D = 0$ random matrix model with $N = 2$ supersymmetry using a manifestly Hermitian supermatrix formalism. They showed that the integration of the fermionic matrix degrees of freedom out of the matrix integral reduces the model exactly to the Penner matrix model, and that in the double-scaling limit the solutions of the string equation in the case of the cubic superpotential exhibit critical behaviour that lies in the same universality class as the Penner model. These ‘mixed’ types of fermionic matrix models are therefore deeply related to $D = 1$ bosonic string theory, but differ from the bosonic matrix models in the type of universal behaviour that they exhibit (e.g. there are 2 distinct universality classes displayed) [60].

It has been conjectured by Makeenko and Zarembo [101] and Ambjørn, Kristjansen and Makeenko [12] that the fermionic one-matrix model also corresponds to a similar sort of statistical theory of triangulated random surfaces, except that now the genus expansion has alternating signs [12]. The resulting convergence of the sum over genera is reflected in the feature of the fermionic matrix model that its partition function and observables are all well-defined since the integrals over Grassmann variables always converge, in contrast to Hermitian matrix models [38, 74]. In fact, one can formally perform the integral (1.62) at finite-$N$ by inserting the matrix-valued delta function (1.76) and using (1.78) as

$$\int \frac{d\lambda}{(2\pi)^4} \int d\psi \; d\bar{\psi} \; e^{\lambda (\phi - \bar{\psi} \bar{\psi})} = \text{det}^N \left( - \frac{\partial}{\partial \phi} \right) \delta(\phi)$$  \hspace{1cm} (1.84)$$

to obtain

$$Z_1 = \text{det}^N \left( \frac{\partial}{\partial \phi} \right) e^{N^2 \text{tr} V(\phi)} \bigg|_{\phi=0}$$  \hspace{1cm} (1.85)$$
Similarly, any correlator at finite-$N$ is given by
\[
\langle \prod_j \text{tr} \left( \bar{\psi} \psi \right)^p_j \rangle = \frac{\int d\psi \ d\bar{\psi} \ e^{N^2 \text{tr} V(\bar{\psi} \psi)} \prod_j \text{tr} \left( \bar{\psi} \psi \right)^p_j}{\int d\psi \ d\bar{\psi} \ e^{N^2 \text{tr} V(\bar{\psi} \psi)}}
\]
\[
= \frac{\det^N \left( \partial_{\phi} \right) \left( \prod_j \text{tr} \phi^p_j \right) e^{N^2 \text{tr} V(\phi)}}{\det^N \left( \partial_{\phi} \right) e^{N^2 \text{tr} V(\phi)}} \bigg|_{\phi=0}
\] \hspace{1cm} (1.86)

For a given potential $V(z)$, the expression (1.85) can be expanded to give a finite polynomial in the parameters of the potential which coincides with the usual perturbative expansion (which is a finite sum because of the nilpotency of the Grassmann variables). To generate a full statistical random surface model with arbitrary numbers of vertices, one needs to take the large-$N$ limit. This is in contrast to the bosonic cases of the last Subsection where the large-$N$ limit was required to make the perturbative expansion a well-defined quantity at each order in $1/N$. However, in spite of this good convergence of the partition function (1.62), it has been shown that this model has a non-trivial phase structure in the infinite $N$ limit [12, 101, 105, 126]. In particular, the adjoint fermion one-matrix model possesses the usual multi-critical behaviour [38, 74] with third order phase transition and string susceptibility with critical exponent $\gamma_{\text{str}} = -1/m, m \geq 2$. It may also have a first order phase transition [105]. Although a first order phase transition is unusual for a matrix model, the phase structure of this model resembles the critical behaviour that occurs near the triple point of a liquid-vapour-solid system. There the vapour phase, for example, can be supercooled to the boundary between the liquid and the solid phase where the phase transition must then occur and be of first order. The existence of 2 critical points in the adjoint fermion 1-matrix model is similar to the occurrence of 2 distinct universality classes in the Gilbert-Perry supermatrix model.

### 1.2.2 Complex Matrix Models

From an analytic point of view, the loop equations of the adjoint fermion matrix model are similar in structure to those of the Hermitian one-matrix model with generalized Penner potential. However, form a combinatorical point of view, there is another bosonic matrix field theory which is similar in structure to the fermionic ones above. Consider the *complex* one-matrix model which is defined by the partition function
\[
Z_C = \int \prod_{i,j} d\phi_{ij} \ e^{-N^2 \text{tr} V(\phi^i \phi)}
\] \hspace{1cm} (1.87)
where the integration is over the space of all $N \times N$ complex-valued matrices $\phi$. The complex matrix model is equivalent to the Hermitian matrix model in the double scaling limit [14, 92] and its complete set of correlators can be written down in closed form expressions [11]. It can also be written as an eigenvalue model since the action in (1.87) depends only on the radial part $\phi^i \phi$ of $\phi$ which is a Hermitian matrix. As a statistical theory of random surfaces, it has been argued that the fat-graphs of the complex matrix model (1.87) represent 'checkered' surfaces which are formed from the doubling of vertex degrees of freedom as discussed above for the fermionic matrix model.

Except for the appropriate minus signs and convergence properties, the perturbative expansion of the model (1.87) yields the same generating function for triangulations of random
surfaces as in the fermionic case because the non-zero Wick contractions occur only among the pairings $\phi^\dagger \phi$. This is a consequence of the much larger unitary and charge conjugation invariances of (1.87)),

$$\phi \to V \phi U^\dagger \quad \text{with} \quad \{U, V\} \in U(N) \otimes U(N) \quad ; \quad \phi \to \phi^\dagger \quad (1.88)$$

which restricts the non-zero correlators to those of invariant functions of $\phi^\dagger \phi$. Thus the combinatorical factors associated with the counting of Feynman diagrams in both cases are the same, and in the case of the complex matrix model it is known that, because of the doubling of degrees of freedom in (1.87), the free energy of the Hermitian matrix model with an even polynomial potential is twice the free energy $\log Z_C$ defined with the same potential. Furthermore, any given correlator of the $N \times N$ complex matrix model can be obtained from that of the $2N \times 2N$ Hermitian matrix model with an even potential and dividing by 2 [11, 92].

As we shall see in the following, the loop equations for the adjoint fermion one-matrix model are derived in a manner similar to those of the complex matrix model. In the latter case the loop equations follow from the same shifts (1.83) as for the Penner matrix models. These correspondences lead to many other similarities between fermionic matrix models and these classes of bosonic matrix models.

1.2.3 Fermionic Matrix Models at $D > 0$

Makeenko and Zarembo also introduced a class of fermionic two-matrix models [101]. The equations of motion for these generalizations are high-degree polynomial equations whose solutions are not as immediate as for the case of the one-matrix models [126]. In these models, as well as the one-matrix models, the $\frac{1}{N}$-expansion of the Schwinger-Dyson equations can be related to discrete Virasoro and $W$-constraints which have recently been used to discuss the integrable structure of matrix models [77, 78, 104] and the relations between matrix models and certain topological quantum field theories such as topological gravity in 2-dimensions [43, 42, 102, 135]. These constraints also provide an alternative way of examining the continuum limit relevant to string theory (see [112] and [114] for recent reviews).

Originally, fermionic matrix models were introduced by Khokhlachev and Makeenko for adjoint fermions [80], and by Migdal for fundamental fermions [109], as another proposal of a model which could be considered as inducing Yang-Mills theory in the continuum limit. This model is the most general fermionic matrix model which is the natural $D$-dimensional generalization of (1.62) minimally coupled to a unitary matrix field [80, 95, 101, 126]. It has been shown that there are several advantages of this fermionic matrix model over its Hermitian counterpart which is the Kazakov-Migdal model of induced QCD [75]. It is by now well-known that if the large-$N$ phase transition in the Kazakov-Migdal model occurs before the one separating the local confinement and perturbative Higgs phases, then one gets normal area law confinement in the intermediate region bounded by these 2 phase transitions [95]. Moreover, the solvability features of the Kazakov-Migdal model are masked by the fact that the extra local $U(1)$ gauge symmetry of the theory must be spontaneously broken before the continuum limit is reached [85, 79, 95]. Furthermore, the Gaussian model in this case is unstable due to an unlimited Bose condensation of scalar particles for an action which is unbounded from below (and if a stabilizing self-interaction is added then it leads to the Higgs phenomenon) [79]. In the fermionic case, however, the perturbative expansion for weakly fluctuating gauge fields resembles that of ordinary QCD and a large-$N$ first order
phase transition occurs with decreasing fermion mass which separates the perturbative regime, associated with the restoration of the area law, from the strong coupling phase with unbroken $U(1)$ gauge symmetry and associated local confinement [80]. Moreover, since fermions can never condense, the Gaussian fermionic action has no instability (or Higgs phase) for any value of the fermion mass (but this does not exclude the existence of a composite Higgs phase associated with a fermion chiral condensate $\langle \bar{\psi} \psi \rangle \neq 0$).

These higher-dimensional adjoint fermion matrix models may also provide important information concerning the $D > 1$ phase of string theory. Some recent works have appeared describing polymer models using fermionic degrees of freedom. Fermionic vector models [127] share many of the random geometry interpretations of the usual scalar vector models which describe the statistical mechanics of randomly branching polymers. However, they generate better defined statistical models and moreover they provide interesting toy examples of many of the features of fermionic matrix models. Supersymmetric matrix models [13],[97]–[99] which combine the adjoint fermion and complex matrix models above contain a dimensional reduction due to the supersymmetry and also therefore describe a statistical theory of discrete filamentary surfaces. Thus the use of fermionic degrees of freedom also enables one to construct more exotic types of matrix models and to exploit their convergence features to describe more complicated combinatorical problems.

1.3 Outline

This Review will focus on the above aspects of adjoint fermion matrix models. We shall discuss in detail the technical features involved in solving these models in the large-$N$ limit. In particular, we shall discuss how to explicitly solve for the observables of these models and examine their phase structures. Although not yet completely understood, we shall at various places indicate what these matrix models may have to do with string theories and other continuum field theories such as Yang-Mills theory. Throughout we shall compare these models to the bosonic matrix models, highlighting the essential differences and similarities.

The structure of this Paper is as follows. We begin in Section 2 by considering, for the sake of illustration, a vector version of (1.62) which is the fermionic analog of the scalar $O(N)$ vector models [124, 40, 41, 138] which have recently been studied as ‘test models’ of concepts for the more complicated Hermitian matrix models. There we show how the fermionic nature of the degrees of freedom can be used to explicitly solve for the observables of such random models, and we show how the construction of the topological expansion carries through explicitly in these cases. The nicest feature of these vector models is that everything can be solved for exactly with relative ease, and the critical behaviour of the more complicated matrix models becomes rather transparent in these simplified versions. We also discuss the relevance of these fermionic vector models for random polymers and establish exactly the alternating nature of their genus expansions.

We then discuss the adjoint fermion one-matrix model (1.62) in more detail in Section 3, and in particular we present the full explicit analysis of its critical behaviour in the case of a symmetric potential. We discuss the physical and mathematical bearings the third and possibly first order phase transitions have [105], and we examine the $\frac{1}{N}$-expansion about the third order multi-critical point and present the argument [12] which indicates that the genus expansion may be a convergent, alternating series version of the Painlevé expansion which otherwise coincides with the genus expansion of the Hermitian one-matrix models (this is
exemplified by the foregoing vector model results). We also show that the large-$N$ expansion of the loop equations in this case can be interpreted as Virasoro constraints which allows one to interpret the fermionic one-matrix model as an integrable system [114].

We then consider the fermionic two-matrix model in Section 4 as the natural next higher-dimensional gauge invariant generalization of (1.62). There we derive the complete sets of loop equations for these models, and hence illustrate that even in the simple cases of symmetric potentials the equations of motion are far too complex for any explicit solution [126]. However, the loop equations for these two-matrix models are quite similar to those of the Hermitian two-matrix model [1, 59, 104, 128], so that one expects the same values of $\gamma_{\text{str}}$ to appear and hence a similar string theoretical interpretation of these higher-dimensional fermionic matrix models. We also show how the $\frac{1}{N}$-expansion of the two-matrix model is related to integrable hierarchies [114].

In Sections 5 and 6 we generalize the adjoint fermion two-matrix model to $D$ dimensions as a lattice gauge theory. We compare this class of lattice gauge theories to both the Kazakov-Migdal model and previous attempts at using fermions in the fundamental representation of the gauge group to model the quark fields. We review the present status of the quark confinement problem for these theories, and in particular the progress thus far in solving the loop equations of the models at strong coupling. As of yet, this is still a highly non-trivial and unsolved problem, and in particular it is not known how to extrapolate the strong coupling solutions to solutions of the loop equations in other phases of the model, specifically where the area law behaviour of Wilson loops holds [80, 95, 101, 126]. Here we shall also analyse in detail the critical behaviour of the Itzykson-Zuber integral (1.45) in both the bosonic and fermionic cases. We shall show that even in the bosonic case it may have a non-trivial phase structure, as has been suggested recently by Makarenko in [96] where it was shown that extended Wilson loops in the Kazakov-Migdal model exhibit a continuum limit due to a singular behaviour of the Itzykson-Zuber correlator of the gauge fields.

Finally, in Section 7 we combine the adjoint fermion one-matrix model with the complex bosonic matrix model of Subsection 1.2.2 above. In certain instances, the model has a supersymmetry between the bosonic and fermionic degrees of freedom which can be exploited to solve complicated combinatorial problems that may be relevant for the dimensionally reduced phases of string theory in target space dimensions $D > 1$. We illustrate this on the particular combinatorial problem of evaluating meander numbers (i.e. the number of foldings of a closed polymer chain). We show how to formulate the loop equations in these models in terms of random variables, a technique which has been exploited recently for the solution of Hermitian matrix models. We also demonstrate how the dimensional reduction of the supersymmetric matrix model reproduces characteristics of a branched polymer theory, and comment on the potential uses of super-matrices for theories of supergravity and superstrings. Generally, throughout the Review we single out the numerous unsolved problems in this field, all of which are interesting problems which certainly deserve future investigation. Although for the most part this Paper is a review, a good portion of the analyses presented in the other Sections is original work.
Scalar $O(N)$-symmetric vector models have been studied over the past few years as an interesting testing ground of ideas for the more complex Hermitian matrix models [40, 41, 58, 116, 124, 138]. Besides this, as mentioned in the previous Section, they describe statistical models of discrete filamentary surfaces relevant for the branched polymer phase of $D > 1$ string theories [8, 15, 17, 31, 55, 115]. We would expect the same to be true of some simplified vector version of the fermionic matrix model (1.62). Therefore, as a preliminary discussion and to illustrate some of the features of statistical ensembles with fermionic degrees of freedom in a context simpler than the matrix theories, we consider a toy model with $N$ Grassmann variables $\psi_i$, $i = 1, \ldots, N$, and $N$ independent conjugate variables $\bar{\psi}_i$. The model is specified by the partition function [126, 127]

$$Z_0 = \int d\psi \; d\bar{\psi} \; e^{NV(\bar{\psi}\psi)} \quad (2.1)$$

where

$$\bar{\psi}\psi \equiv \sum_{i=1}^{N} \bar{\psi}_i\psi_i \quad , \quad (2.2)$$

$V(z)$ is some “potential” function of $z$ and the integration measure is

$$d\psi \; d\bar{\psi} \equiv \prod_{i=1}^{N} d\psi_i \; d\bar{\psi}_i \quad (2.3)$$

The integration over Grassmann variables in (2.1) is well-defined and finite if $e^{NV(z)}$ has a well-defined Taylor expansion to order $N$ in the variable $z$. As we shall show in the following, this simplicity is what makes this model explicitly solvable. The model possesses a continuous symmetry,

$$\psi_i \rightarrow \sum_{j=1}^{N} U_{ij} \psi_j \; , \; \bar{\psi}_i \rightarrow \sum_{j=1}^{N} \bar{\psi}_j (U^{-1})_{ji} \quad \text{with} \quad U \in GL(N, \mathbb{C}) \quad (2.4)$$

parametrized by the Lie group $GL(N, \mathbb{C})$ of invertible complex-valued $N \times N$ matrices. There is a further discrete symmetry under the “chiral” transformation

$$\psi_i \rightarrow \bar{\psi}_i \; , \; \bar{\psi}_i \rightarrow -\psi_i \quad \text{for any} \; i \quad (2.5)$$

Together, these two symmetries restrict the observables of the model to those which are functions only of $\bar{\psi}\psi$. The statistical model is then completely determined once the moments

$$M^k \equiv \langle (\bar{\psi}\psi)^k \rangle \equiv \frac{\int d\psi \; d\bar{\psi} \; (\bar{\psi}\psi)^k \; e^{NV(\bar{\psi}\psi)}}{\int d\psi \; d\bar{\psi} \; e^{NV(\bar{\psi}\psi)}} \quad (2.6)$$

are known. Since nilpotency of the components $\psi_i$ and $\bar{\psi}_i$ imply that

$$\langle (\bar{\psi}\psi)^{N+1} \rangle = 0 \quad (2.7)$$

there are $N$ independent moments $M^1, \ldots, M^N$. Also, symmetry and nilpotency of the vector components $\psi_i$ and $\bar{\psi}_i$ imply that

$$M^k = \frac{N!}{(N-k)!} \langle \prod_{j=1}^{k} \bar{\psi}_j \psi_j \rangle \quad , \quad 0 < k \leq N \quad (2.8)$$

and so the correlators $\langle f(\bar{\psi}\psi) \rangle$ for finite-$N$ are polynomials in the parameters of the potential in (2.1).
2.1 Finite and Infinite $N$ Solutions

The first feature we will stress of the fermionic vector model (2.1) is the extent to which it can be solved at both finite and infinite-$N$. The explicit form of the solution at infinite-$N$ is obtainable even in the scalar models because of the linearity of the vector theories in contrast to the non-linearity of the equations of motion involving matrix fields. The solvability here at finite-$N$ is a consequence of the Grassmann integrations in (2.1). More precisely, the partition function (2.1) at finite $N$ can be formally evaluated by inserting the delta function

$$1 = \int_{-\infty}^{+\infty} dx \, \delta(x - \bar{\psi} \psi)$$

(2.9)

and using the identity

$$\int_{-\infty}^{+\infty} \frac{dy}{2\pi} \int d\psi \, d\bar{\psi} \, e^{iy(x-\bar{\psi} \psi)} = \int_{-\infty}^{+\infty} \frac{dy}{2\pi} \, N!(-iy)^N \, e^{iyx} = N! \left( -\frac{\partial}{\partial x} \right)^N \delta(x)$$

(2.10)

to obtain

$$Z_0 = N! \cdot \left( \frac{\partial}{\partial x} \right)^N e^{NV(x)} \bigg|_{x=0}$$

(2.11)

Similarly, the correlators for $N$ finite are given by

$$M^n = \left( \frac{\partial}{\partial x} \right)^N x^n e^{NV(x)} \bigg|_{x=0}$$

(2.12)

These moments can always be obtained from a (not unique) distribution function $\rho$ defined by

$$M^n = \int d\alpha \, \rho(\alpha) \alpha^n$$

(2.13)

where the integral is over the support of $\rho$ in the complex plane. When $N$ and therefore the number of moments is finite the support of the spectral function $\rho(\alpha)$ is concentrated near the origin in the complex $\alpha$-plane

$$\rho(\alpha) \equiv \langle \delta(\alpha - \bar{\psi} \psi) \rangle = \frac{1}{2\pi} \int dw \, e^{-i\alpha \omega} \langle e^{i\omega \bar{\psi} \psi} \rangle$$

$$= \frac{1}{2\pi} \int dw \, e^{-i\alpha \omega} \left( \frac{\partial}{\partial z} \right)^N e^{i\omega z + NV(z)} \bigg|_{z=0}$$

$$= \frac{1}{Z_0} \left( -\frac{\partial}{\partial \alpha} + NV'(\alpha) \right)^N \delta(\alpha)$$

(2.14)

The spectral density (2.14) completely determines the solution of the random vector model (2.1). The most general potential which we shall consider in this Section is a combination of a logarithm and a polynomial of degree $K \leq \infty$,

$$V(z) = \kappa \log z + \sum_{n=1}^{K} \frac{g_n}{n} z^n$$

(2.15)
When $N$ is infinite, the distribution has an infinite number of moments and the support of the spectral function need no longer be concentrated at the origin. In this case the solution of the vector model simplifies because the correlators factorize,

$$\langle f(\bar{\psi}\psi) \rangle = f(\langle \bar{\psi}\psi \rangle) + \mathcal{O}(1/N)$$  \hspace{1cm} (2.16)

This factorization property follows from the fact that for an arbitrary potential of the form (2.15) the connected correlators are given by

$$\langle (\bar{\psi}\psi)^n \rangle_{\text{conn}} = \frac{1}{N^n} \left( \frac{\partial}{\partial g_i} \right)^n \log Z_0$$  \hspace{1cm} (2.17)

and

$$\log Z_0 = NF_0$$  \hspace{1cm} (2.18)

so that the free energy $F_0$ is of order one at $N \to \infty$. It follows that

$$M^n = (M^1)^n + \mathcal{O}(1/N)$$  \hspace{1cm} (2.19)

and the statistical model in the large-$N$ limit is completely determined by the first moment

$$M^1 = \langle \bar{\psi}\psi \rangle$$  \hspace{1cm} (2.20)

The spectral density in the large-$N$ limit

$$\rho(\alpha) = \lim_{N \to \infty} \sum_{k=0}^{N} \frac{M^k}{k!} \left( -\frac{\partial}{\partial \alpha} \right)^k \delta(\alpha) = e^{-M^1 \frac{\partial^2}{\partial \alpha^2}} \delta(\alpha) = \delta(\alpha - M^1)$$ \hspace{1cm} (2.21)

is a point mass localized at the point $\alpha = M^1$. The expectation value of any observable of the vector model (2.1) at large-$N$ is therefore

$$\langle f(\bar{\psi}\psi) \rangle = f(M^1)$$ \hspace{1cm} (2.22)

Thus the fermionic vector model (2.1) provides a rare example of a statistical theory which is exactly solvable at both finite and infinite $N$. We shall discuss the applications of the fermionic vector model as a random geometry theory [127] in the last Subsection of this Section.

### 2.2 Loop Equations

An alternative way to solve for the moments in a random distribution of variables, which will generalize to the matrix theories we are ultimately interested in, is by the Makeenko-Migdal method of loop equations [107]. Consider the “propagator”

$$\omega(z) \equiv \left\langle \frac{1}{z - \bar{\psi}\psi} \right\rangle$$ \hspace{1cm} (2.23)

The asymptotic expansion of this function is a series in the moments

$$\omega(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\langle (\bar{\psi}\psi)^n \rangle}{z^{n+1}} = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{M^k}{z^{k+1}}$$  \hspace{1cm} (2.24)
When \( N \) is finite, since this series is finite, \( \omega(z) \) has singularities only at the origin in the complex \( z \)-plane. It can be expressed as an integral over the spectral density

\[
\omega(z) = \int d\alpha \frac{\rho(\alpha)}{z - \alpha} \quad (2.25)
\]

where \( \rho(\alpha) \) is the spectral function whose formal solution is given in (2.14) and the integration in (2.25) can be taken over any contour in the complex \( \alpha \)-plane which goes through the origin.

When \( N \) is infinite, the support of the spectral function can be deduced from the analytic structure of \( \omega(z) \). If the support is in a compact region of the complex plane, the propagator has the asymptotic form

\[
\lim_{|z| \to \infty} \omega(z) = 1/z \quad (2.26)
\]

and the integral over \( \alpha \) in (2.25) is on a contour or set of contours in the complex plane which contain the support of \( \rho(\alpha) \). The resulting integral is analytic in the region outside of the support of \( \rho \). The positions of branch and pole singularities of \( \omega \) determines the support of \( \rho \) which in the large-\( N \) limit is generally some contour in the complex plane. The continuous function \( \rho \) itself can be found by computing the discontinuity of \( \omega(z) \) across its support, where

\[
\omega(\beta \pm \epsilon_\perp) = \int d\alpha \frac{\rho(\alpha)}{\beta - \alpha} \mp \frac{\epsilon_\perp}{|\epsilon_\perp|} \pi \rho(\beta) \quad \text{for} \quad \beta \in \text{supp } \rho \quad (2.27)
\]

and \( \epsilon_\perp(\beta) \) is a complex number with infinitesimal amplitude and direction perpendicular to the integration contour in (2.27) at the point \( \beta \).

The loop equation for \( \omega(z) \) follows from the identity

\[
0 = \int d\psi \ d\bar{\psi} \ \frac{\partial}{\partial \psi_i} \left( \psi_j \frac{1}{z - \psi \bar{\psi}} \right) e^{N V(\bar{\psi} \psi)} \quad (2.28)
\]

which is a consequence of the Grassmann integration rules (1.67). Dividing by \( Z_0 \) and expanding out the averages in this equation gives

\[
0 = \delta_{ij} \left\langle \frac{1}{z - \psi \bar{\psi}} \right\rangle + \left\langle \bar{\psi}_i \psi_j \left( \frac{1}{z - \psi \bar{\psi}} \right)^2 \right\rangle + N \left\langle \bar{\psi}_i \psi_j \frac{1}{z - \psi \bar{\psi}} V'(\bar{\psi} \psi) \right\rangle \quad (2.29)
\]

and then summing over \( i = j = 1, \ldots, N \) leads to

\[
\frac{1}{N} \left( z \frac{\partial}{\partial z} + 1 \right) \omega(z) + \omega(z) + \left\langle \frac{1}{z - \psi \bar{\psi}} \bar{\psi} \psi V'(\bar{\psi} \psi) \right\rangle = 0 \quad (2.30)
\]

The correlator in (2.30) involving the potential \( V(\bar{\psi} \psi) \) can be expressed, using (2.25), as an integral over a contour \( C \) in the complex plane which encircles the singularities of \( \omega(z) \) with counterclockwise orientation. The loop equation (2.30) can then be written as the integro-differential equation

\[
\frac{1}{N} \left( z \frac{\partial}{\partial z} + 1 \right) \omega(z) + \omega(z) - \oint_C \frac{d\lambda}{2\pi i} \frac{V'(\lambda)\lambda}{z - \lambda} \omega(\lambda) = 0 \quad (2.31)
\]

When the potential is of the form (2.15), we have

\[
z V'(z) = \sum_{n=0}^{K} g_n z^n \quad (2.32)
\]

28
where \( g_0 = \kappa \). To simplify (2.30) we use the identity

\[
\tilde{\psi}\psi'V(\tilde{\psi}\psi) = zV'(z) + \left(\tilde{\psi}\psi'V(\tilde{\psi}\psi) - zV'(z)\right)
\]

\[
= zV'(z) + (\tilde{\psi}\psi - z) \sum_{n=1}^{K} g_n \sum_{m=0}^{n-1} z^m (\tilde{\psi}\psi)^{n-m-1}
\]  

(2.33)

and obtain the loop equation

\[
\frac{1}{N} \left( z \frac{\partial}{\partial z} + 1 \right) \omega(z) + \omega(z) + zV'(z)\omega(z) = P(z)
\]

where \( P(z) \) is a polynomial of degree \( K - 1 \) which is determined by the first \( K - 1 \) moments \( M^1, \ldots, M^{K-1} \)

\[
P(z) = \sum_{m=1}^{K} g_m \sum_{p=0}^{m-1} \left( (\tilde{\psi}\psi)^{m-p-1} \right) z^p = \sum_{m=1}^{K} g_m \sum_{p=0}^{m-1} M^{m-p} z^p
\]  

(2.35)

This equation can also be obtained from the integro-differential equation (2.31). When the potential is given by (2.32), the contour integral in (2.31) can be evaluated by computing the residues at \( \lambda = z \) and \( \lambda = \infty \) to get

\[
\oint_C \frac{d\lambda}{2\pi i} \frac{V'(\lambda)\lambda}{z - \lambda} \omega(\lambda) = -V'(z)\omega(z) + P(z)
\]  

(2.36)

Generally, when the loop equations are solved, the moments in (2.35), which appear as constants in the equation, must be found self-consistently. Substituting the asymptotic expansion (2.24) into the loop equation (2.34) and equating the coefficients of \( z^p \), we find that the moments are determined in terms of the coupling constants of the potential (2.15) by the set of recursive equations

\[
\sum_{k\geq 1} g_k M^k = 1 - \kappa
\]  

(2.37)

\[
\kappa M^p + \sum_{k\geq 1} g_k M^{k+p} = \left( 1 - \frac{p}{N} \right) M^p \quad \text{for} \quad 1 \leq p \leq N
\]  

(2.38)

The loop equation must be solved with the asymptotic boundary condition (2.26). Its solution is readily found for arbitrary \( N \) by integrating the first order linear ordinary differential equation (2.34) as

\[
\omega(z) = \frac{N}{z} z^{-N} e^{-N V(z)} \int dz \ P(z) z^N e^{N V(z)}
\]  

(2.39)

The overall constant \( N \) is fixed by the asymptotic behaviour (2.26) which can be found by expanding (2.39) in \( 1/z \). The leading term is found by

\[
\lim_{|z| \to \infty} \omega(z) \sim \frac{N}{z} \lim_{|z| \to \infty} \frac{\int_{z}^\infty dw \ P(w) w^N e^{N V(w)}}{z^N e^{N V(z)}}
\]

\[
= \frac{1}{z} \lim_{|z| \to \infty} \frac{N P(z) z^N e^{N V(z)}}{z^N e^{N V(z)}} = \frac{1}{z}
\]  

as expected.
Thus, the linearity of the vector model equations of motion lead to a rare example of a field theory whose loop equations can be solved exactly and explicitly for any $N$. For the adjoint fermion matrix models that we shall soon consider, the loop equations will be non-linear and such a solution at arbitrary $N$ will not be possible, even though the partition function is explicitly computable for finite $N$ as in the vector case. Notice also that the moments appearing in the recursion relations (2.37) and (2.38) can be expressed as

$$M^k = k \frac{\partial}{\partial y_k} F_0, \quad k \geq 1$$

and so the loop equations are equivalent to a system of differential equations in coupling constant space which determine the partition function $Z_0$ of the model. In the matrix generalizations we shall exploit this feature to relate the solutions of the loop equations to integrable hierarchies of differential equations. In the bosonic cases, these appear as Virasoro constraints on a $\tau$-function which is characterized as a symmetry of the string equation. The fact that this is readily observed in the vector model counterparts owes to the simplicity and linearity of these models.

As expected, the loop equations simplify at large-$N$ because of factorization. The propagator (2.24) at $N = \infty$ becomes

$$\omega(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(M^1)^k}{z^{k+1}} = \frac{1}{z - M^1}$$

and the loop equation (2.30) at $N = \infty$ is

$$\omega(z) = \frac{P(M^1, z)}{z V'(z) - 1}$$

where

$$P(M^1, z) = \sum_{m=1}^{K} g_m \sum_{p=0}^{m-1} (M^1)^{m-p-1}z^p$$

The solution (2.42) agrees with that found in the last Subsection. The coincidence of the 2 solutions (2.42) and (2.43) is equivalent to the equations (2.37),(2.38) which determine the single independent moment $M^1$. For a potential of the form (2.15), at large-$N$ the correlator $M^1$ is determined as the solution of the $K$-th order algebraic equation (2.37),

$$M^1 V'(M^1) = 1$$

### 2.3 The Gaussian Model

The simplest model is the Gaussian model which is defined by the potential

$$V(z) = tz$$

The Gaussian potential (2.46) in all of the models we present in this Review will always serve as an interesting consistency check of the formalisms and will moreover present partial solutions for more complicated examples. In this case, the distribution function (2.14) becomes

$$\rho(\alpha) = \left(1 + \frac{1}{Nt} \frac{\partial}{\partial \alpha} \right)^N \delta(\alpha)$$
The $N \to \infty$ limit of (2.47) is

$$\rho(\alpha) = e^{\frac{1}{2} \frac{\partial}{\partial \alpha}} \delta(\alpha) = \delta(\alpha - 1/t)$$  \hspace{1cm} (2.48)

which identifies the first moment of the distribution as

$$M^1 = \langle \bar{\psi} \psi \rangle = 1/t$$  \hspace{1cm} (2.49)

The finite-$N$ loop equation (2.30) in the case of the Gaussian potential (2.46) is the first order ordinary differential equation

$$\frac{z}{N} \omega'(z) + \left( 1 + \frac{1}{N} - tz \right) \omega(z) + t = 0$$  \hspace{1cm} (2.50)

whose solution is

$$\omega(z) = \frac{1}{z} \left( 1 - \frac{1}{N} \frac{\partial}{\partial t} \right)^N \frac{1}{t}$$  \hspace{1cm} (2.51)

where we have used the boundary condition (2.26). In the infinite-$N$ limit (2.51) becomes

$$\omega(z) = \frac{t}{z} e^{-\frac{1}{z} \frac{\partial}{\partial t} \left( \frac{1}{t} \right)} = \frac{t}{z(t - 1/z)}$$  \hspace{1cm} (2.52)

which coincides with the large-$N$ solution (2.43) for the Gaussian potential (2.46). The propagator $\omega(z)$ at both finite and infinite $N$ agrees with the spectral density obtained directly above. Alternatively, we obtain from (2.39) the general solution

$$\omega(z) = \frac{N}{z} \int_0^z dw \frac{w^N t e^{Nw}}{z^N e^{Nzt}} = \frac{1}{z} \sum_{k=0}^N \frac{N!}{(N-k)!} \frac{1}{t^k} N^k \frac{1}{z^k}$$  \hspace{1cm} (2.53)

which identifies the Gaussian moments

$$M^k = \frac{N!}{(N-k)!} \frac{1}{(Nt)^k}$$  \hspace{1cm} (2.54)

as expected from (2.8). In the large-$N$ limit, it is readily seen that $M^k = (M^1)^k$ where the moment $M^1$ is given by (2.49).

### 2.4 Four-Fermi Vector Model

In the simple Gaussian example above, the free energy is analytic in the coupling constant $t$ and there are thus no critical points. In the general case, however, the moment $M^1$, which completely specifies the solution of the vector model at large-$N$, is determined as the solution of a $K$-th order polynomial equation and, from (2.41), the free energy $F_0$ will have a non-analytic cut structure leading to some sort of critical behaviour. For example, the simplest non-Gaussian model is defined by the quadratic potential

$$V(z) = tz + \frac{g}{2} z^2$$  \hspace{1cm} (2.55)
The propagator $\omega(z)$ can be computed formally at finite $N$ by evaluating the integral

$$\omega(z) = \frac{N}{z} z^{-N} e^{-N(tz + g z^2/2)} \int_0^z dw \, w^N (t + gw + gM^1) e^{N(tw + gw^2/2)} \quad (2.56)$$

which, after shifting the integration variable $w$, can be expressed in terms of error functions.

For $p = 1$, the 2 relations (2.37) and (2.38) become

$$tM^1 + gM^2 = 1 \quad , \quad tM^2 + gM^3 - (1 - 1/N)M^1 = 0 \quad (2.57)$$

In the large-$N$ limit, (2.57) yields a quadratic equation for the moment $M^1$,

$$tM^1 + g(M^1)^2 - 1 = 0 \quad (2.58)$$

Note that this quadratic equation also follows from the large-$N$ limit of the generating function (2.56),

$$\omega(z) = \frac{1}{z - M^1} = \frac{t + g(M^1 + z)}{tz + gz^2 - 1} \quad (2.59)$$

Expanding both sides of (2.59) to order $1/z^3$ and equating coefficients, we recover (2.58). The solution of (2.58) which is regular at $g = 0$ (and thus consistently reproduces the Gaussian solution (2.49) when $g \to 0$) is

$$M^1 = \frac{t}{2g} \left( 1 - \sqrt{1 + \frac{4g}{t^2}} \right) \quad (2.60)$$

The $N = \infty$ free energy can be obtained by integrating up the identity $M^1 = \frac{\partial F_0^{(0)}}{\partial t}$. The second set of relations in (2.57) at finite-$N$ then yield a relation among the various partial derivatives of $F_0$ which can be used to recursively construct the large-$N$ expansion of the vector model order by order in $1/N$. This approach was advocated for the scalar version of the four-Fermi model above in [124]. This construction and the physical relevance of the ensuing critical behaviour is the topic of the next Subsection.

### 2.5 Critical Behaviour and the $\frac{1}{N}$-expansion

One of the nicest features of vector models in general is that one can straightforwardly construct the $\frac{1}{N}$-expansion of the free energy explicitly. In the fermionic case, this is an especially useful probe of the characteristics of the more complicated fermionic matrix theories which are hoped to describe 2-dimensional quantum gravity coupled to some sort of fermionic matter placed at the vertices of the discretization. We shall conclude our discussion of fermionic vector models in this Subsection by examining the explicit critical behaviour and double scaling limit of the vector theories described above [127]. This will provide much insight into some general ideas that will be encountered in the matrix models later on where such explicit constructions have yet to be carried out.
2.5.1 Random Polymer Models

We start by examining what sort of random geometry theory the fermionic vector model describes. For completeness, let us begin by briefly reviewing the status of the $O(N)$ vector model with partition function [41, 116, 124]

$$Z_S(t, g; N) = \int_{-\infty}^{+\infty} d\phi_1 \cdots d\phi_N \exp \left\{ -t \sum_{i=1}^{N} \phi_i^2 + \frac{g}{N} \left( \sum_{i=1}^{N} \phi_i^2 \right)^2 \right\}$$  \hspace{1cm} (2.61)

The model (2.61) is invariant under the orthogonal transformation

$$\phi_i \to \sum_{j=1}^{N} S_{ij} \phi_j \text{ with } S \in O(N)$$  \hspace{1cm} (2.62)

The continuous (connected) $SO(N)$ part of the full orthogonal group symmetry in (2.62) is the analog of the $GL(N, \mathbb{C})$ symmetry (2.4), while the discrete reflection symmetry $\phi_i \to -\phi_i$, for any $i$, is the bosonic analog of the chiral symmetry (2.5). The symmetry (2.62) restricts the observables of the theory to those which are functions of

$$\phi^2 \equiv \sum_{i=1}^{N} \phi_i^2$$  \hspace{1cm} (2.63)

When the coupling constants $t$ and $g$ are positive, it is straightforward to show that the formal expression (2.61) counts the number of randomly branching polymers, both those with a tree-like structure and those with arbitrarily many loops. The Feynman graphs produced in this case do not have enough structure to specify a Riemann surface [38] and instead they describe ‘discrete filamentary surfaces’. As we shall see below, this is a consequence of the behaviour of the susceptibility of these models which exhibit a positive string constant indicative of a fragmentation of the surface (into a branched polymer phase in this case) [7]. The fact that the integral (2.61) is divergent is a reflection of the divergence of the statistical sum.

This statistical sum over polymers coincides with the expansion of the free energy

$$F_S \equiv -\frac{1}{N} \log \left( \frac{Z_S(t, g; N)}{Z_S(t, 0; N)} \right)$$  \hspace{1cm} (2.64)

in Feynman diagrams. The propagator is

$$\langle \phi_i \phi_j \rangle = \frac{\int \prod_k d\phi_k \phi_i \phi_j e^{-t\phi^2}}{\int \prod_k d\phi_k e^{-t\phi^2}} = \delta_{ij}$$  \hspace{1cm} (2.65)

the vertex is

$$\langle \phi_i \phi_j \phi_k \phi_l \rangle = \frac{1}{(2t)^2} (\delta_{ij} \delta_{kl} + \delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl})$$  \hspace{1cm} (2.66)

and the free energy is the sum of all connected diagrams with four-point couplings. The standard rules for evaluating the Gaussian integrals as in (2.65) and (2.66) are given by the Wick contraction rules in which each pairing of vector components is assigned a delta-function in the index contraction. The dual graphs to these Feynman diagrams, defined by
associating a vertex (molecule) in the center of each of the scalar loops and lines (bonds) connecting vertices by crossing each of the Feynman 4-point couplings, are random walk diagrams [17, 55, 116, 127]. The number of molecules \( n \) appears as the power in \( N^{n-1} \) and corresponds to the number of index loops of the Feynman graphs. The number of bonds \( b \) is given by the power in \( (g/N^2t)^b \) and corresponds to the number of vertices, or equivalently contacts of these index loops.

The random polymer model generated by (2.64) is therefore of the form

\[
F_S = -\sum_{\mathcal{P}_{b,\ell}} N^{-\ell} \left( \frac{g}{N^2} \right)^b
\]

where the sum is over all polymer graphs \( \mathcal{P}_{b,\ell} \) with \( b \) bonds and

\[
\ell = b - n + 1
\]

loops. Thus the vector model partition function is the generating function for the number of polymer configurations with \( b \) bonds and \( \ell \) loops. (2.67) identifies the parameters of the vector model with the discretized action terms for a random walk model as [8, 9, 15, 115]

\[
\frac{1}{N} = e^\mu \quad , \quad \frac{g}{t^2} = e^{-L}
\]

where \( \mu \) is the fugacity and \( L \) the length of the branched polymer chain. Note that the sum (2.67) includes the self-bonding polymers which are generated by Wick contracting several propagators into single loops (as opposed to multi-loops) and occur in the expansion only for \( \ell \geq 1 \) [116, 127].

From an analytic point of view, the perturbative expansion of the partition function

\[
\frac{Z_S(t; g; N)}{Z_S(t, 0; N)} = \sum_{n \geq 0} \frac{1}{n!} \left( \frac{g}{N} \right)^n \left\langle \left( \phi^2 \right)^{2n} \right\rangle
\]

is completely determined by the Gaussian moments

\[
\left\langle \left( \phi^2 \right)^{2n} \right\rangle = \frac{1}{Z_S(t, 0; N)} \left( \frac{\partial^2}{\partial t^2} \right)^n Z_S(t, 0; N) = t^{N/2} \left( \frac{\partial^2}{\partial t^2} \right)^n t^{-N/2}
\]

as

\[
\frac{Z_S(t; g; N)}{Z_S(t, 0; N)} = \sum_{n=0}^{N_A} \left( \frac{N + 4n - 2}{N - 2} \right)!! \left( \frac{g}{Nt^2} \right)^n
\]

Here we have introduced an “ultraviolet” cutoff \( N_A \in \mathbb{Z}^+ \) to make the partition function well-defined. In the limit \( N_A \to \infty \), the series (2.72) is a non-Borel summable asymptotic series reflecting the divergence of the original integral and also the divergence of the statistical sum. As in the case of random surfaces, even though the series is divergent, if arranged as a power series in \( 1/N \), rather than \( g \), the terms in this series are individually convergent and it is the sum over genera \( \ell \) which is asymptotic. With the cutoff \( N_A \) in (2.72), the partition function is an analytic function of \( N \), but it is only well-defined at \( N = \infty \) when this ultraviolet cutoff is removed.

The random polymer model above can also be generated by the fermionic vector model with partition function

\[
Z_F(t; g; N) = \int d\psi \, d\bar{\psi} \, e^{t\bar{\psi}\psi - \frac{g}{Nt}(|\bar{\psi}|^2)}
\]
As we shall now see, the model (2.73) possesses a random geometry interpretation similar to that of the \( O(N) \) vector model. A main difference with the \( O(N) \) vector model is that the integration over Grassmann variables in the generating function (2.73) for the polymers is a well-defined finite polynomial in the coupling constants \( g \) and \( t \). The dimension \( N \) itself provides a cutoff on the number of terms (and polymers) in the Feynman diagram expansion of (2.73). Here, rather than making the partition function integration well-defined as in the bosonic case, the large-\( N \) limit is needed to generate the full ensemble of randomly-branched chains.

The Feynman diagrams for the fermion vector theory (2.73) have propagator

\[
\langle \bar{\psi}_i \psi_j \rangle = \frac{\int d\bar{\psi} d\bar{\psi}_j e^{\bar{\psi} \gamma \psi_j}}{\int d\bar{\psi} e^{\bar{\psi} \gamma \psi}} = \delta_{ij}
\]  

(2.74)

and the four-Fermi interaction vertex is

\[
\langle \bar{\psi}_i \psi_j \bar{\psi}_k \psi_l \rangle = \frac{1}{i^2} (\delta_{ij} \delta_{kl} - \delta_{il} \delta_{jk})
\]  

(2.75)

These Feynman rules have the same graphical representation as for the \( O(N) \) vector model above with a left-handed orientation for the lines (ingoing lines into a vertex for \( \bar{\psi}_i \) components and outgoing lines for \( \psi_j \) components). Now the graphs are formed from all connected 4-point diagrams which preserve this orientation. The Feynman rules also associate a factor of \(-1\) to each fermion loop in a Feynman graph. The Wick rules for the evaluation of the Gaussian fermionic integrals such as (2.74) and (2.75) associate delta-functions in the indices for each contraction of the form \( \bar{\psi} \psi \) and a minus sign for an interchange in the order of \( \psi \bar{\psi} \). Notice that these Gaussian moments can likewise be obtained from a generating functional

\[
Z_0(\eta, \bar{\eta}) = \frac{\int d\bar{\psi} e^{Nt\bar{\psi} \gamma \psi + \bar{\psi} \gamma \psi \eta}}{\int d\bar{\psi} e^{Nt\bar{\psi} \gamma \psi}}
\]  

(2.76)

where \( \eta \) and \( \bar{\eta} \) are also independent Grassmann-valued vectors, as

\[
\langle \prod_k \bar{\psi}_{i_k} \psi_{j_k} \rangle = \prod_k \frac{\partial}{\partial \eta_{i_k}} \frac{\partial}{\partial \bar{\eta}_{j_k}} Z_0(\eta, \bar{\eta}) \bigg|_{\eta=\bar{\eta}=0} = \prod_k \frac{\partial}{\partial \eta_{i_k}} \frac{\partial}{\partial \bar{\eta}_{j_k}} e^{-\eta \gamma \bar{\eta} / Nt} \bigg|_{\eta=\bar{\eta}=0}
\]  

(2.77)

which also leads to the fermionic Feynman-Wick rules with the appropriate minus signs.

The perturbative expansion of (2.73)

\[
Z_F(t, g; N) = \sum_{n \geq 0} \frac{(-1)^n}{n!} \left( \frac{g}{2N} \right)^n \left\langle (\bar{\psi} \gamma \psi)^{2n} \right\rangle
\]  

(2.78)

is completely determined by the normalized Gaussian moments

\[
\left\langle (\bar{\psi} \gamma \psi)^{2k} \right\rangle = \frac{1}{Z_F(t, 0; N)} \left( \frac{\partial^2}{\partial t^2} \right)^k Z_F(t, 0; N) = t^{-N} \left( \frac{\partial^2}{\partial t^2} \right)^k t^N
\]  

(2.79)

to be

\[
Z_F(t, g; N) = \sum_{n} \frac{(-1)^n}{n!} \frac{N!}{(N-2n)!} \left( \frac{g}{2Nt^2} \right)^n
\]  

(2.80)
where \( N_2 = N/2 \) (respectively \( (N-1)/2 \)) when \( N \) is even (odd). It is straightforward to check that (2.80) coincides with (2.11) for the four-Fermi vector potential. The perturbation series is a finite sum which represents the same sort of random walk distribution as in the scalar theory above except that it only includes polymers with up to \( N_2 \) bonds. From a diagrammatic point of view, the alternating nature of the series arises from the minus signs associated with fermion loops. Term by term, this series can be made identical with the terms of same order in \( g \) in (2.72) by the analytical continuation \( N \rightarrow -N/2 \) in (2.80) (so that \( N_\Lambda = N_2 \) in (2.72)). The factor of 2 is associated with the doubling of degrees of freedom in the fermionic case. Thus, after the substitution \( N \rightarrow N/2 \), the large \( N \) expansion of the fermionic vector model is identical to that of the \( O(N) \) vector model \textit{except} that it is an alternating series in \( 1/N \). The coefficient of \( 1/N^t \) in the former and \( 1/(-N)^t \) in the latter are identical. Now, however, the alternating nature of the fermionic vector series makes its \( N \rightarrow \infty \) limit Borel summable, and as such it defines a better behaved statistical theory.

Notice that the combinatorical factors occurring in the Feynman diagram expansion of this fermionic vector model are identical to those obtained in the \textit{complex} vector model

\[
Z_C(t, g; N) = \int \prod_{i=1}^{N} d\phi_i \, d\phi_i^* \exp \left\{ -t \sum_{i=1}^{N} \phi_i^* \phi_i + \frac{g}{2N} \left( \sum_{i=1}^{N} \phi_i^* \phi_i \right)^2 \right\} \quad (2.81)
\]

where the integration is over the whole of \( \mathbb{C}^N \simeq \mathbb{R}^{2N} \). The model (2.81) is invariant under the unitary transformations

\[
\phi_i \rightarrow \sum_{j=1}^{N} U_{ij} \phi_j \quad \text{with} \quad U \in U(N) \quad (2.82)
\]

This symmetry, along with the discrete charge conjugation symmetry \( \phi_i \rightarrow \phi_i^* \) for any \( i \), restricts the observables of the model to those which are functions only of \( \sum_i \phi_i^* \phi_i \). Although the complex vector model (2.81) exhibits the same doubling of degrees of freedom as in the fermionic case as well the same Wick contraction (non-vanishing only among \( \phi_i^* \) and \( \phi_j \)) and Feynman rules without the minus signs, it still leads to an asymptotic series expansion as in the case of the \( O(N) \) vector model above. In analogy with the matrix case, it can be thought of as describing the statistical mechanics of ‘checkered’ filamentary surfaces.

### 2.5.2 Double Scaling Limit

The large \( N \) expansion of the \( O(N) \) vector model above is a saddle point computation of the integral (2.61). After integration over angular variables, the partition function (2.61) can be written in terms of the radial coordinate in Euclidean \( N \)-space as

\[
Z_S(t, g; N) = \frac{2\pi^{N/2}}{\Gamma(N/2)} \int_0^\infty d\phi \, \phi^{N-1} e^{-t\phi^2 + \frac{-g}{4}\phi^4} \quad (2.83)
\]

In the infinite-\( N \) limit, which counts the tree-graphs (\( \ell = 0 \)), the integral (2.83) can be evaluated using the saddle-point approximation. Rescaling \( \phi \rightarrow \phi/\sqrt{N} \) the stationary condition for the effective action \( -Nt\phi^2 + gN\phi^4 + N\log \phi \) in (2.83) is

\[
2t\phi^2 - 4g\phi^4 = 1 \quad (2.84)
\]
The solution of (2.84) which is regular at $g = 0$ and which minimizes the effective action is

$$\phi_0^2 = \frac{t}{4g} \left( 1 - \sqrt{1 - \frac{4g}{t^2}} \right)$$

(2.85)

The tree-level free energy is then the value of the effective action in (2.83) evaluated at the saddle-point (2.85),

$$F_s^{(0)} \equiv \lim_{N \to \infty} -\frac{1}{N} \log Z_S(t, g; N)$$

$$= \frac{1}{2} \left[ \frac{1}{2} + \frac{t}{4g} \left( t - \sqrt{t^2 - 4g} \right) - \log \left( \frac{t}{4g} - \frac{1}{4g} \sqrt{t^2 - 4g} \right) \right]$$

(2.86)

The free energy (2.86) becomes non-analytic at the critical point

$$g = g_c \equiv t^2/4$$

(2.87)

where the 2 solutions of the quadratic equation (2.84) coalesce. There the minimum of the effective action in (2.83) disappears and it becomes unbounded [40, 41, 58, 138], so that the saddle point solution is no longer valid. There is a second order phase transition at the coupling $g = g_c$ with susceptibility exponent [8, 9, 15, 17, 55, 116]

$$\gamma_{\text{str}}^{(0)} = 1/2$$

(2.88)

This critical point is identified as the “continuum” limit of the random polymer theory where the number of branches, and hence the lengths, of the tree-graphs becomes infinite. The higher-loop contributions (molecular networks) can be found in [124] and their “continuum” limit is associated with an infinite number of molecules, thus tracing out a continuum filamentary surface [8, 9, 15, 17, 55]. Notice that since a negative string susceptibility constant is indicative of a locally 2-dimensional random geometry, the critical exponent (2.88) is inherently related to a dimensionally reduced discretization.

We shall now see that the fermionic vector model (2.73) possesses a similar critical behaviour. To explicitly carry out the $1/N$-expansion of the fermionic vector model, we introduce a scalar Hubbard-Stratonovich field $\varphi$ defined by the identity

$$1 = \int d\varphi \ e^{-\frac{g}{2N}(\varphi + i\bar{\psi}\psi)^2}$$

(2.89)

into the partition function integral (2.73) to write it as

$$Z_F(t, g; N) = \int d\varphi \ e^{-g\varphi^2/2N} \int d\bar{\psi} \ e^{(t - ig\varphi/N)\bar{\psi}\psi} = N! \int d\varphi \ e^{(t - ig\varphi/N)^N} e^{-g\varphi^2/2N}$$

(2.90)

When $N \to \infty$ the integral in (2.90) is determined by the saddle-point value of $\varphi$. Rescaling $\varphi \to \varphi/N$, this can be found from the stationary condition for the effective action

$$S_F(\varphi) \equiv -Ng\varphi^2/2 + N \log (t - ig\varphi)$$

(2.91)

appearing in (2.90) which is

$$tg\varphi - ig\varphi^2 + ig = 0$$

(2.92)
The solution of \( (2.92) \) which is regular at \( g = 0 \) is

\[
\varphi_0 = \frac{t}{2g} \left( 1 - \sqrt{1 - \frac{4g}{t^2}} \right)
\]  

(2.93)

Substituting (2.93) into (2.91) we get the tree-level fermionic free energy

\[
F_F^{(0)} = \frac{1}{2} - \frac{t}{4g} \left( t - \sqrt{t^2 - 4g} \right) - \log \left( \frac{t}{2} + \sqrt{t^2 - 4g} \right)
\]  

(2.94)

Note that the saddle points in (2.85) and (2.93) are related by the correspondence

\[
\theta^2 = \frac{i}{2} \varphi
\]  

(2.95)

which makes more precise the analytical continuation between the bosonic and fermionic models discussed above. Notice also that the free energy (2.94) is related to the free energy (2.86) of the scalar model as

\[
F_F^{(0)} = 1 - 2F_S^{(0)}
\]  

(2.96)

as anticipated since the large-\( N \) limit of the 2 models represents the same combinatorial problem of enumerating tree-graphs.

The higher-loop contributions (which count the polymer networks with a given number of molecules) can be found by carrying out the saddle point calculation of the integral (2.90) to higher orders. For this, we decompose the Hubbard-Stratonovich field as

\[
\varphi = \varphi_0 + \varphi_q
\]  

(2.97)

and expand the action (2.91) in a Taylor series about the saddle-point value (2.93) in terms of the fluctuation fields \( \varphi_q \). Using the saddle-point equation (2.92) when evaluating the higher-order derivatives \( S_F^{(n)}(\varphi_0) \), this Taylor series is found to be

\[
S_F(\varphi) = S_F(\varphi_0) - \frac{N}{2} \left( g + g^2 \varphi_0^2 \right) \varphi_q^2 - N \sum_{n=3}^{\infty} \frac{(-g\varphi_0)^n}{n} \varphi_q^n
\]  

(2.98)

The genus 1 free energy is then obtained from the fluctuation determinant that arises from Gaussian integration over the quadratic part in \( \varphi_q \) of (2.98),

\[
F_F^{(1)} = \frac{1}{2N} \log \left( g + g^2 \varphi_0^2 \right) = \frac{1}{2N} \log \left( 2g - \frac{t^2}{2} + t \sqrt{t^2 - 4g} \right)
\]  

(2.99)

which also agrees with the 1-loop free energy of the \( O(N) \) vector model [124].

The \( \frac{1}{N} \)-expansion of the fermionic free energy also becomes non-analytic at the critical point \( g = g_c = t^2/4 \). It exhibits the same critical behaviour as the \( \phi^4 \) theory above and it therefore lies in the same universality class as this statistical model. Notice that this critical behaviour, and also the free energies above, could have been obtained as well from the results of the previous Subsection with the identification \( \theta^2 \leftrightarrow M^1/2 \) there. It is straightforward to carry out the double-scaling limit of the fermionic vector model in much the same way as in the bosonic case [41, 124, 116]. This limit is associated with the continuum limit of the polymer network at \( N \to \infty \), \( g \to g_c \) in such a way that a coherent contribution from all
orders of the perturbative and $1/N$ expansions is obtained. We approach the critical point $g_c$ by defining a dimensionless “lattice spacing” $a$ by

$$a t^2 = g_c - g$$  \hspace{1cm} (2.100)$$
and taking the continuum limit $a \to 0$. With the rescalings mentioned above, the contribution to an arbitrary $\ell$-loop vacuum diagram with $\ell \geq 1$ is $N^{-\ell}(\sqrt{a})^{-\ell} N^n = N^{1-\ell} (\sqrt{a})^{1-\ell} n$. As shown in [41], the maximum number of vertices that a 4-point polymer diagram with $\ell$ loops can have is $n = 2(\ell - 1)$, so that the most singular behaviour of an $\ell$-loop diagram in the continuum limit $a \to 0$ is $(N a^{3/2})^{1-\ell}$. The proper continuum limit wherein a finite contribution from arbitrary genus polymers is obtained is thus the “double-scaling” limit where $N \to \infty$ and $a \to 0$ in a correlated fashion so that the renormalized “cosmological constant” (or “linear string tension”)

$$\Lambda_R \equiv N a^{3/2}$$  \hspace{1cm} (2.101)$$
remains finite. The double scaling limit enables an explicit construction of the genus expansion of the continuum polymer theory from the vector model.

We can now take the double scaling limit of the partition function (2.73) and write it as a loop expansion in the linear string tension $\Lambda_R$. However, as noted for the $O(N)$ vector model [41], the tree and 1-loop contributions are singular in this limit. The saddle-point value (2.93) can be written in terms of the lattice spacing as

$$g \varphi_0 = 2(1 - 2\sqrt{a})/it$$  \hspace{1cm} (2.102)$$
from which it follows that the genus 0 and 1 free energies (2.94) and (2.99) are given by

$$NF_F^{(0)} \sim -\frac{N}{2} - N \log \left(\frac{t}{2}\right) + 6 N^{1/3} \Lambda^{2/3}_R - \frac{32}{3} \Lambda_R$$

$$NF_F^{(1)} \sim \frac{1}{2} \log \left(\frac{t^2}{2}\right) + \frac{1}{2} \log \left(\frac{2 \Lambda^{1/3}_R}{N^{1/3}}\right)$$  \hspace{1cm} (2.103)$$
in the continuum limit $a \to 0$. The $\Lambda_R$-dependent terms in (2.103) diverge in the double scaling limit and represent a non-universal behaviour of the statistical polymer system. The tree-level and one-loop order Feynman diagrams should therefore be subtracted in the definition of the double-scaling limit leading to a renormalized partition function $Z_R(\Lambda_R, t)$ that only contains contributions from the $\ell$-loop diagrams with $\ell \geq 2$.

This renormalized partition function is obtained by integrating over that part of the action involving $n \geq 3$ vertices in the fluctuation field $\varphi_q$ weighted against the Gaussian form in (2.98). To pick out the finite contribution in the double-scaling limit, we rescale the fluctuation field as $\varphi_q \to 2 N^{1/3} \Lambda^{1/6}_R g_c^{1/2} \varphi_q$ and note that with this rescaling we have

$$- N g \left(g + g^2 \varphi_0^2\right) \varphi_q^2 \to -\frac{1}{2} \varphi_q^2 , \hspace{0.5cm} - N (-g \varphi_0)^n \varphi_q^n \to -\frac{N (g_c)^{n/2}}{2^n} \left(\frac{it}{2}\right)^n N^{-n/3} \Lambda_R^{-n/6} \varphi_q^n$$  \hspace{1cm} (2.104)$$
in the continuum limit $a \to 0$. The $n \geq 4$ vertex terms in (2.104) vanish in the double scaling limit, and therefore the exact renormalized partition function in the double scaling limit is (up to irrelevant normalization factors)

$$Z_R(\Lambda_R, t) = \frac{\int d\varphi_q \ e^{-\frac{1}{2} \varphi_q^2 + \frac{a^2}{12} \Lambda_R^{-1/3} \varphi_q^4}}{\int d\varphi_q \ e^{-\frac{1}{2} \varphi_q^2}} \sum_{k=0}^{\infty} \frac{t^k}{k! (5! 2)^k} N^{-k/2} \Lambda_R^{-k/2} \left(\langle \varphi^{3k} \rangle \right)_{N=1, t=\frac{1}{t}}$$  \hspace{1cm} (2.105)$$
39
The Gaussian moments in (2.105) can be evaluated as in (2.71). The odd moments vanish, while the even moments yield a factor \((3k-1)!!\). Thus the double-scaled renormalized partition function admits the exact genus expansion

\[
Z_R(\Lambda_R,t)=\sum_{\ell=0}^{\infty}(-1)^\ell t^{2\ell} \frac{(6\ell-1)!!}{(2\ell)! (512)^{2\ell}} \Lambda_R^{-\ell}
\]  

(2.106)

The partition function (2.106) has a similar structure as that in the \(O(N)\) vector model where the genus expansion is an asymptotic series with zero radius of convergence [41]. In the fermionic case, however, the genus expansion is an alternating sum, and is therefore Borel summable. The convergence of the sum over genera is easily seen in the integral expression (2.105) where the unbounded cubic term contains a factor of \(i\) which makes the overall integration there finite. The Borel summability of the genus expansion is a feature unique to the fermionic models that does not usually occur for random geometry theories. In this sense, the fermionic vector model represents some novel discretized surface theory in which the topological expansion uniquely specifies the generating function of the statistical theory. The identification \(-\frac{1}{N} = e^\mu\) in the fermionic case suggests that the associated random polymer theory has a complex-valued “fugacity” \(\mu = i\pi + \mu_0\), \(\mu_0 \in \mathbb{R}\), with doubly-degenerate degrees of freedom at each vertex. It would be interesting to give these fermionic properties of the theory a direct interpretation in terms of a random geometry model. From an analytic point of view, the genus sum alternates relative to that of the \(O(N)\) vector model because the saddle-point (2.93) is imaginary in the fermionic vector model (2.73) so that its saddle-point expansion is the analytical continuation \(\phi^0_0 = \frac{1}{2} \varphi_0\) of that for the scalar model.

The above analysis can be straightforwardly generalized to an interaction of the form \(g(\bar{\psi} \psi)^K\), which will then represent a random polymer model with up to \(2K\)-valence vertices. The critical behaviour is the same as that in a \(\phi^{2K}\) scalar vector model and leads to the same susceptibility exponent \(\gamma^{(0)}_{\text{str}} = \frac{1}{2}\), i.e. such a theory of random polymers is universal. To generate more complicated polymer models, for instance those with matter degrees of freedom at the vertices of the discretization [8, 9, 15, 115], one must study vector models with more complicated interactions, such as those which were considered quite generally at the beginning of this Section. To treat such models defined as in (2.1) at \(N = \infty\), we could use the first part of the identity (2.10) to write the partition function as

\[
Z_0 = \frac{(-i)^N N!}{2\pi} \int dz \ dw \ e^{NV(z) + i\bar{w}z + N \log w}
\]  

(2.107)

If the potential is a polynomial of degree \(m\) (\(\kappa = 0, K = m\) in (2.15)), then we can rescale \(z \to z/N\) and the coupling constants \(g_k \to N^k \cdot g_k\) simultaneously so that the effective action in (2.107) is \(NV(z) + iNwz + N \log w\). The integral (2.107) at large-\(N\) is determined by the saddle-point value of this effective action. In the 2-dimensional complex space of the variables \(w\) and \(z\), the stationary conditions are

\[
V'(z) + iw = 0 \quad , \quad iz + 1/w = 0
\]  

(2.108)

which can be combined into the single equation

\[
zV'(z) = 1
\]  

(2.109)

The equation (2.109) is identical to the stationary condition for the \(O(N)\) vector model defined with potential \(V(\phi^2)\) [41, 58, 138]. Thus the critical behaviour of the fermionic vector
The model (2.1) is the same as that for the $O(N)$ vector model with the same polynomial potential (2.15). The genus expansion is generated by the 2-dimensional saddle-point evaluation of the integral (2.107). The imaginary saddle-point values given by (2.108) will lead to an alternating genus expansion in the double-scaling limit for the fermionic theory, leading to a Borel summable polymer model, in contrast to the scalar case. In this case the critical point is again that point in coupling constant space where the function $zV'(z)$ vanishes. This point coincides with the simple pole of residue 1 in the propagator $\omega(z)$ in (2.43) and the solutions of the loop equations in (2.45) cease to exist. For a potential of the form (2.15) with $\kappa = 0, K = m$, we can adjust the coupling constants in such a way that the critical point is a zero of $zV'(z)$ of order $m$. The leading singularity of the free energy will then be $a^{(m+1)/m}$ [41, 116] which leads to the critical susceptibility exponent (c.f. (1.36))

$$\gamma_{\text{str}}^{[0]} = 1 - 1/m , \quad m = 2, 3, \ldots$$

(2.110)

This is the multicritical series for generalized random polymer systems in dimension $D \geq 0$ [8, 9, 15, 115] which interpolates between the Cayley tree at $m = \infty$ with $\gamma_{\text{str}}^{[0]} = 1$ and the ordinary random walk we discussed earlier at $m = 2$ with $\gamma_{\text{str}}^{[0]} = \frac{1}{2}$. In the $O(N)$ vector models, the former case would represent a phase of bosonic string theory (as in the Penner model) in target space dimension $D \geq 1$ while the latter case would represent a phase of pure 2-dimensional quantum gravity. In the general case, the potential (2.15) leads to discrete filamentary surfaces which have vertices of even valence up to $2m$. The $N^0$-component of the vector model free energy represents the self-avoiding random walk (i.e. no loops, $\ell = 0$), and it can be computed by dividing the statistical sum by $N$ and then taking the $N \to 0$ limit. This method of isolating the constant configurations in a random surface model is known as the ‘replica trick’ and it will be encountered again in Section 7. It would be interesting to determine precisely what physical systems the fermionic vector models represent in the continuum limit.

Given the convergence properties of the fermionic vector models, they can be combined with bosonic models to obtain supersymmetric-type vector theories representing new sorts of generating functions for random geometry theories [127]. Some supersymmetric generalizations of the $O(N)$ vector model have been studied in [30, 106, 117, 125] and it would be interesting to find physical applications of these models. We shall discuss some of these supersymmetric theories in Section 7. The main lessons we wish to draw here concerning our “toy model” analysis of this Section is that random geometry models involving fermionic degrees of freedom admit solutions analogous to those of the more conventional bosonic theories, except that the overall models have better convergence properties and lead to better defined statistical theories. In particular, the Borel summability will be argued later on to hold as well in the adjoint fermion one-matrix models. As we shall see, this is expected to be only true for odd polynomial potentials [12] as it is only in that case that the matrix model possesses a chiral symmetry and imaginary endpoints for the support of the spectral distribution (i.e. an analytical continuation of a Hermitian spectral density). In the case of the simpler fermionic vector models the partition function is always invariant under chiral transformation of the fermionic vector components. Furthermore, in the case of fermionic matrix models the correspondence with a scalar theory is more complicated – a polynomial fermion model can be analytically continued to a Hermitian matrix model with a generalized Penner potential. It should therefore represent a Borel summable generating function for the virtual Euler characteristics of the discretized moduli spaces of Riemann surfaces (rather than just the generating function for a random surface triangulation itself). In the vector case,
the fermionic model represents the same type of random surface theory as the corresponding $O(N)$ vector field theory. The results of the vector model analysis above therefore clarify and confirm many of the matrix model arguments that are presented.

3 Adjoint Fermion One-matrix Models

In this Section we will analyse in detail the adjoint fermion one-matrix model (1.62). It can be viewed as a $D = 0$ dimensional quantum field theory of a self-interacting Dirac fermion which transforms under the adjoint action of a “colour” gauge group. The model possesses the symmetry

$$
\psi \rightarrow U \psi V^{-1}, \quad \bar{\psi} \rightarrow V \bar{\psi} U^{-1} \quad \text{with} \quad \{U, V\} \in GL(N, \mathbb{C}) \otimes GL(N, \mathbb{C})
$$

(3.1)

In spite of this large degree of symmetry, it is not possible to diagonalize a matrix with anticommuting elements. Thus, unlike the more familiar Hermitian one-matrix models [20, 25], the model (1.62) cannot be written as a statistical theory of eigenvalues. Nevertheless, in the large-$N$ limit it shares many of the properties of such a theory. Furthermore, the large degree of symmetry restricts the observables to those which are essentially invariant functions of $\bar{\psi}\psi$.

The chiral transformation

$$
\psi \rightarrow \bar{\psi}, \quad \bar{\psi} \rightarrow -\psi
$$

(3.2)

is the analog of the reflection symmetry $\phi \rightarrow -\phi$ in a Hermitian 1-matrix model [25]. The invariant traces transform under (3.2) as

$$
\text{tr} (\bar{\psi}\psi)^k \rightarrow (-1)^{k+1} \text{tr} (\bar{\psi}\psi)^k
$$

(3.3)

Thus, (3.2) is a symmetry of the model when the potential is an odd polynomial. It is interesting that the analog in Hermitian matrix models is the reflection symmetry when the potential there is an even polynomial and in that case one expects the eigenvalue distribution to be symmetric and all odd moments of the distribution vanish, $\langle \text{tr} \phi^{2k+1} \rangle = 0$ [25] $^1$. In the present model, this symmetry introduces the feature that all even moments vanish,

$$
\langle \text{tr} (\bar{\psi}\psi)^{2k} \rangle = 0
$$

(3.4)

When $N$ is finite, because of the anticommuting property of the elements of $\psi$ and $\bar{\psi}$, there is an integer $k_0 \leq N^2$ such that

$$
\langle \text{tr} (\bar{\psi}\psi)^{k_0} \rangle = 0
$$

(3.5)

There are also a finite number of non-zero correlators of the form $\langle \Pi_{i=1}^n \text{tr} (\bar{\psi}\psi)^{k_i} \rangle$. In the large-$N$ limit, correlators of the matrix model factorize

$$
\langle \text{tr} f(\bar{\psi}\psi) \text{tr} g(\bar{\psi}\psi) \rangle = \langle \text{tr} f(\bar{\psi}\psi) \rangle \langle \text{tr} g(\bar{\psi}\psi) \rangle + \mathcal{O}(1/N^2)
$$

(3.6)

$^1$Such symmetric matrix models are usually referred to as ‘reduced’ matrix models.
This factorization property follows from the existence of a finite large-$N$ limit for the correlators $\langle \text{tr} (\bar{\psi}\psi)^k \rangle$ for arbitrary polynomial potential $V(\bar{\psi}\psi) = \sum_{n \geq 1} \frac{\theta_n}{n}(\bar{\psi}\psi)^n$, since then the connected correlators are given by

$$
\langle \text{tr} (\bar{\psi}\psi)^p \text{tr} (\bar{\psi}\psi)^k \rangle_{\text{conn}} = \langle \text{tr} (\bar{\psi}\psi)^p \text{tr} (\bar{\psi}\psi)^k \rangle - \langle \text{tr} (\bar{\psi}\psi)^p \rangle \langle \text{tr} (\bar{\psi}\psi)^k \rangle
$$

\begin{equation}
= \frac{1}{N^2} p \frac{\partial}{\partial p} \langle \text{tr} (\bar{\psi}\psi)^k \rangle \sim \frac{1}{N^2} \tag{3.7}
\end{equation}

and

$$
\langle \text{tr} (\bar{\psi}\psi)^k \rangle = \frac{1}{N^2} \frac{\partial}{\partial k} \log Z_1 \tag{3.8}
$$

Factorization and symmetry imply that the large-$N$ limit of the model is completely characterized by the set of correlators $\langle \text{tr} (\bar{\psi}\psi)^k \rangle$.

When $N$ is finite, the moment generating function

$$
\omega(z) = \langle \text{tr} \left( \frac{1}{z - \bar{\psi}\psi} \right) \rangle = \sum_{k=0}^{N^2} \langle \text{tr} (\bar{\psi}\psi)^k \rangle \frac{1}{z^{k+1}}
$$

has singularities only at the origin in the complex $z$-plane. The potential $V(z)$ is now a source for the inverse Laplace transformation of the Wilson loop $\text{tr} \frac{z}{z - \bar{\psi}\psi}$,

$$
\text{tr} V(\bar{\psi}\psi) = \int_{0-i\infty}^{0+i\infty} \frac{dz}{2\pi i} V(z) \text{tr} \frac{z}{z^2 - \bar{\psi}\psi} \tag{3.10}
$$

The set of moments can always be obtained from a (not unique) distribution function $\rho$ with support in the complex plane

$$
\langle \text{tr} (\bar{\psi}\psi)^k \rangle = \int d\alpha \rho(\alpha) \alpha^k \quad \text{with} \quad \int d\alpha \rho(\alpha) = 1 \tag{3.11}
$$

The support of $\rho$ can be deduced from the position of the singularities of $\omega$ in (3.9). When $N$ and therefore the number of moments is finite the support of $\rho$ is concentrated near the origin in the complex plane just as in (2.14)

$$
\rho(\alpha) = \langle \text{tr} (\delta(\alpha - \bar{\psi}\psi)) \rangle = \sum_{k=0}^{N^2} \frac{1}{k!} \langle \text{tr} (\bar{\psi}\psi)^k \rangle \left( -\frac{\partial}{\partial \alpha} \right)^k \delta(\alpha) \tag{3.12}
$$

In the large-$N$ limit, the spectral function $\rho(\alpha)$ can be a function with support on some contour in the complex plane. The distribution function $\rho$ is the analog in the fermionic matrix model of the density of eigenvalues in Hermitian one-matrix models as the quantity which completely specifies the solution of the model in the infinite $N$ limit [25].

The generating functions for the connected correlators are

$$
\omega_n(z_1, \ldots, z_n) = \langle \text{tr} \left( \frac{1}{z_1 - \bar{\psi}\psi} \right) \left( \frac{1}{z_2 - \bar{\psi}\psi} \right) \ldots \left( \frac{1}{z_n - \bar{\psi}\psi} \right) \rangle_{\text{conn}} \tag{3.13}
$$

and in the Hermitian case they are associated with the (inverse Laplace transforms of) the sum over discretized open surfaces with $n$ boundaries [74]. When the potential is a polynomial
The single-loop correlator \( \omega_1(z) \equiv \omega(z) \) as in (2.25) is analytic in \( z \) away from the support of \( \rho \) in the complex plane. The distribution function can be determined as before by computing the discontinuity (2.27) of \( \omega(z) \) across its support. Notice that since the signs of the actions in (1.62) and (1.75) are opposite (see (1.80)), the connected correlators of the fermionic matrix model alternate in sign relative to those of the generalized Hermitian Penner model (1.75). This indicates that the large-\( N \) genus expansion of the fermionic matrix model (1.62) is an alternating series. As we shall see, this feature will follow from the different boundary conditions that must be used to define (1.62) and (1.75).

### 3.1 Loop Equations

The loop equation for the single-loop correlator \( \omega(z) \) follows from the identity

\[
\int d\psi \, d\bar{\psi} \, \frac{\partial}{\partial \psi_{ij}} \left[ \left( \psi \frac{1}{z - \bar{\psi}} \right)_{k\ell} \right] e^{N^2 \operatorname{tr} V(\bar{\psi})} = 0
\]  

In contrast to Hermitian matrix models [74], the identity (3.16) is exact for fermionic matrices. Dividing by \( Z_1 \) in (3.16) and expanding out the expectation values gives

\[
0 = \delta_{ik} \left[ \left( \frac{1}{z - \bar{\psi}} \right)_{ji} \right] - \left[ \psi_{k\ell} \left( \frac{1}{z - \bar{\psi}} \right)^2 \right]_{ji} - N \left[ \left( \psi \frac{1}{z - \bar{\psi}} \right)_{k\ell} \left( \bar{\psi} V'(\bar{\psi}) \right)_{ji} \right]
\]  

(3.17)

Setting \( i = k, j = \ell \) and summing over \( i, j = 1 \ldots, N \) then leads to

\[
-z \left( \omega(z)^2 + \omega_2(z, z) \right) + 2\omega(z) + \oint_{C} \frac{d\lambda}{2\pi i} \frac{V'(\lambda)\lambda}{z - \lambda} \omega(\lambda) = 0
\]  

(3.18)

where the contour \( C \) encircles the cut (and possibly pole) singularities of \( \omega(z) \) with counterclockwise orientation and (3.18) should again be solved with the boundary condition (2.26). When the potential is a polynomial of degree \( K \) as in (2.15) (with \( \kappa = 0 \)), then the contour integral in (3.18) can be obtained as in (2.36) and the loop equation (3.18) becomes

\[
-z\omega(z)^2 + (2 - zV'(z)) \omega(z) + V'(z) + P(z) = \omega_2(z, z)
\]  

(3.19)

where \( P(z) \) is a polynomial of degree \( K - 2 \)

\[
P(z) = \sum_{k=2}^{K} g_k \sum_{p=0}^{k-2} \left[ \operatorname{tr} (\bar{\psi} \psi)^{k-1-p} \right] z^p
\]  

(3.20)

\[\text{Note that with this definition we have } \mathcal{L}(z)V(\omega) = \frac{1}{z-\omega} \text{ which acts as a delta-function when integrated along the imaginary axis as in (3.10).}\]
Note that the loop equation (3.18) can also be derived from the Schwinger-Dyson equations expressing the invariance of the partition function (1.62) under arbitrary changes of variables. Under the field transformations

$$\psi \to \psi \left(1 + \epsilon \frac{1}{z(z - \psi \bar{\psi})}\right), \quad \bar{\psi} \to \bar{\psi}, \quad (3.21)$$

where $\epsilon$ is an infinitesimal parameter, the integration measure in (1.62) changes by

$$d\psi \, d\bar{\psi} \to d\psi \, d\bar{\psi} \left(1 - \epsilon \left[N^2 \left(\text{tr} \frac{1}{z - \psi \bar{\psi}}\right)^2 - 2N \text{tr} \frac{1}{z(z - \psi \bar{\psi})}\right]\right) \quad (3.22)$$

and then the invariance of (1.62) to first order in $\epsilon$ under the transformations (3.21) leads directly to (3.18). Notice also that the field transformation (3.21) is similar to the shift (1.83) used to derive the loop equations of the Penner matrix model [12].

Factorization implies that the connected correlators are all suppressed by factors of $1/N^2$ and the term on the right-hand side of the loop equation (3.19) vanishes in the large-$N$ limit. Then the loop equation has the solution

$$\omega(z) = \frac{1}{z} - \frac{V'(z)}{2} + \frac{1}{z} \sqrt{1 + \left(\frac{zV'(z)}{2}\right)^2 + zP(z)} \quad (3.23)$$

where the sign of the square root is chosen to satisfy the asymptotic boundary condition (2.26). The branches of the square root must be placed so that it is negative near the origin in order to cancel the pole at $z = 0$. If the potential is a polynomial of order $K$, then the solution (3.23) in general will possess a square root singularity with $K$ branch cuts and the spectral density $\rho$ will have $K$ contours in its support.

The simplest solution of the model is the one-cut solution which assumes that the singularities of $\omega(z)$ consist of only a single square root branch cut, so that the distribution function $\rho$ has support only on one arc in the complex plane with endpoints at some complex values $a_1$ and $a_2$. The simplest one-cut solution of the homogeneous part of the equation

$$\omega(z + \epsilon) + \omega(z - \epsilon) = 2/z - V'(z) \quad (3.24)$$

is $\sqrt{(z - a_1)(z - a_2)}$. Dividing (3.24) through by this function gives the discontinuity equation

$$\frac{\omega(z + \epsilon)}{\sqrt{(a_1 - z)(a_2 - z)}} - \frac{\omega(z - \epsilon)}{\sqrt{(a_2 - z)(z - a_1)}} = \frac{V'(z) - 2/z}{\sqrt{(a_2 - z)(z - a_1)}} \quad (3.25)$$

from which it follows that the one-cut solution for $\omega(z)$ can be represented in the form

$$\omega(z) = \int_{C_z} \frac{dw}{4\pi i} \frac{V'(w) - 2/w}{z - w} \sqrt{\frac{(z - a_1)(z - a_2)}{(w - a_1)(w - a_2)}} \quad (3.26)$$

where the closed contour $C_z$ encloses the support of the spectral function but not the point $w = z$. The absence of any terms regular in $z$ in (3.26) follows from the large-$|z|$ behaviour of the 1-loop correlator.
The endpoints of the cut can then be found by expanding (3.26) in $\frac{1}{z}$ and imposing the asymptotic boundary condition (2.26) on the solution (3.26), which leads to the two equations

$$0 = \oint_C \frac{dw}{2\pi i} \frac{V'(w) - 2/w}{(w - a_1)(w - a_2)} = \oint_C \frac{dw}{2\pi i} \frac{V'(w)}{\sqrt{(w - a_1)(w - a_2)}} + \frac{2}{\sqrt{a_1 a_2}}$$

(3.27)

$$2 = \oint_C \frac{dw}{2\pi i} \frac{w V'(w) - 2}{\sqrt{(w - a_1)(w - a_2)}} = \oint_C \frac{dw}{2\pi i} \frac{w V'(w)}{\sqrt{(w - a_1)(w - a_2)}}$$

(3.28)

In particular these equations show that the points $a_1$ and $a_2$ cannot lie on the real axis. If they did, the solution (3.26) would have a pole at $z = 0$ and the contour $C$ would encircle the origin. But then (3.27) and (3.28) would have no real solutions. Notice also that in this case the degree-$2K$ polynomial that appears under the square root in (3.23) must have $K - 1$ double roots. This yields $K - 1$ conditions that fully determine the polynomial $P(z)$ in (3.20).

To determine the precise location of the support contour of $\rho$ in the complex plane, we first use the observation [101] that the large-$N$ equation (3.19) for the single-loop correlator is identical to the loop equation for the generalized Penner model (1.75). It follows that the matrix models (1.62) and (1.75) are equivalent at any order of the $\frac{1}{N}$-expansion and therefore belong to the same universality class [12, 101]. Notice, however, that this does not imply that all observables in the 2 models are the same. In particular, the endpoints $a_1$ and $a_2$ of the one-cut ansatz in the fermionic case are complex-valued. The same is true of the spectral distribution function where in addition the requirement of positivity of $\rho$ is lost in the fermionic case. Nevertheless, the solution at $N = \infty$ is the same in both models and this fact can be used to derive some important properties of the one-cut solution for the fermionic one-matrix model.

In particular, in the large-$N$ limit the spectral density therefore obeys the saddle-point equation [25]

$$\frac{2/\beta - V'(\beta)}{2} = \int d\alpha \frac{\rho(\alpha)}{\beta - \alpha} , \quad \beta \in \text{supp} \rho$$

(3.29)

Note that this equation can be obtained from the discontinuity (2.27),(3.24) of the loop correlator (3.23), and it also follows from the local minimization condition $\frac{\partial F_0}{\partial \rho} = 0$ for the free energy

$$F_0 = \lim_{N \to \infty} -\frac{1}{N^2} \log Z_P = \int d\alpha \rho(\alpha) (V(\alpha) - 2 \log \alpha) + \int \int d\alpha d\beta \rho(\alpha) \rho(\beta) \log(\alpha - \beta)$$

(3.30)

with respect to the distribution function $\rho$. Note the change in sign of the fermionic free energy relative to the Hermitian one (compare (1.75) and (1.80)). The double integral in (3.30) is evaluated by integrating up the saddle-point equation (3.29). This introduces a logarithmic divergence at $\beta = 0$ arising from the Penner potential in (1.75) which we remove by subtracting from (3.30) the Gaussian free energy $F_G$ defined by setting $V(\bar{\psi} \psi) = g_1 \bar{\psi} \psi$ in (1.62)

$$F_0 - F_G = \frac{1}{2} \int d\alpha \rho(\alpha) (V(\alpha) - 2 \log \alpha) + \int d\alpha \rho(\alpha) \log \alpha$$

(3.31)

where we have ignored terms independent of the general potential couplings $g_k, k > 1$, in (3.30).

The support contour of $\rho$ can now be determined from the David primitive function [37]

$$G(w) = \int_{a_1}^{w} dz \left( \frac{2}{z} - V'(z) - 2\omega(z) \right)$$

(3.32)
The branch points of $\omega(z)$ in (3.23) (i.e. the solutions to (3.27) and (3.28)) fix the endpoints of the support of $\rho$, but not its position in the complex plane. From (3.23) and the analogy above with the Hermitian Penner matrix model (for which $\rho$ is positive and real-valued) it follows that the support of $\rho$ is an arc connecting $a_1$ to $a_2$ in the complex plane along which $G(w)$ is purely imaginary and which can be embedded in a region where $\text{Re } G(w) < 0$ \footnote{This first property of $G(w)$ follows from the Hermitian matrix model definition $\rho(\lambda) \equiv (\frac{d\lambda(x)}{dx})^{-1} > 0$. The second property follows from the fact that global variations along $\text{supp } \rho$ of the planar free energy $F_0$ in (3.30) are proportional to $G$, $\delta F_0 \propto \delta G$, so that positivity of the real part of $\delta F_0$ ensures global stability of the ground state solution (3.30) determined by $\rho$.}. This feature, however, depends strongly on the boundary conditions used in (1.75) \cite{37}.

The general $n$-loop correlators (3.13) in the spherical approximation can also be found by applying the loop insertion operators (3.15) $n - 1$ times to $\omega(z)$ as prescribed by (3.14). For example, applying the differential operator $\mathcal{L}(z)$ to the boundary conditions (3.27) and (3.28) we can evaluate $\mathcal{L}(z) a_i$ as

$$ (\mathcal{L}(z)a_1) \cdot \oint \frac{dw}{2\pi i} \frac{V'(w) - 2/w}{(w-a_1)^{3/2}(w-a_2)^{1/2}} \frac{1}{(z-a_1)^{3/2}(z-a_2)^{1/2}} \quad (3.33) $$

where we have used the identity

$$ (\mathcal{L}(w)V'(z)) = \frac{\partial}{\partial z} \frac{1}{w-z} \quad (3.34) $$

and $\mathcal{L}(z)a_2$ is obtained from (3.33) by interchanging $a_1$ and $a_2$. Then applying $\mathcal{L}(w)$ to the one-cut solution (3.26) we arrive at the two-loop correlator

$$ \omega_2(z,w) = \frac{1}{N^2} \oint C_{z,w} \frac{dw'}{4\pi i} \frac{1}{(z-w')(w-w')^2} \frac{(w' - a_1)(w' - a_2)}{(z-a_1)(z-a_2)} $$

$$ = \frac{1}{N^2} \frac{1}{4(w-z)^2} \left( \frac{(w-a_1)(z-a_2) + (w-a_2)(z-a_1)}{\sqrt{(w-a_1)(w-a_2)(z-a_1)(z-a_2)}} - 2 \right) \quad (3.35) $$

(3.35) is identical to the 2-loop correlator of the Hermitian one-matrix model [11]. Therefore all the multi-loop correlators (3.13) for $n \geq 2$ are the same as those in the Hermitian 1-matrix model with the same polynomial potential $V$ (and when $V$ is odd in the fermionic case these correlators are the same as those for a Hermitian model with a symmetric potential). As we shall discuss in Subsection 3.4, this indicates a certain equivalence between the genus expansions of the fermionic and Hermitian one-matrix models. As (3.13) represents the complete set of operators for the fermionic matrix model, the loop equation (3.18) therefore determines the complete set of equations of motion of the model. Notice that the 2-loop correlator (3.35) depends on the potential $V$ in (1.62) only implicitly through the endpoints $a_1$ and $a_2$ (but not explicitly). This is not so for the higher-order multi-loop correlators of the matrix model [11]. The system of standard Schwinger-Dyson equations for the connected correlators of the model $\langle \prod_{j=1}^3 \text{tr } (\tilde{\psi}\tilde{\psi}^j) \rangle_{\text{conn}}$ can now be obtained by expanding the multiloop correlators (3.13) in powers of $\frac{1}{z_1}, \ldots, \frac{1}{z_n}$ and using the loop equation (3.18).
The solution of the model for the Gaussian potential (2.46) is
\[ \omega(z) = \frac{1}{z} - \frac{t}{2} + \frac{1}{2z} \sqrt{4 + t^2 z^2} \]  
(3.36)
and the distribution function is
\[ \rho(\alpha) = \frac{t}{2\pi i} \sqrt{1 + \frac{4}{t^2 \alpha^2}} \quad ; \quad \alpha \in \text{supp } \rho \]  
(3.37)
This is the analog in the fermionic case of the Wigner semi-circle law for a Gaussian distribution of Hermitian random matrices [25]. As mentioned already, one of the crucial features of the fermionic models, as compared to Hermitian models, is that the endpoints of the support region of the spectral distribution function lie off of the real axis and the support contour is in general embedded in some region of the complex plane. In the present case the endpoints are situated on the imaginary axis at \( \pm 2i/t \), and to satisfy (2.25) and the normalization condition \( \int d\alpha \rho(\alpha) = 1 \) the support contour connecting the points \( \pm 2i/t \) must be chosen so that it avoids the origin in the complex \( \alpha \)-plane.

To determine the precise support contour for \( \rho \) which connects these points, we evaluate the David function (3.32)
\[
G(z) = -\int_{\frac{2}{|t|}}^{z} \frac{dw}{w} \sqrt{4 + t^2 w^2} \\
= -\sqrt{4 + t^2 z^2} - \text{sgn}(t) \log \left( \frac{\sqrt{4 + t^2 z^2} - 2}{\sqrt{4 + t^2 z^2} + 2} \right) - i\pi \text{ sgn } t
\]  
(3.38)
where the branch cut of the square roots in (3.38) is taken to be the straight line joining the points \( \pm 2i/t \). Notice that \( \text{Re } G(z) \to \mp \infty \) as \( z \to \pm \infty \) and \( \text{Re } G(z) \to +\infty \) as \( z \to 0 \). A careful study of the equation \( \text{Re } G(z) = 0 \) and of the region where \( \text{Re } G(z) < 0 \) shows that the support contour of \( \rho \) cannot cross the imaginary axis for \( |\text{Im } z| > 2/|t| \) and that it crosses the real axis at some non-zero values of order \( \pm 1/|t| \). The regions \( \text{Re } G(z) < 0 \) are to the right of these crossing points (but note that \( \text{Re } G(z) \) changes sign across \( \text{supp } \rho \)). Thus the support contour of (3.37) can be taken to be the counterclockwise oriented half circle of radius \( 2/|t| \) in the first and fourth quadrants of the complex \( \alpha \)-plane. It is easy to verify that with this definition of \( \rho \) the equations (3.11) and (2.25) are indeed satisfied. Again, since the free energy is analytic in the couplings for this simple Gaussian case (i.e. a free fermion field theory), there is no critical behaviour in this model.

### 3.3 Critical Behaviour of a Non-Gaussian Model

We now analyse some non-trivial polynomial potentials at large-\( N \) and discuss the ensuing phase structure of the model.
3.3.1 The Cubic Potential

The simplest symmetric case is the cubic potential

\[ V(z) = tz + \frac{g}{3}z^3 \]  

(3.39)

for which

\[ \omega(z) = \frac{1}{z} - \frac{t}{2} - \frac{g z^2}{2} + \frac{1}{2z} \sqrt{g^2 z^6 + 2tgz^4 + (t^2 + 4g\xi)z^2 + 4} \]  

(3.40)

where \( \xi \) is the as yet unknown correlator

\[ \xi = \langle \text{tr} \bar{\psi}\psi \rangle \]  

(3.41)

In this case the vanishing of all even moments, \( \int da \rho(a)\alpha^{2k} = 0 \), implies that the endpoints of the support contour of the continuous function \( \rho \) lie in the complex plane and are symmetric on reflection through the origin. Furthermore, an application of Wick’s theorem shows that the series (3.9) in the odd moments is alternating.

Generically the square root in \( \omega(z) \) has three branch cuts, so that in the general case the distribution function \( \rho \) will have three disjoint and symmetric (about the origin) support contours. The one-cut solution for (3.40) takes the form

\[ \omega(z) = \frac{1}{z} - \frac{t}{2} - \frac{g z^2}{2} + \frac{g z^2}{2z} \sqrt{z^2 + 4/b^2} \]  

(3.42)

where comparing the polynomial coefficients in (3.42) with those of (3.40) shows that the parameter \( b \) and the correlator \( \xi \) are determined by the two equations

\[ \pm b^3 - tb^2 + 2g = 0 \]  

(3.43)

\[ b^3 - (t^2 + 4g\xi)b \pm 8g = 0 \]  

(3.44)

The sign ambiguity here can be eliminated by requiring that at \( g = 0 \) the correct Gaussian value \( b(g = 0, t) = t \) for \( b \) be attainable. This is the boundary condition that is relevant for an interpretation of this matrix model as a discretized random surface theory, i.e. for a consistent perturbative expansion of the model in the coupling constant \( g \). It means that we take the positive sign in the above equations. The choice of negative sign yields solutions with boundary conditions at \( g = 0 \) appropriate to generalized Penner models [12] (e.g. they yield real-valued endpoints for \( \text{supp} \rho \)). This can also be seen directly from the contour integrals (3.27) and (3.28) by computing the residues at \( \infty \). For any odd polynomial potential, (3.28) is an identity since there is no residue at infinity, while for the potential (3.39) the equation (3.27) yields exactly (3.43) with the sign ambiguity arising from the possible choices of sign of the square root \( \sqrt[3]{a_1a_2} \).

We assume henceforth that \( t \) is a positive constant. The 3 solutions of (3.43) are

\[ b_0(x, t) = \frac{t}{3} \left( \beta^{1/3}(x) + \beta^{-1/3}(x) + 1 \right) \]  

(3.45)

\[ b_{\pm}(x, t) = \frac{t - b_0(x, t)}{2} \pm \frac{i\sqrt{3}t}{6} \left( \beta^{1/3}(x) - \beta^{-1/3}(x) \right) \]  

(3.46)
where
\[
\beta(x) = 2x - 1 + 2\sqrt{x(x-1)}
\]  
(3.47)
and we have introduced the dimensionless scaling parameter
\[
x = 1 - 27g/2x^3 \equiv 1 - g/g_c
\]  
(3.48)

When \(x \leq 0\) (\(g \geq g_c \equiv 2^{9/7}\)) or \(x \geq 1\) (\(g \leq 0\)), \(\beta(x)\) is a monotone real-valued function with \(\beta(x) \geq 1\) for \(x \geq 1\) and \(\beta(x) \leq -1\) for \(x \leq 0\). In the region \(0 < x < 1\) (\(0 < g < g_c\)), \(\beta(x)\) is a unimodular complex-valued function. The function (3.45) is always real-valued and the region \(0 < x < 1\) is the region wherein all 3 roots (3.45), (3.46) of the cubic equation (3.43) are real. These 3 roots can all be obtained from (3.45) by choosing the 3 inequivalent cube roots of \(\beta(x)\). For \(x \notin (0, 1)\) the solutions (3.46) are complex.

For the fermionic matrix model, where the distribution function \(\rho\) can be complex-valued, there is no immediate reason to disregard generic complex-valued endpoints for the support of \(\rho\). However, the free energy (3.31) for the cubic potential (3.39) up to terms independent of \(b\) and \(g\) is
\[
F_b(x, t) - F_c(t) = \frac{t(3b(x, t) - t)}{6b^2(x, t)} + \log |b(x, t)|
\]  
(3.49)
where we have used the spectral density determined by (2.27) and (3.42) as
\[
\rho(\alpha) = \frac{1}{2\pi i} \left( b + g\alpha^2 \right) \sqrt{1 + \frac{4}{b^2\alpha^2}} ; \; \alpha \in \text{supp } \rho
\]  
(3.50)
with \(b(g, t)\) given by (3.45) and (3.46) (or equivalently the moments \(\xi\) and \(\langle \text{tr} (\bar{\psi}\psi)^3\rangle\) determined by expanding (3.42) to order \(1/\pi\)). In arriving at (3.49) we have used the boundary conditions (3.43) and (3.44), and the fact that in the difference between the two logarithmic integrations in (3.31) only the residue at \(\alpha = 0\) survives. The free energy (3.49) is complex-valued for the complex values (3.46) of \(b(x, t)\) for \(x \notin (0, 1)\). Such a free energy corresponds to an unstable state and we therefore consider only the real-valued solutions to (3.43). The support contour on which (3.50) is defined is again found from the David function (3.32) which for the cubic potential (3.39) has the same qualitative properties as (3.38) [105]. Thus the support contour in (3.50) can be taken as the counterclockwise oriented half-circle of radius \(2/|b|\) in the first and fourth quadrants of the complex \(\alpha\)-plane. The boundary condition (3.44) now follows from evaluating the correlator \(\xi = \int d\alpha \; \rho(\alpha)\alpha\) with this distribution function.

There are 2 critical points in this large-\(N\) matrix model, at \(g = 0\) and \(g = g_c\), which separate 3 phases determined by the analytic structure of the function (3.47), i.e. the one-cut solution is a non-analytic function of \(x\) about \(x = 0\) and \(x = 1\) where it acquires a square root branch cut. For \(x \geq 1\) the solution
\[
b_0^+(x, t) = \frac{t}{3} \left[ 1 + \left( 2x - 1 + 2\sqrt{x(x-1)} \right)^{1/3} + \left( 2x - 1 + 2\sqrt{x(x-1)} \right)^{-1/3} \right]
\]  
(3.51)
of (3.43) satisfies the Gaussian boundary condition \(b_0(x = 1, t) = t\). When \(x \leq 0\) the real solution for \(b\) is
\[
b_0^-(x, t) = \frac{t}{3} \left[ 1 - \left( 2x - 1 + 2\sqrt{x(x-1)} \right)^{1/3} - \left( 2x - 1 + \sqrt{x(x-1)} \right)^{-1/3} \right]
\]  
(3.52)
As \( x \) is varied between 0 and 1, \( \beta(x) \) has modulus one and phase which varies from \( \pi \) to 0, i.e. \( \beta(x) = e^{i\phi(x)} \) where

\[
\phi(x) = \arctan \left( \frac{2\sqrt{x(1-x)}}{2x-1} \right) \in [0, \pi]
\]  

(3.53)

The arctangent function in (3.53) is well-defined only up to an integral multiple of 2\( \pi \), and the three real solutions for \( b \) are

\[
b^{(n)}(x, t) = \frac{t}{3} \left[ 1 + 2\cos \left\{ \frac{1}{3} \arctan \left( \frac{2\sqrt{x(1-x)}}{2x-1} \right) + \frac{2n\pi}{3} \right\} \right]
\]  

(3.54)

where \( 0 < x < 1 \) and \( n = 0, 1, 2 \). The branch which matches (3.52) is the one with \( n = 1 \), whereas the branch which matches (3.51) is the one with \( n = 0 \). The branch with \( n = 2 \) does not connect with either solution. Any of these 3 branches can be used to define the one-cut solution (3.42). The free energy (3.49) is positive for all \( x \in (0, 1) \) for the \( n = 0 \) branch, negative for all \( x \in (0, 1) \) for the \( n = 1 \) branch, and for the \( n = 2 \) branch it is positive for \( 0 < x < \frac{1}{2} \) and flips sign for the rest of the interval at \( x = \frac{1}{2} \). Thus the \( n = 1 \) branch in (3.54) is the ground state solution in the region \( 0 < x < 1 \).

The free energy associated with this stable one-cut solution is discontinuous across \( g = 0 \), and thus with this choice of branch in the regime \( 0 < x < 1 \) the Gaussian point of this matrix model is a critical point of a first order phase transition. The other one-cut solution which is a perturbation of the Gaussian solution is metastable but can still be thought of as a valid solution of the model since the energy barrier between the stable and metastable one-cut solutions is infinite at \( N = \infty \) (the height of the barrier is of order \( N^2 \)). Ordinarily, the infinite energy barrier prevents tunneling and also a phase transition from occurring [37]. However, if we restrict attention to one-cut solutions and follow them over the range of \( g \), we must encounter a discontinuity of the free energy somewhere, i.e. a first order phase transition.

This is similar to the situation in the Hermitian one-matrix model with symmetric polynomial potential of degree 6 [72]. There a phase transition occurs due to an infinite volume effect, as opposed to a large-\( N \) effect where the only possibilities could be second or third order phase transitions. There is also the possibility that the loop correlator (3.40) evolves into a three-cut phase at \( g = 0 \), corresponding to a third order phase transition (see below), but there is no immediate indication of this since in the fermionic case the spectral measure \( \rho(a) dx \) need not be positive. This possibility is also suggested by the exact form (3.40) of the loop correlator. Although the one-cut ansatz (3.42) is insensitive to a change in sign of \( g \), the analytic properties of (3.40) are affected by the passage through \( g = 0 \) (i.e. the sign of the square root flips in order to satisfy the boundary condition (2.26)). The model therefore cannot be analytically continued to negative values of \( g \), and the resulting peculiarities in the \( \frac{1}{N} \)-expansion of the model are related to the occurrence of complex-valued endpoints for the

\[ g_c \]
support of the distribution function $\rho$. The existence of 3 phases in this matrix model and the possibility of a first order phase transition at the Gaussian point $g = 0$ are completely unlike what occurs in the conventional polynomial Hermitian matrix models [38, 74] or in Penner models [12, 28, 130]. Notice, however, that the free energy (3.49) with the choice of stable branch for $x \in (0,1)$ is continuous across the critical point $g = g_c$.

The scaling behaviour of the matrix model in the vicinity of its critical points is determined by the string susceptibility

$$\chi(g, t) = -\frac{1}{N^2} \frac{\partial^2 \log Z_1}{\partial g^2} = -\frac{1}{3} \frac{\partial}{\partial g} \left\langle \text{tr} (\tilde{\psi} \psi)^3 \right\rangle$$  \hspace{1cm} (3.55)

where the correlator $\left\langle \text{tr} (\tilde{\psi} \psi)^3 \right\rangle$ can be read off from the $\frac{1}{z^s}$ coefficient of the large-$z$ expansion of (3.42). The critical exponent $\gamma_{str}^{(i)}$ at each critical point $g_c^{(i)}$ is defined by the leading non-analytic behaviour of (3.55) [38]

$$\chi(g, t) \sim_S (g - g_c^{(i)})^{-\gamma_{str}^{(i)}} \ \text{as} \ \ g \to g_c^{(i)}$$  \hspace{1cm} (3.56)

where $\sim_S$ denotes the most singular part of the function in a neighbourhood of the critical point. In terms of the scaling variable (3.48), the susceptibility (3.55) is

$$\chi(x, t) = \frac{1}{3} \frac{\partial}{\partial x} \left( \frac{10(x - 1)}{9b^6(x, t)} - \frac{4}{g_c b^3(x, t)} \right)$$

$$= \frac{972}{t^6(\beta^2 \beta^3 + \beta^1 \beta^3 + 1)^6} \left[ 1 - 8x + 8x^2 + 4\sqrt{\frac{x}{x - 1}} \left( 1 - 3x + 2x^2 \right) \right]$$  \hspace{1cm} (3.57)

From (3.57) we find that the leading singular parts of the susceptibility near each of the two critical points $g = g_c$ and $g = 0$ are respectively

$$\chi(x, t) \sim_S -\frac{15552}{t^6} \sqrt{x} \ \text{as} \ \ x \to 0$$  \hspace{1cm} (3.58)

$$\chi(x, t) \sim_S \frac{11648}{3t^6} \sqrt{x - 1} \ \text{as} \ \ x \to 1$$  \hspace{1cm} (3.59)

Both critical points therefore have string constant

$$\gamma_{str} = -1/2$$  \hspace{1cm} (3.60)

which coincide with those of the usual $m = 2$ quantum gravity models [74].

In particular, this shows that with the choice of stable branch for $x \in (0,1)$ the phase transition at the non-zero critical coupling $g = g_c$ is of third order. Notice that the spectral density at this critical point is given by

$$\rho_s(\alpha) = \frac{g_c}{2\pi i \alpha} \left( \alpha^2 + \frac{27b_c}{2\beta^3} \right) \sqrt{\alpha^2 + \frac{4}{b_c^2}}$$  \hspace{1cm} (3.61)

---

5In polynomial Hermitian matrix models criticality is the result of $m$ zeroes of $\rho(\alpha)$ coalescing with one of the endpoints of supp $\rho$ with string constant $\gamma_{str} = -1/m$. For generalized Penner models, in addition to this multi-critical behaviour there are critical points with $\gamma_{str} = 0$ for which logarithmic scaling violations occur [46] and which are the result of the coalescence of two endpoints of supp $\rho$ [12, 28, 130]. In these cases, the multi-critical coupling constants are negative and separate the unique one-cut phase from a multi-cut phase [25, 38]. In the simplest symmetric case of a quartic plus quadratic interaction, the endpoint of the one-cut solution obeys a quadratic equation whose 2 solutions coalesce at criticality (as in the vector model of Section 2 above).
where \( b_c(t) = b(g_c, t) \). For either the \( n = 0 \) or \( n = 2 \) branches in (3.54) we find \( b_c(t) = 2t/3 \), and therefore a zero of \( \rho(\alpha) \) at criticality coalesces with each of the symmetrical endpoints of its support. The critical point \( g = g_c \) therefore enjoys all of the properties of a conventional \( m = 2 \) multi-critical point [38, 74]. It represents a third order phase transition with string susceptibility exponent \( \gamma_{\text{str}} = -1/2 \), at criticality the zeroes of the spectral distribution function coalesce with the endpoints of the cut, and the scaling behaviour of functions near this critical point coincides with that of the Hermitian one-matrix model with symmetric quartic potential. The 2 critical points of the fermionic one-matrix model, which arise as those points in parameter space where the cubic equation (3.43) which determines the one-cut solution with real free energy exists\(^6\), while in the phase \( x \in (0, 1) \) a multi-cut solution as well as the multi-branch one-cut solutions can in addition exist. The scaling behaviours (3.58) and (3.59) indicate that the 2 transitions into the multi-cut phase would both be of third order, while the transitions into the stable one-cut phase are of first and third order. The existence of a single-cut or multi-cut phase in the region \( 0 < x < 1 \) is determined by which one of these 2 possibilities is in fact the vacuum state. It would be interesting to investigate this point further, although there is no immediate way to determine the various parameters of the three-cut ansatz due to the appearance of the unknown correlator \( \xi \) in (3.40). The appearance of this unknown variable is another one of the distinguishing analytic features of the fermionic matrix models.

### 3.3.2 General Polynomial Potentials

We now discuss the critical behaviour associated with higher order potentials. For simplicity we consider the chirally symmetric case where the potential (2.15) is a generic odd polynomial, i.e. \( \kappa = g_{2k} = 0 \) for all \( k \) in (2.15), with \( K = \deg V > 3 \) an odd integer. The solution for the loop correlator is then

\[
\omega(z) = \frac{1}{z} - \sum_{k=1}^{K+1} \frac{g_{2k-1} z^{2(k-1)}}{2} + \frac{1}{2z} \left( 4 + \sum_{k, m=1}^{K+1} g_{2k-1} g_{2m-1} z^{2(k+m-1)} + \sum_{k, m=1}^{K+1} g_{2(k+m)-1} \xi_{2m} z^{2k} \right)^{1/2} \tag{3.62}
\]

where \( \xi_{2m} \) are the as yet unknown moments \( \xi_{2m} = \langle \text{tr} \ (\bar{\psi}\psi)^{2m-1} \rangle \). The one-cut solution for (3.62) takes the form

\[
\omega(z) = \frac{1}{z} - \sum_{k=1}^{K+1} \frac{g_{2k-1} z^{2(k-1)}}{2} + \frac{1}{2z} \left( g_K z^{K-1} + \sum_{k=1}^{K-3} a_{2k} z^{2k} + b \right) \sqrt{z^2 + 4/b^2} \tag{3.63}
\]

where we have fixed the sign in front of the endpoint parameter \( b \) by the same convention as before. The one-cut ansatz (3.63) along with the general solution (3.62) together involve

\(^6\)The solution of a Riemann-Hilbert problem is always unique [25].
$K-1$ unknown parameters - $b$, the $(K-3)/2$ polynomial coefficients $a_{2k}$, and the $(K-1)/2$ correlators $\xi_{2k}$.

These parameters can be found by comparing the various polynomial coefficients of (3.62) with those of (3.63), which leads to the set of equations

$$b^3 + 8a_2 - b \left( g_1 + \sum_{k=1}^{K-1} g_{2k+1} \xi_{2k} \right) = 0$$ (3.64)

$$2b^3 + 8ba_4 + 4a_2^2 - b^2 \left( 2g_1g_3 + \sum_{k=1}^{K-3} g_{2k+3} \xi_{2k} \right) = 0$$ (3.65)

$$2bg_K a_{K-5} + 8a_{K-3} - b \sum_{k=1}^{K-2} g_{2k-1} g_{2K-2k-3} = 0$$ (3.66)

$$2b^2 g_K a_{K-3} + 4a_{K-3}^2 - b^2 \sum_{k=1}^{K-1} g_{2k-1} g_{2K-2k-1} = 0$$ (3.67)

$$2b^2 a_{K-3} + 8g_K - b \left( g_K \xi + \sum_{k=1}^{K-3} g_{2k-1} g_{2K-2k} \right) = 0$$ (3.68)

$$2b^3 g_K + 8bg_K a_2 - b^2 \sum_{k=1}^{K+1} g_{2k-1} g_{2K-2k+2} = 0$$ (3.69)

When $K > 7$ we have in addition the sets of equations

$$2b^3 a_{2(m-1)} + 8ba_{2m} + 4a_2 a_{2(m-1)} + \sum_{k=1}^{m-2} \left( b^2 a_{2k} a_{2(m-k-1)} + 4a_{2k} a_{2(m-k)} \right)$$

$$-b^2 \left( \sum_{k=1}^{m} g_{2k-1} g_{2(m-k)+1} + \sum_{k=1}^{K+1-2m} g_{2(m+k)-1} \xi_{2k} \right) = 0$$ (3.70)

for $3 \leq m \leq \frac{K-3}{2}$, and

$$2b^2 g_K a_{2m-K-1} + 8ba_{2m-K+1} + b^2 \sum_{k=1}^{m-2} a_{2k} a_{2(m-k-1)}$$

$$+ 4 \sum_{k=1}^{m-1} a_{2k} a_{2(m-k)} - b^2 \sum_{k=1}^{m} g_{2k-1} g_{2(m-k)+1} = 0$$ (3.71)

for $\frac{K+3}{2} \leq m \leq K-3$.

(3.64)–(3.71) yield a complete set of equations for the $K-1$ unknown coefficients of the one-cut solution (3.63) in terms of the coupling constants of the potential (2.15). The parameter $b$ can alternatively be found from the contour integral (3.27) which leads to a $K$-th order equation for $b$

$$b^K + \sum_{k=1}^{K+1} \frac{(-1)^k 2^{k-1} (2k - 3)!!}{(k-1)!} g_{2k-1} b^{2k+1} = 0$$ (3.72)
Since $K$ is odd this equation always has a real solution, and as before the one-cut solution can always be constructed. The spectral density is

$$\rho(\alpha) = \frac{1}{2\pi i} \left( g_K \alpha^{K-1} + \sum_{k=1}^{[K/2]} a_{2k} \alpha^{2k} \right) \sqrt{1 + \frac{4}{b^2 \alpha^2}} ; \quad \alpha \in \text{supp} \rho$$  \hspace{1cm} (3.73)

The first $(K-1)/2$ moments of this distribution function are given by the solutions to (3.64)–(3.71).

In general (3.72) will acquire multiple real roots at some coupling $g_{c_{2k-1}}$ which will be a critical point of a third order phase transition with string constant $\gamma_{\text{str}} = -1/2$. We expect that the phase with multiple real roots will be bounded by other coupling constant values so that the model will contain several critical points corresponding possibly to different order phase transitions. Notice that since the potential now depends on more parameters, we can adjust them in such a way that $m-1$ regular zeroes of (3.73) coalesce with a cut end-point $\pm 2i/b$ at criticality for the same critical coupling $g_{c_{2k-1}}$, i.e. so that

$$\rho_c(\alpha) \sim (\alpha^2 + 4/b^2 c)^{m-1/2}$$  \hspace{1cm} (3.74)

where we neglect possibly other irrelevant zeroes. $g_{c_{2k-1}}$ will then be an $m$-th order multi-critical point [38, 74] with susceptibility exponent

$$\gamma_{\text{str}} = -1/m$$  \hspace{1cm} (3.75)

### 3.4 The Topological Expansion

The fermionic one-matrix model possesses a novel critical behaviour which includes the usual multi-critical behaviour that occurs in Hermitian one-matrix models, and furthermore, in the case of the simplest symmetric potential, there may also be a first order phase transition at zero coupling. This would imply that the perturbative expansion of the theory in $g$ has a preferred direction through values of the coupling with a definite sign (corresponding to $-\text{sgn} \, t$). It means that perturbation theory near the Gaussian point $g = 0$ does not correctly reflect the properties of the theory when $g$ is small and negative and this fact is important for the interpretation of the fermionic one-matrix model as a statistical theory of discretized random surfaces (because then the model cannot be continued to values of couplings with $\text{sgn} \, g = -\text{sgn} \, t$). As mentioned before, this effect seems to be merely an artifact of the fermionic nature of the matrix degrees of freedom here. The other critical point, which is the usual $m$-th order multi-critical point with third order phase transition and string susceptibility with critical exponent $\gamma_{\text{str}} = -1/m$, gives the continuum limit of the topological genus expansion relevant to string theory. In the Hermitian case, this continuum limit corresponds to $D \leq 1$ modes of pure 2-dimensional quantum gravity [38, 74]. We shall now examine the topological $\frac{1}{N}$-expansion of the fermionic one-matrix model, which determines explicitly the number of 't Hooft diagrams of a given genus, and present the argument [12] indicating why one expects that this results in a genus expansion which is an alternating series but otherwise coincides with the usual Painlevé expansion [38] (but otherwise no conclusive argument is available as of yet).

The crucial point of the argument is the identification of the fermionic matrix model (1.62) with the generalized Penner model (1.75) and the fact that the multi-critical points of the
adjoint fermion model belong to the same universality class as those of the Hermitian one-matrix model with the same potential $V$ [12, 38, 101]. In particular, the value of $\gamma_{\text{str}}$ for the $m$-th multi-critical points is the same to all genera (i.e. powers of $1/N^2$) in the 2 models [12]. However, we expect that the fermionic genus series alternates relative to the Hermitian case because the branch points of the one-loop correlator (3.23) are complex-valued in the fermionic case. Using the usual identification with the Penner matrix model, it follows that the moment functions

$$M_k = \oint_C \frac{dw}{2\pi i} \frac{V'(w) - 2/w}{(w - a_2)^{k+1/2}(w - a_1)^{1/2}}, \quad J_k = \oint_C \frac{dw}{2\pi i} \frac{V'(w) - 2/w}{(w - a_2)^{1/2}(w - a_1)^{k+1/2}} \quad (3.76)$$

defined for $k \geq 0$ can be used to determine the critical points of the matrix model [14]. We restrict attention as above to a generic chirally symmetric potential, so that $a_2 = - a_1 = iy$ where $y = 2/|b| \in \mathbb{R}^+$. From (3.23) and (3.26) it follows that the 1-cut ansatz for the loop correlator is

$$\omega(z) = \frac{1}{z} - V'(z) + M(z)\sqrt{z^2 + y^2} \quad (3.77)$$

where

$$M(z) = \oint_C \frac{dw}{2\pi i} \frac{1}{z - w} \frac{V'(z) - V'(w) + 2/w - 2/z}{\sqrt{w^2 + y^2}} \quad (3.78)$$

and $zM(z)$ is a polynomial of degree $K - 1$. The algebraic coefficients of (3.78) can be found by calculating the residue of the contour integral there at infinity (see the last Subsection). At an $m$-th multi-critical point, $m - 1$ zeroes of $M(z)$ coalesce with the branch point $y$. From (3.26) we then have

$$M_k \propto \frac{\partial^{k-1} M(z)}{\partial z^{k-1}} \bigg|_{z = iy}, \quad k \geq 1 \quad (3.79)$$

and so the condition for the $m$-th multi-critical point is equivalent to the requirement that [14]

$$M_1(iy_c) = M_2(iy_c) = \ldots = M_{m-1}(iy_c) = 0, \quad M_m(iy_c) \neq 0 \quad (3.80)$$

where $y_c$ denotes the endpoint parameter $y$ at criticality.

In the case at hand, the moment functions in (3.76) are related by

$$M_k = (-1)^k J_k \quad (3.81)$$

and moreover we have

$$M_k = (-i)^k \tilde{M}_k \quad (3.82)$$

where

$$\tilde{M}_k = \sum_{n \geq 0} (-1)^n g_{2n+1} y^{2n+1} \oint_C \frac{dw}{2\pi i} \frac{1}{(w - y)^{k+1/2}(w + y)^{1/2} + \frac{2}{y(-y)^k}} \quad (3.83)$$

is real-valued. In particular, the 0-th order moment function is

$$M_0 = \sum_{n \geq 0} (-1)^{n+1} g_{2n+1} \frac{(2n)!}{2^{2n}(n!)^2} y^{2n} + \frac{2}{y} \quad (3.84)$$

and the higher order moments are related to it by

$$M_k = \frac{2^k}{(2k - 1)!} \frac{\partial^k}{\partial a_2^k} M_0 \bigg|_{a_2 = -a_1 = iy}, \quad k \geq 1 \quad (3.85)$$
We also introduce the cut parameter
\[ d = a_2 - a_1 \] (3.86)
which in the present case is purely imaginary
\[ d = 2iy \equiv id \] (3.87)

The free energy \( F_P = \frac{1}{N^2} \log Z_P \) admits the genus expansion
\[ F_P = \sum_{h=0}^{\infty} \frac{1}{N^{2h}} F_h \] (3.88)
where the genus 0 free energy \( F_0 \) is given by (3.31). From (3.14) it follows that the one-loop correlator \( \omega(z) \) has the genus expansion
\[ \omega(z) = \sum_{h=0}^{\infty} \frac{1}{N^{2h}} \omega^{(h)}(z) \] (3.89)
where
\[ \omega^{(h)}(z) = N^2 L(z) F_h \] (3.90)
The higher genus contributions to the single-loop correlator are obtained by iterating the genus zero contribution \( \omega^{(0)}(z) \) (i.e. the \( N = \infty \) solution to (3.18)). Substituting the genus expansion (3.89) into (3.18) and equating the \( \frac{1}{N^{2h}} \) coefficients, we obtain an iterative equation for \( h \geq 1 \)
\[ -z \left( 2\omega^{(0)}(z) \omega^{(h)}(z) + \sum_{h' = 1}^{h-1} \omega^{(h')}(z) \omega^{(h-h')}(z) + N^2 L(z) \omega^{(h-1)}(z) \right) + 2 \omega^{(h)}(z) + \oint_{C} \frac{d\lambda}{2\pi i} \frac{V'(\lambda)\lambda}{z - \lambda} \omega^{(h)}(\lambda) = 0 \] (3.91)
which determines \( \omega^{(h)}(z) \) entirely in terms of \( \omega^{(h')}(z) \) with \( h' < h \). The solution \( \omega^{(h)}(z) \) to (3.91) can be expressed in terms of \( 2 \cdot (3h - 1) \) lower moment functions (3.76) [14].

For instance, to find \( \omega^{(1)}(z) \) in the one-cut phase, from (3.35) we have
\[ L(z) \omega^{(0)}(z) = \omega_2(z, z) = \frac{1}{N^2 16(z - a_1)^2(z - a_2)^2} \] (3.92)
and so (3.91) for \( h = 1 \) yields
\[ \omega^{(1)}(z) = \frac{1}{\sqrt{(z - a_1)(z - a_2)}} \oint_{C} \frac{dw}{2\pi i} \frac{1}{(w - z)M(w)} \frac{(a_1 - a_2)^2}{16(w - a_1)^2(w - a_2)^2} \] (3.93)
which is unambiguous provided that it is analytic at the zeroes of \( M(z) \). After some algebra it can be written as [14]
\[ \omega^{(1)}(z) = \frac{1}{8dM_1} \frac{1}{\sqrt{(z - a_1)(z - a_2)^3}} - \frac{1}{8dJ_1} \frac{1}{\sqrt{(z - a_1)^3(z - a_2)}} + \frac{1}{16M_1} \left( \frac{1}{\sqrt{(z - a_1)(z - a_2)^5}} - \frac{M_2}{8dM_1 \sqrt{(z - a_1)(z - a_2)^3}} \right) + \frac{1}{16J_1} \left( \frac{1}{\sqrt{(z - a_1)^5(z - a_2)}} + \frac{J_2}{8dJ_1 \sqrt{(z - a_1)^3(z - a_2)}} \right) \] (3.94)
In the case of the cubic potential (3.39) we have

\[ M(z) = (g z^2 + b)/2z \]  

so that (3.93) is

\[ \omega^{(1)}(z) = \frac{y^2 z}{8g(z^2 + p^2)\sqrt{z^2 + y^2}} \left( \frac{y}{(z^2 + y^2)^2} - \frac{p}{(p^2 + y^2)^2} \right) \]  

which is manifestly analytic at \( z^2 = -p^2 = -b/g \). After some rewriting, it can be checked that the terms in (3.94) can all be expressed in terms of the loop insertion operator \( \mathcal{L}(z) \) acting on some quantities, and that the genus 1 free energy determined from (3.90) is [14]

\[ F_1 = \frac{1}{24} \log M_1 + \frac{1}{24} \log J_1 + \frac{1}{6} \log d \]  

This method of iterative solution can be carried out order by order in the \( \frac{1}{N} \)-expansion.

The general structure of the terms in the topological expansion (3.88) as found from (3.90) has been studied in detail by Ambjørn-Chelklov-Kristjansen-Makeenko [14] and they have shown that the higher-genus terms can be written symbolically in the form

\[ F_h = - \sum_{\alpha_i, \beta_j > 1} \frac{1}{\alpha_i, \beta_j} f_h(\alpha_1, \ldots, \alpha_s; \beta_1, \ldots, \beta_t | \alpha, \beta, \gamma) f_h(\alpha_1, \ldots, \alpha_s; \beta_1, \ldots, \beta_t; \alpha, \beta, \gamma) \]  

where \( h \geq 1 \) and

\[ f_h(\alpha_1, \ldots, \alpha_s; \beta_1, \ldots, \beta_t; \alpha, \beta, \gamma) = \frac{M_{\alpha_1} \cdots M_{\beta_t} J_{\beta_1} \cdots J_{\beta_t}}{M_1^h J_1^d} \]  

The brackets in (3.98) are rational functions of the non-negative integers \( \alpha, \beta \) and \( \gamma \), the indices \( \alpha_i \) and \( \beta_j \) lie in the interval \([2, 3h - 2]\), and \( h - 1 \leq \gamma \leq 4h - 4 \). The prime on the sum in (3.98) means that the summation is over the sets of indices obeying the restrictions

\[ s - \alpha \leq 0 \quad s = \alpha \iff s = \alpha = 0 \quad \ell - \beta \leq 0 \quad \ell = \beta \iff \ell = \beta = 0 \]  

\[ \alpha - s + \beta - \ell = 2h - 2 \quad \sum_{i=1}^{s} (\alpha_i - 1) + \sum_{j=1}^{t} (\beta_j - 1) + \gamma = 4h - 4 \]  

The first relation in (3.101) follows from the invariance of the partition function \( Z_P = e^{\sum_N N^{2-h} F_h} \) under simultaneous rescalings of \( N \) and the spectral density \( \rho, N \to \lambda \cdot N, \rho \to \frac{1}{\lambda} \rho \). The second relation in (3.101) follows from the invariance of \( Z_P \) under the rescalings \( N \to \lambda^2 \cdot N, g_j \to \lambda^{j-2} \cdot g_j \). The restriction of the integer \( \gamma \) to the range \( h - 1 \leq \gamma \leq 4h - 4 \) is a consequence of the double-scaling limit of the Hermitian one-matrix model [14] in which

\[ F_h \sim \Lambda_R^{(2-\gamma_m)(1-h)} \]  

where \( \Lambda_R \sim x \) is the renormalized cosmological constant.

In the Hermitian case, the higher genus coefficients can be related to intersection indices on the moduli space and the virtual Euler characteristics of the discretized moduli space of compact Riemann surfaces [14]. This follows from the identification of the Hermitian one-matrix model with the Kontsevich and Kontsevich-Penner matrix models which all have
the same double-scaling limits. In this way, the topological expansion can be related to
the intersection indices represented by Kontsevich matrix models which relate the 1-matrix
models to topological gravity [87] (see the next Subsection) in terms of a discretization of
moduli space similar to that used to define the virtual Euler characteristic for Penner matrix
models (c.f. Subsection 1.2.1). Given the genus \( h \) contribution to the free energy (3.88) the
genus \( h \) contribution to any connected correlator can be found from (3.14). We refer to [14]
for the technical details of this iterative determination of the free energy (3.98) and its relation
to the Kontsevich matrix model.

Substituting the relations (3.81), (3.82) and (3.87) into (3.99) we find
\[ f_h = i^{(\alpha+\beta-\sum_{i=1}^n \alpha_i - \sum_{j=1}^m \beta_j - \gamma)} \bar{f}_h \]  
(3.103)
where \( \bar{f}_h \) is real-valued and is defined from \( f_h \) by replacing \( M_k, J_k \) and \( d \) by \( \bar{M}_k, \bar{J}_k = (-1)^k \bar{M}_k \) and \( \bar{d} \) in (3.99). Using the restrictions (3.101) we then find that
\[ F_h = (-1)^{h-1} \bar{F}_h \]  
(3.104)
where \( \bar{F}_h \) is real-valued and is defined from \( F_h \) by replacing \( f_h \) by \( \bar{f}_h \) in (3.98). (3.104) shows
that the fermionic free energy alternates in sign according to genus in some sense. To show
that it alternates relative to the Hermitian case, we need to determine the scaling behaviour
of \( \bar{M}_k \) near the critical point.

For example, for the cubic potential (3.39), consider the moment functions (3.76) near the
\( m = 2 \) multi-critical point \( g_c = 2t^3/27 \). Expanding the boundary condition (3.27) to leading
order in \( y - y_c \), after some algebra we find that it can be expressed as
\[ 3M_2(iy_c)y_c(y - y_c)^2 = -4x \]  
(3.105)
which is similar to the scaling behaviour in the Hermitian one-matrix model with the same
odd polynomial potential (i.e. (1.75) without the logarithm term) [12, 38]. Moreover, the
moment functions of interest have the leading order scaling behaviours
\[ \bar{M}_2 = \frac{2}{y_c^3} - g_c = -2g_c < 0 \quad , \quad \bar{M}_1 = \bar{M}_2(iy_c)(y - y_c) = -2g_c(y - y_c) > 0 \]  
(3.106)
and
\[ \bar{d}_c = 2y_c > 0 \]  
(3.107)

Since (3.106) and (3.107) have the same signs as \( M_1, M_2 \) and \( d \) for the corresponding
asymmetric Hermitian model [12, 38], it follows from (3.104) that the topological expansion
(3.88) about the \( m = 2 \) multi-critical point is an alternating series relative to that of the
Hermitian one-matrix model with the same potential. Aside from this alternating nature, the
genus expansion (3.88) resembles the usual Painlevé expansion [38] (for suitable normalization
of the cosmological constant). It is expected that this is also true for general higher-order
multi-critical points, and thus it is conjectured that the genus expansion (3.88) about an \( m \)-th
order multi-critical point in the scaling limit alternates according to
\[ F_h = (-1)^{h-1} F_h^H \]  
(3.108)

\[ ^7 \text{This critical point and the value } y_c \text{ can also be found from the moment condition (3.80).} \]
where $F_h^H$ is the genus $h$ contribution to the free energy of an $m$-th multi-critical Hermitian model obtained from a symmetric potential. Being an alternating series the topological expansion of chirally symmetric fermionic one-matrix models may be Borel summable and thus these matrix models provide some novel worldsheet discretization of the string theory. However, it should be expected that the double-scaling limit of the fermionic one-matrix model differs from that of the Hermitian one-matrix model by more than just signs. For instance, the alternating nature of the genus expansion above is no longer true for generic (non-chirally symmetric) fermionic potentials $V(\bar{\psi}\psi)$ [12] (essentially because the cut endpoints do not lie on the imaginary axis in these cases), whereas in the Hermitian case it is well-known that the generic matrix model free energy coincides (modulo a factor of 2) in the continuum limit with that of a reduced matrix model [14, 38, 92]. The topological expansion of the fermionic one-matrix model therefore represents a rather unusual random surface theory which deserves future investigation.

### 3.5 Virasoro Algebra Constraints and Integrable Hierarchies

The loop equation (3.18) for any polynomial potential $V(\bar{\psi}\psi) = \sum_{k \geq 0} g_k (\bar{\psi}\psi)^k$ can be represented as a set of discrete Virasoro constraints imposed on the partition function (1.62). From (3.14) and (3.15) it follows that the expansion of (3.18) in $1/z$ can be written as

$$\frac{1}{Z_1} \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} L_n Z_1 = 0$$

(3.109)

where the differential operators

$$L_n = \sum_{k \geq 1} k g_k \frac{\partial}{\partial g_{k+n}} + \frac{1}{N^2} \sum_{k=0}^{n} \frac{\partial^2}{\partial g_k \partial g_{n-k}} + 2 \frac{\partial}{\partial g_n}$$

(3.110)

generate the discrete $c = 0$ Virasoro algebra

$$[L_n, L_m] = (n - m) L_{n+m}$$

(3.111)

The loop equation (3.18) is therefore represented by the Virasoro constraints

$$L_n Z_1 = 0 \ , \ n \geq 0$$

(3.112)

These Virasoro constraints resemble those of the complex one-matrix model (1.87) [92]. Notice also that by definition we have the additional constraint

$$\frac{\partial}{\partial g_0} Z_1 = N^2 Z_1$$

(3.113)

It is instructive to see precisely what symmetry of the fermionic matrix model the operators (3.110) represent for each $n$. They are associated with the invariance of the partition function (1.62) under the infinitesimal shifts

$$\psi \to \psi + e \psi (\bar{\psi}\psi)^n \ , \ n \geq 1 \ ; \ \bar{\psi} \to \bar{\psi}$$

(3.114)
of the fermionic variables, under which the potential $V$ in (1.62) changes by

$$V(\tilde{\psi}\psi) = \sum_{k \geq 0} g_k (\tilde{\psi}\psi)^k \rightarrow \sum_{k \geq 0} g_k \left(\tilde{\psi}\psi + \epsilon(\tilde{\psi}\psi)^{n+1}\right)^k$$

$$= V(\tilde{\psi}\psi) + \epsilon \sum_{k > n} (g_k + (k - n)g_{k-n}) (\tilde{\psi}\psi)^k$$

This variation in the partition function (1.62) is represented by the action of the operators

$$L^1_n = \sum_{k \geq 1} kg_k \frac{\partial}{\partial g_{k+n}}$$

on the partition function $Z_1$. The operators $L^1_n$ represent the “classical” invariance of $Z_1$ and are the usual classical generators of the Virasoro algebra (3.111) (or, more precisely, of the Borel subalgebra of the full Virasoro algebra). The integration measure in (1.62) under the shifts (3.114) changes by

$$d\psi\ d\tilde{\psi} \rightarrow d\psi\ d\tilde{\psi} \cdot \det \left[ \frac{\partial(\psi + \epsilon(\tilde{\psi}\psi)^n)}{\partial\psi} \right]$$

$$\sim d\psi\ d\tilde{\psi} \cdot \left( 1 + \epsilon \text{ tr} \frac{\partial(\tilde{\psi}\psi)^n}{\partial\psi} \right)$$

$$= d\psi\ d\tilde{\psi} \cdot \left( 1 + 2\epsilon \text{ tr} (\tilde{\psi}\psi)^n + \epsilon \sum_{k=0}^n \text{ tr} (\tilde{\psi}\psi)^k \text{ tr} (\tilde{\psi}\psi)^{n-k} \right)$$

which is represented by the action of the last 2 terms in (3.110) on $Z_1$ and which can be thought of as encoding the “quantum corrections” to the classical Virasoro generators (3.116). The fermionic Virasoro operators (3.110) differ from the standard bosonic ones in the last derivative operator $2\frac{\partial}{\partial g_n}$, which is absent in the scalar cases [92].

The loop equations of the fermionic matrix model represent the full set of Ward identities (or equations of motion) of the model. The completeness of these sets of equations is reflected in the fact that the Virasoro operators (3.110) form a closed algebra (3.111). The advantage of representing the loop equations of the matrix model in terms of Virasoro constraints is that it represents an invariant formulation of the partition function $Z_1$, i.e. $Z_1$ is determined as a solution of this set of compatible differential equations. From this point of view one can now compare these with other solutions to the Virasoro constraints, for example those which arise naturally from free fermion or free boson conformal field theory [61], or Kontsevich integrals [38, 112, 114]. In the former case this implies a certain string theoretical duality, between 2-dimensional world-sheets and the spectral surfaces which are associated to the configuration space of the string theory. The analysis of the uniqueness of such representations of the Virasoro constraints partitions the solutions into universality classes which can be used to relate these models to the multicomponent KP and Toda-chain integrable hierarchies [77]. In this case they also yield an alternative way of studying the double-scaling and continuum limits of the discretized random surface theories [43, 102]. Furthermore, the intersection numbers on the (compactified) moduli space of compact Riemann surfaces in 2-dimensional topological gravity are known to be related to a number of recursion relations which are equivalent to the Virasoro constraints of Hermitian 1-matrix models in the continuum limit [43, 87, 135]. The similarities between the Virasoro constraints in the fermionic case and
the bosonic ones are another indication of the relation between the adjoint fermion 1-matrix models and 2-dimensional quantum gravity. It still remains an unsolved problem, however, as to what this precise connection really is.

Nonetheless, the loop equations are precisely just the above set of Virasoro constraints imposed on the partition function \( Z_1 \). In the continuum limit, this Virasoro symmetry would then represent the underlying conformal invariance of the associated random surface theory, and it allows one to identify the proper continuum (double-scaling) limit partition function with the correct continuum Virasoro algebra invariance [92]. It would be interesting to relate the continuum loop equations of the fermionic one-matrix model more precisely to the techniques of exactly solvable integrable systems (e.g. the KdV or KP hierarchies) [19, 49, 67], and also to Witten’s approach [42, 135] which is based on the interpretation of the double scaling limit of the matrix model as a topological quantum field theory so that the problem is reduced to the calculation of intersection indices on moduli space. It would also be very interesting to find a relation, based on the Virasoro algebra constraints, between the adjoint fermion one-matrix model and conformal field theory (e.g. in the Hermitian case the one-matrix model can be represented in terms of correlation functions of a \( D = 1 \) conformal field theory [112, 114], i.e. a Gaussian field theory of a free scalar field).

### 4 Adjoint Fermion Two-matrix Models

The simplest higher-dimensional generalization of the \( D = 0 \) dimensional model (1.62) is a non-dynamical gauge theory minimally coupled to a fermionic two-matrix ensemble. The partition function of the two-matrix model is

\[
Z_2 = \int [dU] \int d\psi \ d\bar{\psi} \ d\chi \ d\bar{\chi} \ e^{S[\psi, \bar{\psi}, \chi, \bar{\chi}; U]} \tag{4.1}
\]

with action

\[
S[\psi, \bar{\psi}, \chi, \bar{\chi}; U] = N^2 \ tr \left( \bar{\psi} U \chi U^\dagger + \bar{\chi} U^\dagger \psi U + V(\bar{\psi} \psi) + \bar{V}(\bar{\chi} \chi) \right) \tag{4.2}
\]

where \([dU]\) is Haar measure on the unitary group \( U(N) \) and \( V \) and \( \bar{V} \) are independent potentials. The fermion fields \( \psi, \bar{\psi}, \chi \) and \( \bar{\chi} \) are independent \( N \times N \) Grassmann-valued matrices and the partition function (4.1) describes staggered self-interacting Dirac fermions which interact with a gauge field on a \( D = \frac{1}{2} \) dimensional lattice (i.e. a single link shared by 2 fermions) and which transform under the adjoint representation of the gauge group \( U(N) \). Using a gauge transformation (see below) the unitary matrices in (4.1) can be eliminated and one is left with the natural fermionic analog of a Hermitian two-matrix model [38, 101]. Just as the fermionic one-matrix model provides some novel random theory of discretized quantum gravity, the adjoint fermion two-matrix model (4.1) will yield some novel random theory of discretized gravity interacting with some type of matter. It may be that these matter fields are not restricted by the \( D = 1 \) conformal barrier as they are in the Hermitian cases, and one might therefore obtain a matrix model representation of strings in \( D > 1 \) dimensional target spaces. Alternatively, the Grassmann integrals over the fermion matrix fields can be performed leaving an induced gauge theory. This will be discussed in the next Section where we consider the generalizations of (4.1) to arbitrary dimensions, i.e. the analog of the Kazakov-Migdal model [75] using (adjoint) fermions [80, 95, 101, 109, 110, 126].
The model defined by (4.1) possesses a number of symmetries which follow from the invariance properties of the Haar measure. It is invariant under the gauge transformation

$$U \rightarrow V^\dagger U W \quad , \quad (\psi, \bar{\psi}) \rightarrow (V^\dagger \psi V, V^\dagger \bar{\psi} V) \quad , \quad (\chi, \bar{\chi}) \rightarrow (W^\dagger \chi W, W^\dagger \bar{\chi} W)$$

where \(\{V, W\} \in U(N) \otimes U(N)\). The model also has the \(U(1)\) gauge-symmetry

$$U \rightarrow Z U$$

where \(Z\) is an element of the center of \(U(N)\), i.e. a unimodular complex number. This symmetry implies that any correlator of the theory must contain the same number of \(U\) and \(U^\dagger\) matrices. The \(U(1)\) phase invariance of (4.1)

$$(\psi, \bar{\psi}) \rightarrow (e^{i\theta} \psi, e^{-i\theta} \bar{\psi}) \quad , \quad (\chi, \bar{\chi}) \rightarrow (e^{i\theta} \chi, e^{-i\theta} \bar{\chi})$$

leads to fermion number conservation in the gauge theory, i.e. any correlator of the matrix model must contain the same number of fermion and conjugate matrices.

In the symmetric case where \(V = \bar{V}\) the charge conjugation

$$U \rightarrow U^\dagger \quad , \quad (\psi, \bar{\psi}) \leftrightarrow (\chi, \bar{\chi})$$

is a symmetry of the two-matrix model and it implies equality of a large number of correlators of the model. When both potentials \(V\) and \(\bar{V}\) in (4.2) are odd polynomials, the chiral transformation

$$(\psi, \bar{\psi}) \rightarrow (\bar{\psi}, -\psi) \quad , \quad (\chi, \bar{\chi}) \rightarrow (\bar{\chi}, -\chi)$$

is a symmetry and as before it implies that all even \(\bar{\psi}\psi\) and \(\bar{\chi}\chi\) moments vanish,

$$\langle \text{tr} (\bar{\psi}\psi)^{2k} \rangle = \langle \text{tr} (\bar{\chi}\chi)^{2k} \rangle = 0$$

where the normalized averages are now with respect to the statistical ensemble (4.1). In the \(\mathbb{Z}_2\)-symmetric case when \(V = -\bar{V}\) is an odd polynomial potential the charge conjugation

$$U \rightarrow U^\dagger \quad , \quad (\psi, \bar{\psi}) \rightarrow (\chi, -\bar{\chi}) \quad , \quad (\chi, \bar{\chi}) \rightarrow (-\psi, \bar{\psi})$$

is an invariance of the model (4.1) and it relates the non-vanishing \(\bar{\psi}\psi\) and \(\bar{\chi}\chi\) moments by

$$\langle \text{tr} (\bar{\psi}\psi)^{2k+1} \rangle = -\langle \text{tr} (\bar{\chi}\chi)^{2k+1} \rangle$$

Another important symmetry in this case is the composition of the charge conjugation invariance (4.9) and the chiral symmetry (4.7) which leads to a mixed symmetry among the more general correlators of the two-matrix model (4.1). As before, even though the fermionic matrices in (4.1) cannot be diagonalized the fermionic two-matrix model leads to solutions analogous to those of Hermitian two-matrix models.

### 4.1 Loop Equations

The most general generating functions for the correlators of the two-matrix model (4.1) are those which generate all possible observables respecting the symmetries (4.3)–(4.5). We therefore introduce the even-even two-point correlator

$$G(z, w) = \left\langle \text{tr} \left( \frac{1}{z - \bar{\psi}\psi} U \frac{1}{w - \bar{\chi}\chi} U^\dagger \right) \right\rangle$$

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and the odd-odd two-point correlators

\[ \mathcal{H}(z, w) = \left\langle \text{tr} \, \frac{1}{z - \bar{\psi} \psi} \frac{1}{U} \frac{1}{w - \bar{\chi} \chi} U^\dagger \right\rangle \]  

(4.12)

\[ \mathcal{K}(z, w) = \left\langle \text{tr} \, \frac{1}{z - \bar{\psi} \psi} \bar{\psi} U \chi \frac{1}{w - \bar{\chi} \chi} U^\dagger \right\rangle \]  

(4.13)

We also have the usual generating functions for the \( \bar{\psi} \psi \) and \( \bar{\chi} \chi \) moments

\[ \omega(z) = \left\langle \text{tr} \, \frac{1}{z - \bar{\psi} \psi} \right\rangle \, , \, \bar{\omega}(w) = \left\langle \text{tr} \, \frac{1}{w - \bar{\chi} \chi} \right\rangle \]  

(4.14)

When \( N \) is finite these functions are all analytic in the punctured complex plane.

The even-even correlator has the asymptotic expansions

\[ \mathcal{G}(z, w) = \frac{\omega(z)}{w} + \sum_{n=1}^{N^2} \frac{\mathcal{G}_n(z)}{w^{n+1}} = \frac{\bar{\omega}(w)}{z} + \sum_{n=1}^{N^2} \frac{\bar{\mathcal{G}}_n(w)}{z^{n+1}} \]  

(4.15)

where

\[ \mathcal{G}_n(z) = \left\langle \text{tr} \, \frac{1}{z - \bar{\psi} \psi} U (\bar{\chi} \chi)^n U^\dagger \right\rangle \, , \, \bar{\mathcal{G}}_n(w) = \left\langle \text{tr} \, (\bar{\psi} \psi)^n U \frac{1}{w - \bar{\chi} \chi} U^\dagger \right\rangle \]  

(4.16)

Similarly, the odd-odd correlators have the asymptotic expansions

\[ \mathcal{H}(z, w) = \sum_{n=0}^{N^2} \frac{\mathcal{H}_n(z)}{w^{n+1}} = \sum_{n=0}^{N^2} \frac{\bar{\mathcal{H}}_n(w)}{z^{n+1}} \, , \, \mathcal{K}(z, w) = \sum_{n=0}^{N^2} \frac{\mathcal{K}_n(z)}{w^{n+1}} = \sum_{n=0}^{N^2} \frac{\bar{\mathcal{K}}_n(w)}{z^{n+1}} \]  

(4.17)

where

\[ \mathcal{H}_n(z) = \left\langle \text{tr} \, \frac{1}{z - \bar{\psi} \psi} U (\bar{\chi} \chi)^n \bar{\chi} U^\dagger \right\rangle \, , \, \bar{\mathcal{H}}_n(w) = \left\langle \text{tr} \, \psi(\bar{\psi})^n \frac{1}{w - \bar{\chi} \chi} U^\dagger \right\rangle \]  

(4.18)

and

\[ \mathcal{K}_n(z) = \left\langle \text{tr} \, \frac{1}{z - \bar{\psi} \psi} \bar{\psi} U \chi (\bar{\chi} \chi)^n U^\dagger \right\rangle \, , \, \bar{\mathcal{K}}_n(w) = \left\langle \text{tr} \, (\bar{\psi} \psi)^n \bar{\psi} U \chi \frac{1}{w - \bar{\chi} \chi} U^\dagger \right\rangle \]  

(4.19)

The various symmetries (4.6)–(4.9) imply some noteworthy relations among these generating functions. For example, in the symmetric case \( V = \bar{V} \) we have the symmetries

\[ \mathcal{G}(z, w) = \mathcal{G}(w, z) \, , \, \mathcal{K}(z, w) = -\mathcal{H}(-z, -w) \, , \, \omega(z) = \bar{\omega}(z) \]  

(4.20)

When the potential \( V = -\bar{V} \) is an odd polynomial, we have the symmetries

\[ \mathcal{G}(z, w) = \mathcal{G}(-w, -z) \, , \, \mathcal{H}(z, w) = \mathcal{H}(w, z) \, , \, \mathcal{K}(z, w) = -\mathcal{H}(-z, -w) \]  

(4.21)

and the vanishing of the even moments in this case further implies that

\[ \omega(z) + \bar{\omega}(z) = 2/z \]  

(4.22)
An important observable when the two-matrix model (4.1) is viewed as an induced gauge theory is the pair correlator of the gauge fields

\[
\frac{1}{N} C_{ij} \delta_{kl} \delta_{jk} = \langle U_{ij} U_{kl}^\dagger \rangle
\]

(4.23)

where the delta functions on the left-hand side of (4.23) arise from the gauge invariance of (4.1). Unitarity implies that it obeys the sum rule

\[
\frac{1}{N} \sum_{i,j=1}^{N} C_{ij} = \frac{1}{N} \sum_{j=1}^{N} C_{ij} = 1
\]

(4.24)

In the Hermitian case, when \( V = \tilde{V} \) the pair correlator \( C_{ij} = N \langle |U_{ij}|^2 \rangle \) can be computed from the double discontinuity of the scalar version of the generating function (4.11) using

\[
G^H(z, w) \equiv \left\langle \operatorname{tr} \frac{1}{z - \phi} U \frac{1}{w - \phi} U^\dagger \right\rangle = \int d\alpha \frac{d\beta}{\rho(\alpha) \rho(\beta)} \frac{C(\alpha, \beta)}{(z - \alpha)(w - \beta)}
\]

(4.25)

since there it depends only on the moments of powers of the Hermitian fields \([47, 95]\). This is not the case for the adjoint fermion matrix ensemble (4.1) because it involves a larger set of fermionic correlators than just those of the type \( \langle \operatorname{tr} (\tilde{\psi} \psi)^n \rangle \) and \( \langle \operatorname{tr} (\tilde{\chi} \chi)^n \rangle \) (for instance one needs to know the correlators of the form \( \langle \operatorname{tr} \tilde{\psi}^n \psi^n \rangle \)). There does not appear to be any direct way to generate these observables from the loop equations. Moreover, there is no known analog of the Itzykson-Zuber formula \([71]\) for the integral

\[
I[\psi, \chi] = \int [dU] e^{N^2 \operatorname{tr} (\psi U \chi U^\dagger)}
\]

(4.26)

when \( \psi \) and \( \chi \) are Grassmann-valued matrices that transform under the adjoint representation of \( U(N) \). In the Hermitian case the knowledge of the explicit form of (4.26) at least allows one to formally determine \( C_{ij} \) using saddle-point methods \([47]\).

For the fermionic matrix chain (4.1), factorization and symmetry imply that the correlators (4.11)–(4.13) and (4.23) generate the complete set of observables of the model at \( N = \infty \). The loop equations for the correlators (4.11)–(4.13) can now be derived as before and they will involve sets of mixed equations for the even-even correlator with either of the odd-odd correlators. The loop equations involving the generating function (4.12) are as follows. The first one follows from the identity

\[
\int [dU] \int d\psi \, d\tilde{\psi} \, d\chi \, d\tilde{\chi} \frac{\partial}{\partial \psi_{ij}} \left[ \left( \psi \frac{1}{z - \psi \tilde{\psi}} U \frac{1}{w - \chi \tilde{\chi}} U^\dagger \right)_{kl} \right] e^{S[\psi, \tilde{\psi}, \chi, \tilde{\chi}, U]} = 0
\]

(4.27)

Expanding (4.27) into averages and summing over \( i = k, j = \ell \) as before leads to

\[
0 = \left\langle \operatorname{tr} \frac{1}{z - \psi \tilde{\psi}} U \frac{1}{w - \chi \tilde{\chi}} U^\dagger \right\rangle + \left\langle \operatorname{tr} \psi \frac{1}{z - \psi \tilde{\psi}} \tilde{\psi} \operatorname{tr} \frac{1}{z - \psi \tilde{\psi}} U \frac{1}{w - \chi \tilde{\chi}} U^\dagger \right\rangle
\]

\[
+ \left\langle \operatorname{tr} \psi \frac{1}{z - \psi \tilde{\psi}} U \frac{1}{w - \chi \tilde{\chi}} U^\dagger \right\rangle + \left\langle \operatorname{tr} \psi \frac{1}{z - \psi \tilde{\psi}} U \frac{1}{w - \chi \tilde{\chi}} U^\dagger V'(\tilde{\psi} \psi) \psi \right\rangle
\]

(4.28)

which at \( N = \infty \), when factorization holds, gives

\[
(2 - z \omega(z)) G(z, w) + H(z, w) + \oint_C \frac{d\lambda}{2\pi i} \frac{V'(\lambda)}{z - \lambda} G(\lambda, w) = 0
\]

(4.29)
where the contour $C$ encircles the singularities of $G(z, w)$ with counterclockwise orientation in the complex $z$-plane. Writing the same sort of equation for $\bar{\chi}$ instead of $\psi$

$$\int [dU] \int d\psi \; d\bar{\psi} \; d\chi \; d\bar{\chi} \frac{\partial}{\partial \psi_{ij}} \left[ \left( \frac{1}{z - \psi \psi} U \frac{1}{w - \bar{\chi} \bar{\chi}} \right)_G \right]_{kl} e^{s[\psi, \bar{\psi}, \chi, \bar{\chi}; U]} = 0$$

leads to the $N = \infty$ loop equation

$$(2 - w \bar{\omega}(w)) G(z, w) + H(z, w) + \oint_C \frac{d\lambda}{2\pi i} \frac{V'(\lambda) \lambda}{w - \lambda} G(z, \lambda) = 0 \quad (4.31)$$

Two more loop equations involving the two-point correlators (4.11) and (4.12) follow first from the identity

$$\int [dU] \int d\psi \; d\bar{\psi} \; d\chi \; d\bar{\chi} \frac{\partial}{\partial \psi_{ij}} \left[ \left( \frac{1}{z - \psi \psi} U \frac{1}{w - \bar{\chi} \bar{\chi}} \right)_G \right]_{kl} e^{s[\psi, \bar{\psi}, \chi, \bar{\chi}; U]} = 0 \quad (4.32)$$

which leads to

$$\omega(z) H(z, w) - \omega(z) + w G(z, w) - \oint_C \frac{d\lambda}{2\pi i} \frac{V'(\lambda) \lambda}{z - \lambda} H(\lambda, w) = 0 \quad (4.33)$$

at large-$N$. Finally, we analogously have the identity

$$\int [dU] \int d\psi \; d\bar{\psi} \; d\chi \; d\bar{\chi} \frac{\partial}{\partial \chi_{ij}} \left[ \left( \frac{1}{z - \psi \psi} U \frac{1}{w - \bar{\chi} \bar{\chi}} \right)_G \right]_{kl} e^{s[\psi, \bar{\psi}, \chi, \bar{\chi}; U]} = 0 \quad (4.34)$$

which yields the large-$N$ loop equation

$$\bar{\omega}(w) H(z, w) - \bar{\omega}(w) + z G(z, w) - \oint_C \frac{d\lambda}{2\pi i} \frac{V'(\lambda) \lambda}{w - \lambda} H(z, \lambda) = 0 \quad (4.35)$$

A similar set of loop equations involving the odd-odd two-point correlator $\mathcal{K}(z, w)$ instead of $H(z, w)$ can also be derived as above with the obvious modifications.

These loop equations could also have been obtained from the Schwinger-Dyson equations expressing the invariance of the fermionic integration measure under arbitrary changes of variables and the symmetries of the Haar measure $[dU]$. For example, denote the generators of $U(N)$ by $T^a$, $a = 1, \ldots, N^2$. They obey the completeness and normalization relations

$$\sum_{a=1}^{N^2} T^a_{ij} (T^a)_{kl} = N \delta_{il} \delta_{jk} \quad , \quad \text{tr} \; T^a T^b = \delta^{ab} \quad (4.36)$$

We can then represent the Grassmann-valued matrices $\psi$ as

$$\psi = \sum_{a=1}^{N^2} T^a \psi^a \quad \text{where} \quad \psi^a = \text{tr} \; T^a \psi \quad (4.37)$$

The first loop equation (4.29) then follows from the invariance of the integration measure in the vanishing correlator

$$\left\langle \text{tr} \; T^a \psi \frac{1}{z - \psi \psi} U \frac{1}{w - \bar{\chi} \bar{\chi}} U^\dagger \right\rangle \equiv 0 \quad (4.38)$$

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under the field transformation
\[ \psi \rightarrow \psi + e^a T^a \]  \hspace{1cm} (4.39)
with \( e^a \) infinitesimal parameters. The vanishing of (4.38) is a consequence of the Grassmann integrations in (4.1) and gauge invariance. (4.29) now follows explicitly by performing the shift (4.39), using the invariance of the integration measure to get
\[
\frac{1}{Z_2} \int [dU] \int d\psi \overline{d\psi} d\chi \overline{d\chi} T^a_{ik} \frac{\partial}{\partial \psi_i} \left\{ \left( \frac{1}{z - \psi} U - \frac{1}{w - \overline{\chi}} \overline{U} \right) \right\} e^\epsilon \tilde{T}^a_{\tilde{V}} = 0 \hspace{1cm} (4.40)
\]
and then summing over \( a = 1, \ldots, N^2 \) using (4.36) and calculating the derivatives \( \frac{\partial}{\partial \tilde{\tilde{\psi}}} \). The other 3 loop equations follow from the invariance conditions analogous to (4.38), (4.39) [101].

The loop equations (4.29) and (4.33) can be combined to give an equation which determines the one-loop correlator \( \omega(z) \). Substituting into these loop equations the asymptotic expansions (4.15) and (4.17) in \( 1/w \) and equating the coefficients of \( \frac{1}{w^{n+1}} \) we get recursive relations determining \( G_n(z) \) and \( H_n(z) \) in terms of \( \omega(z) \)
\[
H_n(z) = (z\omega(z) - 2) G_n(z) - \oint_C \frac{d\lambda}{2\pi i} \frac{V'(\lambda) \lambda}{z - \lambda} G_n(\lambda) \hspace{1cm} (4.41)
\]
\[
G_{n+1}(z) = \oint_C \frac{d\lambda}{2\pi i} \frac{V'(\lambda) \lambda}{z - \lambda} H_n(\lambda) - \omega(z)H_n(z) \hspace{0.5cm}, \hspace{0.5cm} G_0(z) \equiv \omega(z) \hspace{1cm} (4.42)
\]
The equation determining \( \omega(z) \) now follows from expanding the loop equation (4.35) in \( 1/w \) using (4.15) and (4.17) and keeping only the leading-order term in \( 1/w \),
\[
z\omega(z) = 1 + \oint_C \frac{d\lambda}{2\pi i} \tilde{V}'(\lambda) \lambda H(z, \lambda) \hspace{1cm} (4.43)
\]
When the potential \( \tilde{V} \) is a polynomial of degree \( \tilde{K} \)
\[
\tilde{V}(w) = \sum_{n=1}^{\tilde{K}} \frac{\tilde{g}_n}{n} w^n \hspace{1cm} (4.44)
\]
the equation (4.43) becomes
\[
z\omega(z) = 1 + \sum_{k=1}^{\tilde{K}} \tilde{g}_k H_{k-1}(z) \hspace{1cm} (4.45)
\]
and it involves \( \tilde{K} \) unknown functions which are determined by the recursive equations (4.41) and (4.42). The equations that determine \( \omega(z) \) in this case lead to a \( (2\tilde{K}) \)-th order polynomial equation for the one-loop correlator \( \omega(z) \). When in addition the potential \( V \) is a polynomial (2.15), the contour integrals in (4.41) and (4.42) are
\[
\oint_C \frac{d\lambda}{2\pi i} \frac{V'(\lambda) \lambda}{z - \lambda} G_n(\lambda) = -V'(z)zG_n(z) + G_n(z) \hspace{1cm} (4.46)
\]
\[
\oint_C \frac{d\lambda}{2\pi i} \frac{V'(\lambda) \lambda}{z - \lambda} H_n(\lambda) = -V'(z)zH_n(z) + H_n(z) \hspace{1cm} (4.47)
\]
where \( G_n(z) \) and \( H_n(z) \) are polynomials of degree \( K - 1 \)
\[
G_n(z) = \sum_{m=1}^{K} g_m \sum_{p=0}^{m-1} \tilde{g}_{m-p-1,n} z^p, \hspace{0.5cm} H_n(z) = \sum_{m=1}^{K} g_m \sum_{p=0}^{m-1} \tilde{H}_{m-p-1,n} z^p \hspace{1cm} (4.48)
\]
67
and

\[ G_{m,n} = \langle \text{tr} (\bar{\psi} \psi)^m U(\bar{\chi} \chi)^n U^\dagger \rangle \quad , \quad H_{m,n} = \langle \text{tr} \, \psi (\bar{\psi} \psi)^m U(\bar{\chi} \chi)^n U^\dagger \rangle \quad (4.49) \]

are the coefficients of the asymptotic expansions

\[ G_n(z) = \sum_{m=0}^{\infty} \frac{G_{m,n}}{z^{m+1}} \quad , \quad H_n(z) = \sum_{m=0}^{\infty} \frac{H_{m,n}}{z^{m+1}} \quad (4.50) \]

An identical set of asymptotic equations determining \( \tilde{\omega}(w) \) can also be written down and they correspond to interchanging tilde and un-tilde quantities in the above in the obvious way.

Once the single-loop correlators are known they can be substituted back into the original loop equations and the even-even and odd-odd two-point correlators can be found. In the case of the polynomial interactions above, the contour integrals appearing in the loop equations can be determined as

\[ \oint \frac{d\lambda}{2\pi i} \frac{V'(\lambda)\lambda}{z - \lambda} G(\lambda, w) = -V'(z) z G(z, w) + \sum_{m=1}^{K} g_m \sum_{p=0}^{m-1} \tilde{G}_{m-p-1}(w) z^p \quad (4.51) \]

\[ \oint \frac{d\lambda}{2\pi i} \frac{\tilde{V}'(\lambda)\lambda}{w - \lambda} G(z, \lambda) = -\tilde{V}'(w) w G(z, w) + \sum_{m=1}^{K} g_m \sum_{p=0}^{m-1} \tilde{G}_{m-p-1}(z) w^p \quad (4.52) \]

and similarly for the integrals involving \( H(z, w) \).

### 4.2 The Gaussian Model

The asymmetric Gaussian potential

\[ V(z) = -\tilde{V}(z) = mz \quad (4.53) \]

describes a free Dirac fermion of mass \( m \) on a lattice in \( D = \frac{1}{2} \) dimensions. (4.45) is then

\[ z\omega(z) = 1 - mH_0(z) \quad (4.54) \]

where from (4.41)

\[ H_0(z) = (z\omega(z) - 2)\omega(z) + mz\omega(z) - m \quad (4.55) \]

The equation determining \( \omega(z) \) is quadratic and it is the same as the loop equation for the Gaussian one-matrix model (2.46) with \( t \equiv t_+ \) where

\[ t_+ = m \pm 1/m \quad (4.56) \]

The one-loop correlator \( \omega(z) \) is therefore given by (3.36) with this definition of \( t \). The correlator \( \tilde{\omega}(w) \) is then determined by (4.22).

The even-even correlator \( G(z, w) \) can now be determined by subtracting the loop equation (4.31) from (4.29), using (4.51) and (4.52), and substituting in the one-loop correlators obtained above. We find

\[ G(z, w) = \frac{(t_+ + t_-) \left( \frac{1}{2} + \frac{1}{w} + \frac{1}{2u} \sqrt{4 + t_+^2 w^2} - \frac{1}{2w} \sqrt{4 + t_+^2 w^2} \right)}{t_-(z + w) + \sqrt{4 + t_+^2 z^2} + \sqrt{4 + t_+^2 w^2}} \quad (4.57) \]
which when substituted back into (4.29) gives the odd-odd two-point correlator

\[
\mathcal{H}(z, w) = \frac{t_+(z + w) - \sqrt{4 + t_+^2 z^2} - \sqrt{4 + t_+^2 w^2}}{t_-(z + w) + \sqrt{4 + t_-^2 z^2} + \sqrt{4 + t_-^2 w^2}}
\]

(4.58)

The remaining generating function \( \mathcal{K}(z, w) \) is given by (4.21). It is easy to see that these correlators are non-singular for any \( z \) and \( w \) and obey the appropriate symmetries (4.21).

### 4.3 Cubic Interaction Model

The simplest non-Gaussian model is associated with the asymmetric cubic potential

\[
V(z) = -\bar{V}(z) = mz + \frac{g}{3}z^3
\]

(4.59)

In this case the asymptotic equation (4.45) is

\[
z\omega(z) = 1 - m\mathcal{H}_0(z) - g\mathcal{H}_2(z)
\]

(4.60)

where the unknown functions in (4.60) are found by combining (4.41) and (4.42) together using (4.46) and (4.47) to generate the 3 equations

\[
\mathcal{H}_0(z) = -m - gz^2 - g\bar{z}\xi - \omega(z)\left[2 - z(\omega(z) + m + g\bar{z}^2)\right]
\]

(4.61)

\[
\mathcal{H}_1(z) = -\mathcal{H}_0(z)(2 - z(\omega(z) + m + g\bar{z}^2))(\omega(z) + m + g\bar{z}^2) + (m + g\bar{z}^2)\xi
\]

\[
+ g\bar{z}\mathcal{H}_{0,0}(1 - z(\omega(z) + m + g\bar{z}^2)) - g\mathcal{G}_{2,1}
\]

(4.62)

\[
\mathcal{H}_2(z) = \mathcal{H}_0(z)(2 - z(\omega(z) + m + g\bar{z}^2))^2(\omega(z) + m + g\bar{z}^2)^2 - \mathcal{H}_1(z)(2 - z(\omega(z) + m + g\bar{z}^2))
\]

\[
\times(\omega(z) + m + g\bar{z}^2) - g\bar{z}\mathcal{G}_{1,2} - g\mathcal{G}_{2,2} + g\mathcal{H}_{1,1}(2 - z(\omega(z) + m + g\bar{z}^2))
\]

(4.63)

where as before \( \xi = \langle \text{tr } \bar{\psi}\psi \rangle \). In arriving at (4.61)–(4.63) we have used the \( \mathbb{Z}_2 \) charge-conjugation and chiral symmetries of the potential (4.59) to deduce from (4.21) that \( \mathcal{H}_0(z) = -\mathcal{H}_0(-z) \) so that \( \mathcal{H}_{1,0} = \mathcal{H}_{0,1} = 0 \) here. These same symmetries also imply that \( \mathcal{G}_{2,1} = -\mathcal{G}_{1,2} \). Also, we have expanded the asymptotic loop equation (4.42) for \( n = 1 \) in powers of \( 1/z \) to find that \( \mathcal{H}_{0,0} = \mathcal{G}_{1,1} \). Then combining the 4 equations (4.60)–(4.63) we find that the one-loop correlator \( \omega(z) \) is determined by a complicated 6-th order equation

\[
0 = \left[2g\{2 - z(\omega(z) + m + g\bar{z}^2)\}^2(\omega(z) + m + g\bar{z}^2)^2 + m\right]
\]

\[
\times\left[\{z(\omega(z) + m + g\bar{z}^2) - 2\}\omega(z) - (m + g\bar{z}^2)z - g\bar{z}\xi\right]
\]

\[-g(\omega(z) + m + g\bar{z}^2)\{2 - z(\omega(z) + m + g\bar{z}^2)\}\left[(m + g\bar{z}^2)\xi\right]
\]

\[+ g\bar{z}\mathcal{H}_{0,0}(1 - z(\omega(z) + m + g\bar{z}^2)) - g\mathcal{G}_{2,1} + g^2z\mathcal{G}_{2,1} - g^2\mathcal{G}_{2,2}
\]

\[+ g^2\{2 - z(\omega(z) + m + g\bar{z}^2)\}\mathcal{H}_{1,1} + z\omega(z) - 1
\]

(4.64)

The solvability features of fermionic multi-matrix models are even worse compared to Hermitian multi-matrix models where a degree \( K \) polynomial potential leads to a \( K \)-th order
polynomial equation for the one-loop correlator [95]. Here the order of the equation doubles due to the doubling of degrees of freedom in the fermionic case, just like in the fermionic one-matrix models, which can be understood from the result (1.85) which required 2 Hermitian matrices for its derivation. There does not appear to be any way to directly solve the 6-th order equation (4.64), nor does there seem to be any way of reducing it to a lower degree equation. Thus it is not possible to directly study the cut structure of \( \omega(z) \) and determine the ensuing phase structure of the two-matrix model as before.

### 4.4 A Penner-type Model

In the Hermitian case there is still an exactly solvable non-Gaussian multi-matrix model involving a non-polynomial interaction [94,96],[118]. In certain limiting cases it can be reduced to a polynomial interaction. The analog of this in the fermionic case is the asymmetric logarithmic interaction

\[
V(z) = -\tilde{V}(z) = mz + g \log(\bar{g} - z)
\]  

(4.65)

In this case there is an extra boundary condition imposed on the loop equations by requiring that the residue of the simple pole at \( z = \bar{g} \) vanish.

The asymptotic equation (4.43) for the potential (4.65) reads

\[
z \omega(z) = 1 - m \mathcal{H}_0(z) - g \mathcal{H}(z, \bar{g})
\]  

(4.66)

where the function \( \mathcal{H}_0(z) \) is determined from (4.41) as

\[
\mathcal{H}_0(z) = \left[ 2 - z \left( \omega(z) - m - \frac{g}{z - \bar{g}} \right) \right] \omega(z) - \left( m + \frac{g \bar{g}}{z - \bar{g}} \omega(\bar{g}) \right)
\]  

(4.67)

and the function \( \mathcal{H}(z, \bar{g}) \) is determined by letting \( w \to \bar{g} \) in the loop equations (4.29) and (4.31) which lead to

\[
\mathcal{H}(z, \bar{g}) = \left( \frac{2 - z \left( \omega(z) - m - \frac{g}{z - \bar{g}} \right) \left( \omega(z) - \frac{g}{z - \bar{g}} \mathcal{H}(\bar{g}, \bar{g}) \right) - \bar{g} \left( m \bar{\omega}(\bar{g}) + \frac{g \bar{g}}{z - \bar{g}} \mathcal{G}(\bar{g}, \bar{g}) \right)}{\left( \omega(z) - m - \frac{g}{z - \bar{g}} \right) \left( 2 - z \left( \omega(z) - m - \frac{g}{z - \bar{g}} \right) \right)} \right) + \bar{g}
\]  

(4.68)

Substituting (4.67) and (4.68) into (4.66) leads to a complicated quartic equation for the one-loop correlator \( \omega(z) \)

\[
0 = m \left[ \left( 2 - z \left( \omega(z) - m - \frac{g}{z - \bar{g}} \right) \right) \omega(z) - \left( m + \frac{g \bar{g}}{z - \bar{g}} \omega(\bar{g}) \right) \right] + z \omega(z) - 1
\]

\[
+ g \left[ \left( 2 - z \left( \omega(z) - m - \frac{g}{z - \bar{g}} \right) \right) \omega(z) - \frac{g}{z - \bar{g}} \mathcal{H}(\bar{g}, \bar{g}) - \bar{g} \left( m \bar{\omega}(\bar{g}) + \frac{g \bar{g}}{z - \bar{g}} \mathcal{G}(\bar{g}, \bar{g}) \right) \right]
\]  

(4.69)

In the Hermitian case the extra boundary condition at \( z = \bar{g} \) leads to a quadratic equation for \( \omega(z) \) [94, 95]. Here, again because of the doubling of the fermionic degrees of freedom, we obtain a quartic equation which is not directly amenable to a multi-branch solution but which is nonetheless explicitly solvable. It is not clear, however, how to determine the precise cut structure or the existence of phase transitions with these solutions due to the complicated structure of their square root branches. Thus the only exactly solvable fermionic multi-matrix models are the trivial Gaussian ones.
4.5 The Genus Expansion and $W$-algebra Constraints

Although the loop equations in Subsection 4.1 were derived at infinite $N$, it is possible to examine the structure of the $1/N$-expansion of the fermionic two-matrix model by including the irreducible correlators (of order $1/N^2$) using the loop insertion operator (3.15) in the asymmetric case $V \neq \bar{V}$. The extensions of the loop equations (4.29) and (4.33) to finite $N$ are (see (4.28))

\[
(2 - z \omega(z)) \mathcal{G}(z, w) + \mathcal{H}(z, w) - \mathcal{L}(z) \mathcal{G}(z, w) + \oint_c \frac{d \lambda}{2\pi i} \frac{V'(\lambda) \lambda}{z - \lambda} \mathcal{G}(\lambda, w) = 0
\]

(4.70)

\[
\omega(z) \mathcal{H}(z, w) - \omega(z) + w \mathcal{G}(z, w) + \mathcal{L}(z) \mathcal{H}(z, w) - \oint_c \frac{d \lambda}{2\pi i} \frac{V'(\lambda) \lambda}{z - \lambda} \mathcal{H}(\lambda, w) = 0
\]

(4.71)

so that the recurrence relations (4.41) and (4.42) at finite $N$ become

\[
\mathcal{H}_n(z) = (z \omega(z) - 2) \mathcal{G}_n(z) - \oint_c \frac{d \lambda}{2\pi i} \frac{V'(\lambda) \lambda}{z - \lambda} \mathcal{G}_n(\lambda) + \mathcal{L}(z) \mathcal{G}_n(z)
\]

(4.72)

\[
\mathcal{G}_{n+1}(z) = \oint_c \frac{d \lambda}{2\pi i} \frac{V'(\lambda) \lambda}{z - \lambda} \mathcal{H}_n(\lambda) - \omega(z) \mathcal{H}_n(z) - \mathcal{L}(z) \mathcal{H}_n(z)
\]

(4.73)

We can introduce the analogs of the loop insertion operator (3.15) for the correlators $\mathcal{G}_n(z)$ and $\mathcal{H}_n(z)$ by

\[
\mathcal{H}_n(z) = \frac{1}{Z_2} \mathcal{L}_{2n+1}(z) Z_2
\]

(4.74)

where

\[
\mathcal{L}_{2n+1}(z) = \sum_{k \geq -n} \frac{1}{z^{k+n}} \mathcal{W}_k^{(2n)}
\]

(4.75)

and

\[
\mathcal{G}_n(z) = \frac{1}{Z_2} \mathcal{L}_{2n}(z) Z_2
\]

(4.76)

where

\[
\mathcal{L}_{2n}(z) = \sum_{k \geq -n} \frac{1}{z^{k+n+1}} \mathcal{W}_k^{(2n+1)}
\]

(4.77)

Here $\mathcal{W}_k^{(m)}$ are some differential operators determined by the general potential $V(\bar{\psi} \psi) = \sum_{k \geq 0} g_k (\bar{\psi} \psi)^k$. From (3.15) it follows that

\[
\mathcal{L}_0(z) = N^2 \mathcal{L}(z) , \quad \mathcal{W}_k^{[1]} = \frac{\partial}{\partial g_k}
\]

(4.78)

To determine the operators $\mathcal{W}_k^{(m)}$ for $m \neq 1$, we substitute (4.74)–(4.77) into (4.72) and (4.73) and equate the coefficients of the $1/z^{k+n}$ terms. We then find that the operators $\mathcal{W}_k^{(m)}$ obey the recurrence relations

\[
\mathcal{W}_k^{(2n)} = \frac{1}{N^2} \sum_{m=0}^{k+n} \frac{\partial}{\partial g_m} \mathcal{W}_{k-m}^{(2n-1)} - 2\mathcal{W}_k^{(2n+1)} - \sum_{m \geq 1} mg_m \mathcal{W}_{m+k}^{(2n+1)} , \quad k \geq -n
\]

(4.79)

\[
\mathcal{W}_k^{(2n+1)} = \sum_{m \geq 1} mg_m \mathcal{W}_{m+k}^{(2n-2)} - \frac{1}{N^2} \sum_{m=0}^{k+n-1} \frac{\partial}{\partial g_m} \mathcal{W}_{k-m}^{(2n-2)} , \quad k \geq -n
\]

(4.80)
For example, the relations (4.78)–(4.80) imply that the first couple of $\mathcal{W}$-operators are

$$
\mathcal{W}^{(0)}_k = \frac{1}{N^2} \sum_{m=0}^{k} \frac{\partial^2}{\partial g_m \partial g_{k-m}} - \sum_{m \geq 1} m g_m \frac{\partial}{\partial g_{k+m}} - 2 \frac{\partial}{\partial g_k}
$$

$$
\mathcal{W}^{(3)}_k = \frac{1}{N^2} \sum_{m \geq 1} m g_m \sum_{i+j=k+m} \frac{\partial^2}{\partial g_i \partial g_j} - \sum_{m,n \geq 1} m n g_m g_n \frac{\partial}{\partial g_{k+m+n}}
\quad - 2 \sum_{m \geq 1} m g_m \frac{\partial}{\partial g_{k+m}} - \frac{1}{N^4} \sum_{i+j+m=k} \frac{\partial^3}{\partial g_i \partial g_j \partial g_m} + \frac{2}{N^2} \sum_{i+j=k} \frac{\partial^2}{\partial g_i \partial g_j}
\quad + \frac{1}{N^2} \sum_{m \geq 1} m g_m \sum_{i+j=k+1} \frac{\partial^2}{\partial g_i \partial g_{j+1-k}} + \frac{1}{N^2} \frac{k(k+1)}{2} \frac{\partial}{\partial g_k}
$$

Notice that the operators $\mathcal{W}^{(1)}_k$ generate a $U(1)$ Kac-Moody algebra and the operators $\mathcal{W}^{(0)}_k$ coincide with generators $L_k$ of the Virasoro algebra that we encountered in Subsection 3.5 above. The $\mathcal{W}$-operators above therefore resemble the generators of the conventional $W_k$-algebras. Again, these operators differ from the $\mathcal{W}$-operators that appear in Hermitian multi-matrix models [43] by extra derivative operators. In these latter models the algebra generated by the $\mathcal{W}$-operators coincide in the double-scaling limit with the canonical continuum $W$-algebras and lead to new symmetries of the underlying conformal matrix models [112, 114].

If $\tilde{g}_k = 0$ in (4.45) for all $k \neq n$ and $\tilde{g}_n = \frac{1}{n}$ for some $n \neq 0$, then the $\frac{1}{k}$-expansion of (4.45) can be written as

$$
\frac{1}{Z_2} \sum_{k=1-n}^{\infty} \frac{1}{\tilde{z}_{k+n}} \left( \mathcal{W}^{(2n-2)}_k - \mathcal{W}^{(1)}_{k+n} \right) Z_2 = 0
$$

and so the loop equations of the fermionic two-matrix model (4.1) are equivalent to a set of discrete $W$-constraints imposed on the partition function $Z_2$

$$
\mathcal{W}^{(2n-2)}_k Z_2 = \mathcal{W}^{(1)}_{k+n} Z_2 , \quad k \geq 1 - n
$$

The action of the $W$-algebra generators in (4.84) represent the Ward identities of the two-matrix model (4.1) associated with the infinitesimal fermionic variable changes

$$
\chi \rightarrow \chi + \epsilon \chi (\bar{\psi} \psi)^n , \quad \bar{\chi} \rightarrow \bar{\chi}
$$

$$
\psi \rightarrow \psi + \epsilon \psi (\bar{\psi} \psi)^n \left( V'(\bar{\psi} \psi) - \bar{\chi} \chi \right) , \quad \bar{\psi} \rightarrow \bar{\psi}
$$

under which the potential $V$ transforms as

$$
V(\bar{\psi} \psi) \rightarrow V(\bar{\psi} \psi) - \epsilon V'(\bar{\psi} \psi)^2 (\bar{\psi} \psi)^n + \epsilon (\bar{\psi} \psi)^{n+1}
$$

These $W$-constraints are the two-matrix analogs of the Virasoro constraints (3.109)–(3.111). In the Hermitian case they are the basis of the integrable hierarchy structure in multi-matrix models and generalized Kontsevich models [77, 104, 112, 114]. Thus the fermionic multi-matrix models, in addition to providing some novel generalization of the $(p,q)$ conformal models of string theory [32, 38, 50], also admit the conformal algebraic structure that relates their integrability features to KdV and generalized Toda-chain hierarchies [38, 77, 112, 114]. Since the discrete $W$-constraints here again represent the full set of Schwinger-Dyson equations of the model, the solutions of the matrix model can be represented in more convenient
forms relevant to conformal string theory and the double-scaling continuum limit string theory can be obtained from the Borel subalgebras of the continuum $W_{1+\infty}$-algebras [112, 114]. These hierarchical structures should be quite different though than in the conventional matrix models because of the convergence properties of the fermionic partition functions. From the fermionic matrix models some novel integrable hierarchies may therefore emerge. Moreover, the $W$-constraints relate the Hermitian multi-matrix models to more general types of topological field theories coupled to topological gravity [42, 135]. This might make a more explicit connection between matrix models and the recent implementations of the ideas of representing topological intersection indices in terms of matrix models using twisted $N = 2$ superconformal quantum field theories [44, 57, 91].

These $W$-symmetries contain the usual Kac-Moody and Virasoro algebra invariances which characterize conformal field theory and string theory [61]. This once again indicates the connection between the adjoint fermion matrix models and string theory, and it would be interesting to use the $W$-constraints to find other “physical” (e.g. conformal field theoretic) representations of the fermionic matrix models which makes this connection complete. The appearance of the new $W$-algebra symmetries of the loop equations (beyond the usual Kac-Moody and Virasoro ones) may have severe implications for their extensions to higher-dimensional continuum string and gauge theories where loop equations exist [92, 107].

5 Induced Gauge Theories and Mean Field Theory for Higher Dimensional Matrix Models

In this Section, we shall introduce a class of higher dimensional matrix models related to the Kazakov-Migdal model of induced gauge theory, which can be formulated with either bosonic or fermionic adjoint matter fields. We will concentrate on the fermionic case. We discuss the relation of this model to other lattice gauge theories, such as the so-called adjoint model [26, 69, 100]. In this Section, we review the mean field analysis of these models and postpone detailed discussion of the loop equation approach to the next Section. Using loop equations, we shall see that the Gaussian model, and some others in principle, are exactly solvable in the strong coupling regime. In this Section we shall present some speculations about the structure of the theory in the weak coupling region. Since no exact results are available, we appeal to mean field theory techniques. The main idea is that there could be a phase transition in the Itzykson-Zuber integral, which drastically modifies the solution of the model in the weak coupling regime. The motivation is to look for solutions of the Kazakov-Migdal model which have a continuum limit resembling quantized Yang-Mills theory. It also can be used to say something about whether the Kazakov-Migdal model can be obtained as a limit of more conventional lattice gauge theory and whether the continuum theory which describes Yang-Mills theory can be obtained in that limit.

In the present Section, we shall use mean field theory as an intuitive introduction to the subject of induced gauge theory. Although our central purpose here is to discuss fermionic models, we devote some space to comparison with similar results for Hermitian adjoint models.
5.1 The Kazakov-Migdal Model with Adjoint Fermion Fields

Consider a $D$-dimensional oriented hypercubic lattice $\mathcal{L}^D \subset \mathbb{R}^D$ with sites $x$ and links $\ell = \langle x, y \rangle$ connecting nearest neighbour sites $x$ and $y$. At each site there are fermion fields $\psi_j(x)$ and their conjugates $\bar{\psi}_j(x)$, $j = 1, \ldots, N_f$, which are independent $N \times N$ Grassmann-valued matrices describing $N_f$ flavours of fermions with $N$ colours. These fields also have an implicit spin index $\mu = 1, \ldots, 2^{[D/2]}$ (i.e. $\bar{\psi}$ and $\psi$ are actually $N \times (2^{[D/2]}N)$ and $(2^{[D/2]}N) \times N$ matrix fields, respectively) which labels the spinor representation of the Euclidean group in $D$-dimensions. On each link there is a gauge generalization of (4.1) to $D$-dimensions and $N_f$ fermion flavours is the lattice gauge theory

$$Z_D = \int \prod_{(x,y) \in \mathcal{L}^D} \left[ du(x,y) \right] \int \prod_{x \in \mathcal{L}^D} d\psi_j(x) \ d\bar{\psi}_j(x) \ e^{S_F[\psi, \bar{\psi} ; U]} \tag{5.1}$$

where

$$S_F[\psi, \bar{\psi} ; U] = \sum_{j=1}^{N_f} \sum_{x \in \mathcal{L}^D} N^2 \ tr \left( V[\bar{\psi}_j(x)\psi_j(x)] \right)$$

$$- \sum_{\ell=1}^{D} \left[ \bar{\psi}_j(x) \mathcal{P}_\ell^- U(x, x + \ell) \psi_j(x + \ell) U^\dagger(x, x + \ell) \right. \right.$$ 

$$\left. + \bar{\psi}_j(x + \ell) \mathcal{P}_\ell^+ U^\dagger(x + \ell, x) \psi_j(x) U(x + \ell, x) \right] \tag{5.2}$$

is the gauge-invariant lattice fermion action [86, 133]. It includes the usual fermion kinetic term and fermion-gauge minimal coupling in the last sum through the gauge-covariant lattice derivative for Dirac fermions which transform in the adjoint representation of the colour gauge group $U(N)$. Here

$$\mathcal{P}_\ell^\pm = r \pm \gamma_\ell \tag{5.3}$$

are the usual projection operators acting on the spin components in (5.2), so that $r = 0$ for chiral fermions while $r = 1$ for Wilson fermions [86, 133], and $\gamma_\ell$ are the gamma-matrices which generate the $D$-dimensional Euclidean Dirac algebra

$$\{\gamma_\ell, \gamma_\ell'\} = 2\delta_{\ell\ell'} \tag{5.4}$$

The potential is the usual Dirac potential

$$V(\bar{\psi}\psi) = m\bar{\psi}\psi + V_{\text{im}}(\bar{\psi}\psi) \tag{5.5}$$

with $m$ the bare fermion mass.

The action (5.2) differs from that of ordinary Wilson lattice gauge theory [133] in that the kinetic term for the gauge field

$$S_W[U] = \sum_{\ell \in \mathcal{L}^D} \frac{N^2}{g^2} \ Re \ W[U; \Box] \tag{5.6}$$

Recall that in lattice gauge theory a gauge field $U(x, x + \ell) \sim e^{iA_\ell(x)}$ (with $A_\ell(x)$ the continuum gauge field) is a function on lattice links, so that the curvature $F \sim dA + [A, A]$ is a function on plaquettes of the lattice. Consequently, (5.6) coincides with the usual Yang-Mills action $F^2$ in the continuum limit where the (unit) lattice spacing goes to zero, or equivalently $g \to 0$. 

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is absent. Here $\square$ denotes the elementary plaquettes of the lattice and for any oriented contour $C \in \mathcal{L}^D$, $W[U; C]$ is the Wilson line operator associated with $C$, i.e. the trace of the counterclockwise-oriented path-ordered product of the link operators along $C$,

$$W[U; C] = \text{tr} \ P \prod_{(x,y) \in C} U(x, y) = W[U; -C]^*$$

(5.7)

where $-C$ denotes the contour $C$ with the opposite orientation. However, because of the local $U(1)$ gauge-invariance (4.4) of the matrix model (5.1) (applied to a single link), the expectation value of any fundamental representation Wilson loop\footnote{Note that for the fundamental representation $N$ of $U(N)$, whose action on matrices $M$ is defined as $M \to U \cdot M, U \in U(N)$, we have the group-theoretical decomposition}

$$N \otimes \tilde{N} = A \otimes 1$$

where $\tilde{N}$ is the complex conjugate representation of $N$, and $A$ and 1 are the adjoint and trivial (singlet) representations of $U(N)$, respectively. This means that invariant traces in the adjoint representation are related to (ordinary) invariant traces in the fundamental representation by

$$\text{tr}_A U = | \text{tr} U |^2 - 1/N^2$$

(5.11)
If we consider in addition the large-$N_f$ limit correlated with the large-mass expansion so that $m/N_f^{1/4} \to 1$ as $m, N_f \to \infty$, then all terms in (5.11) except the leading-order contributions in $1/m$ from single plaquettes are suppressed by factors of order $N_f^{-1/2}$. Then the induced action $S_{\text{ind}}[U]$ becomes the single-plaquette adjoint action

$$S_A[U] = N^2 \beta_A \sum_{\Box \in \mathcal{D}} |W[U; \Box]|^2$$

with coupling constant

$$\beta_A = \frac{2^{2r-1} N_f (1 + 2r^2 - r^4)}{m^4}$$

(5.12)

Thus the adjoint fermion matrix model (5.1) in these limits induces a single-plaquette lattice gauge theory built from traces in the adjoint representation of the gauge group (as opposed to the more conventional fundamental representation as in (5.6)).

Notice that for a matrix chain, associated with dimension $D \leq 1$, the gauge fields $U(x, y)$ in (5.1) can be absorbed by a local gauge transformation (4.3) and the model (5.1) reduces to a fermionic multi-matrix model. In particular the case $D = 1$ is associated with an infinite matrix chain while $D = \frac{1}{2}$ corresponds to the two-matrix model of the last Section. The generic $D > 1$ model is the fermionic analog of the Kazakov-Migdal model which is defined using heavy scalar fields in the adjoint representation of the gauge group and which has recently been proposed as a model for (induced) QCD [75, 85, 95]. If the lattice gauge theory (5.1) is to indeed have a continuum limit which reproduces the characteristic properties of QCD, then it must have a phase transition into a phase where Wilson loop observables obey an area law, the characteristic feature of quark confinement. This phase structure can be studied from various points of view, such as a mean field analysis of the induced gauge theory (5.11) and loop equations for the matrix model (5.1). These approaches, as well as the possibility of using (5.1) as a model for strings in $D > 1$ dimensions, will be discussed in this and the next Section.

### 5.2 Phase Transitions in Unitary Matrix Integrals at Infinite $N$

Before embarking on a detailed analysis of the critical behaviour of the matrix model (5.1) when it is viewed as an induced gauge theory, we need some preliminary results concerning the phase structure of unitary matrix models.

#### 5.2.1 The Fundamental Model

Consider first the “fundamental" unitary matrix model [21, 68] with partition function

$$Z_F(\alpha) = \int [dU] \ e^{N^2 \text{tr} \ [\alpha^* U + \alpha U^\dagger]}$$

(5.14)

This integral can be computed explicitly in the large $N$ limit to get the free energy [68],

$$F_F(\alpha) = -\lim_{N \to \infty} \frac{1}{N^2} \log Z_F(\alpha) = \begin{cases} -|\alpha|^2 & |\alpha| \leq 1/2 \\ -2|\alpha| + \frac{1}{2} \log 2|\alpha| + 3/4 & |\alpha| \geq 1/2 \end{cases}$$

(5.15)
Because of the invariance of the integration measure under a change in phase of the unitary matrices, the free energy depends only on the modulus of $\alpha$. There is a third order phase transition at $|\alpha| = 1/2$. We shall call the phase with $|\alpha| \leq 1/2$ the “strong coupling phase” and that with $|\alpha| \geq 1/2$ the “weak coupling phase”.

From this expression, connected correlators of $\text{tr} \mathbf{U}$ and $\text{tr} \mathbf{U}^\dagger$ can be obtained by taking derivatives with respect to $\alpha$ and $\alpha^*$. For example,

$$\langle \text{tr} \mathbf{U} \rangle_F \equiv -\frac{\partial}{\partial \alpha^*} F_F(\alpha) = \begin{cases} \alpha & |\alpha| \leq 1/2 \\ \frac{\alpha}{|\alpha|} \left( 1 - \frac{1}{|\alpha|} \right) & |\alpha| \geq 1/2 \end{cases} \quad (5.16)$$

and

$$\langle |\text{tr} \mathbf{U}|^2 \rangle_F - |\langle \text{tr} \mathbf{U} \rangle_F|^2 = 0 \quad (5.17)$$

The latter result, which is correct in both the weak and strong coupling phases, is a result of factorization, i.e., the expectation value of a product of any two functions of $\mathbf{U}$ and $\mathbf{U}^\dagger$ which are separately invariant under unitary conjugation, $\mathbf{U} \rightarrow \mathbf{WUW}^\dagger$ and $\mathbf{U}^\dagger \rightarrow \mathbf{WU}^\dagger \mathbf{W}^\dagger$, factorizes into the product of expectation values.

The result (5.15) can be obtained explicitly [68] by using the invariance of the integrand and measure in the partition function under conjugation by unitary matrices to diagonalize $\mathbf{U}$ and $\mathbf{U}^\dagger$ and to write the partition function as an integral over the eigenvalues. The infinite $N$ limit is then obtained by saddle point integration which can be done explicitly. It is straightforward to check the small and large $|\alpha|$ limits. First, consider small $|\alpha|$. One can Taylor expand the integrand in $|\alpha|$ to obtain

$$Z_F(\alpha) = \int [d\mathbf{U}] \left\{ 1 + N^2 \text{tr} \left( \alpha^* \mathbf{U} + \alpha \mathbf{U}^\dagger \right) + \frac{1}{2!} \left( N^2 \text{tr} \left( \alpha^* \mathbf{U} + \alpha \mathbf{U}^\dagger \right) \right)^2 + \frac{1}{3!} \left( N^2 \text{tr} \left( \alpha^* \mathbf{U} + \alpha \mathbf{U}^\dagger \right) \right)^3 + \ldots \right\} \quad (5.18)$$

Using the explicit integrals over the normalized Haar measure (see Subsection 1.1.1) we obtain

$$Z_F(\alpha) = 1 + N^2 |\alpha|^2 + \frac{1}{2} N^4 |\alpha|^4 + \ldots \quad (5.19)$$

so that

$$F_F(\alpha) = -|\alpha|^2 + \ldots \quad (5.20)$$

which agrees with the strong coupling phase in (5.15).

Also, when $|\alpha|$ is large, we perform a saddle point integration. Consider the variation $\delta \mathbf{U} = i \mathbf{U} \mathbf{H}$ and $\delta \mathbf{U}^\dagger = -i \mathbf{H} \mathbf{U}^\dagger$ where $\mathbf{H}$ is a Hermitian matrix. The exponent in the integrand of the partition function is maximized by matrices which satisfy the equation

$$U_0 = \frac{\alpha}{\alpha^*} U_0^\dagger \quad (5.21)$$

or equivalently

$$U_0 = \frac{\alpha}{|\alpha|} \mathbf{I} \quad (5.22)$$
Then, defining \( U = U_0 e^{iH} \), \( U^\dagger = e^{-iH} U_0^\dagger \), the partition function is approximated as\(^{10}\)

\[
Z_F(\alpha) = e^{2N^2|\alpha|} \int dH \ e^{-N^2|\alpha| \text{ tr } H^2} = e^{2N^2|\alpha| - \frac{N^2}{2} \log(|\alpha|) + \ldots}
\]

(5.23)

which agrees with (5.15) to the leading orders. It is interesting that, up to normalization, it agrees with (5.14) completely. This means that, in the “weak coupling” phase, the large \( N \) limit of the integral is given exactly by its semiclassical (spherical) approximation. This is already known to be the case for the Itzykson-Zuber integral (at finite-\( N \)) with Hermitian matrices \(^{[85, 114]}\), and it would be interesting to know whether this happens in other closely related cases.

It is possible to use a more sophisticated semiclassical approximation to the above integral. For this, we identify all extrema of the action (maxima, minima and saddle points). It is easy to see that they are given by

\[
U_0 = \alpha \begin{pmatrix}
\pm 1 & 0 & 0 & 0 & \ldots \\
0 & \pm 1 & 0 & 0 & \ldots \\
0 & 0 & \pm 1 & 0 & \ldots \\
0 & 0 & 0 & \pm 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots
\end{pmatrix}
\]

(5.24)

The semiclassical limit is given by

\[
Z_F(\alpha) = \sum_{U_0} e^{2N^2|\alpha| \text{ tr } U_0} \int dH \ e^{-N^2|\alpha| \text{ tr } H^2 U_0}
\]

\[
= e^{2N^2|\alpha| (N|\alpha|)} e^{-\frac{N^2}{2}} + N e^{2N(N-2)|\alpha|((-N|\alpha|)} e^{-\frac{N^2}{2}} + \ldots
\]

\[
+ \binom{N}{m} e^{2N(N-2m)|\alpha|((-1)^m N|\alpha|)} e^{-\frac{N^2}{2}} + \ldots
\]

\[
= e^{2N^2|\alpha| (N|\alpha|)} e^{-\frac{N^2}{2}} \left(1 + (-1)^{\frac{N^2}{2}} e^{-2N|\alpha|}\right)^N
\]

(5.25)

This (exact) partition function is real when \( N \) is even. In the large \( N \) limit, it reduces to (5.23).

\(^{10}\)Here we use the fact that a Hermitian Gaussian matrix integral is easily evaluated to be

\[
\int dH \ e^{-N^2m^2 \text{ tr } H^2} = \prod_{i=1}^N \int dH_{ii} e^{-N \frac{m^2}{2} H_{ii}^2} \prod_{1 \leq i < j \leq N} \int d^2 H_{ij} e^{-N m^2 |H_{ij}|^2}
\]

\[
= (2\sqrt{2})^N \left(\frac{\pi}{N m^2}\right)^{N^2/2}
\]

for \( m \in \mathbb{R} \).
5.2.2 The Adjoint Model

We are primarily interested in the model with partition function

\[ Z_A(\beta) = \int [dU] \ e^{N^2 \beta |\text{tr} \ U|^2} \]  

(5.26)

This model has, besides the symmetry under conjugation of \( U \) by unitary matrices exhibited by (5.14), a symmetry under redefining \( U \) by an element of the center of the unitary group, \( U \rightarrow e^{i\theta} U \). This symmetry guarantees that a correlator vanishes unless it contains the same number of \( U \)'s and \( U^\dagger \)'s. In particular, we have (c.f. eq. (4.23))

\[ \langle U_{jk} U_{kl} \rangle_A = N \langle | \text{tr} \ U |^2 \rangle_A \delta_{ij} \delta_{kl} \]  

(5.27)

The partition function \( Z_A \) can be obtained as a Gaussian integral transform of \( Z_F \) as

\[ Z_A(\beta) \sim \int d\eta \ e^{-N^2 |\eta|^2/\beta} Z_F(\eta) \]  

(5.28)

In the infinite \( N \) limit, this integral can be performed by saddle point integration. It is necessary to find the value of \( \eta \) which minimizes the normalized free energy function

\[ F_A(\beta) = \min_{\eta} \left\{ \begin{array}{ll}
(\beta^{-1} - 1)|\eta|^2 & |\eta| \leq 1/2 \\
\beta^{-1}|\eta|^2 - 2|\eta| + \frac{1}{2} \log 2|\eta| + 3/4 & |\eta| \geq 1/2
\end{array} \right. \]  

(5.29)

When \( \beta < 1, |\eta| = 0 \). When \( \beta > 1, |\eta| = \frac{1}{2} \beta \left( 1 + \sqrt{1 - 1/\beta} \right), \) so that

\[ F_A(\beta) = \left\{ \begin{array}{ll}
0 & \beta \leq 1 \\
\frac{\beta}{4} (1 - \sqrt{1 - 1/\beta})^2 - \beta + 3/4 + \frac{1}{2} \log \left( \beta (1 + \sqrt{1 - 1/\beta}) \right) & \beta \geq 1
\end{array} \right. \]  

(5.30)

This solution has the interesting property that

\[ \langle | \text{tr} \ U |^2 \rangle_A \equiv -\frac{\partial}{\partial \beta} F_A(\beta) = \left\{ \begin{array}{ll}
0 & \beta < 1 \\
\frac{1}{2} \left( 1 + \sqrt{1 - 1/\beta} \right)^2 & \beta \geq 1
\end{array} \right. \]  

(5.31)

This expectation value is discontinuous at \( \beta = 1 \) and the phase transition there is of first order.

This behavior can be checked by perturbation theory in both the strong and weak coupling regions. In strong coupling, we expand the exponents in the integrand in \( \beta \),

\[ \langle | \text{tr} \ U |^2 \rangle_A = \frac{\int [dU] \ | \text{tr} \ U |^2 \left( 1 + N^2 \beta \text{tr} \ (U + U^\dagger) + \ldots \right)}{\int [dU] \ (1 + N^2 \beta \text{tr} \ (U + U^\dagger) + \ldots)} \]  

(5.32)

which, using the explicit integrals for the Haar measure correlation functions given in Subsection 1.1.1, we can easily see is zero to order \( \beta^2 \). In the large \( \beta \) limit, we use the saddle point method. The variation of the action gives the saddle point equation

\[ U \text{tr} \ U^\dagger = U^\dagger \text{tr} \ U \]  

(5.33)

which has solution

\[ U_0 = e^{i\theta} I \]  

(5.34)

and

\[ Z_A(\beta) \approx e^{\beta N^2} \int dH \ e^{-\beta(N^2 \text{tr} \ H^2 - N^3 \text{tr} \ H)} = e^{N^2(\beta - \frac{1}{2} \log \beta)} \]  

(5.35)

which agrees with an asymptotic expansion of (5.30).
Finally, we consider the model with a mixed action,

$$Z_M(\alpha, \beta) = \int [dU] \ e^{N^2 \beta \langle \text{tr} \ U \rangle^2 + N^2 \text{tr} \ [\alpha U + \alpha U^\dagger]}$$  \hspace{1cm} (5.36)$$

This action has a term which explicitly breaks the symmetry under phase transformation of $U$ and $U^\dagger$. The mixed model (5.36) is invariant under the unitary transformations $U \to VUW$ where either $V$ is taken as the matrix that multiplies the $n$-th row of $U$ by $-1$ and $W$ multiplies the $n$-th column by $-1$, or when $V$ interchanges the $m$-th and $n$-th rows while $W$ interchanges the $m$-th and $n$-th columns. This symmetry combined with the invariance of (5.36) under conjugation by unitary matrices leads to the identities

$$\langle U_{jk}U_{kl} \rangle_M = \frac{N^2(N-1)-1}{N^2-1} \left( \langle | \text{tr} \ U |^2 \rangle_M - 1 \right) \delta_{jk} \delta_{kl} + \frac{N^2}{N^2-1} \left( 1 - \langle | \text{tr} \ U |^2 \rangle_M \right) \delta_{ij} \delta_{kl}$$  \hspace{1cm} (5.37)

$$\langle U_{ij} \rangle_M = \langle \text{tr} \ U \rangle_M \delta_{ij}$$  \hspace{1cm} (5.38)

The partition function can be obtained by the offset Gaussian transform

$$Z_M(\alpha, \beta) \sim \int d\eta \ d\eta^* \ e^{-N^2 \eta^2/\beta + N^2 \eta^* \alpha + \eta \alpha^* - |\eta|^2 N^2/\beta} \ Z_F(\eta)$$  \hspace{1cm} (5.39)

In the large $N$ limit, this can again be found by minimizing the free energy function

$$F_M(\alpha, \beta) = \min_{\eta} \left\{ (\beta^{-1} - 1) |\eta|^2 - \frac{\eta^\alpha + \alpha^* \eta}{\beta} + \frac{|\eta|^2}{\beta} \right\} \text{ for } |\eta| \leq \frac{1}{2}$$

$$= \min_{\eta} \left\{ (\beta^{-1} - 1) |\eta|^2 - \frac{2 |\eta| + \frac{1}{2} \log 2 |\eta| - \frac{\eta^\alpha + \alpha^* \eta}{\beta} + \frac{|\eta|^2}{\beta} + \frac{3}{4} }{ } \right\} \text{ for } |\eta| \geq \frac{1}{2}$$  \hspace{1cm} (5.40)

The minima occur at $\eta = \alpha/(1 - \beta)$ when $|\eta| \leq 1/2$, which is the region $|\alpha| \leq (1 - \beta)/2$. In the other region, $|\eta| = \frac{1}{2}(\beta + |\alpha| + \sqrt{(\beta + |\alpha|^2 - \beta)})$ which exists where $|\alpha| \geq \sqrt{\beta - \beta}$ which is always satisfied in the region $|\alpha| \geq \frac{1}{2}(1 - \beta)$. Thus, the solution for the free energy is

$$\frac{1}{N^2} F_M(\alpha, \beta) = \frac{1}{(1 - \beta)} \text{ for } |\alpha| \leq \frac{1}{2}(1 - \beta), 0 \leq \beta \leq 1$$

$$= \frac{1}{4} \beta \left( (\beta + |\alpha| + \sqrt{(\beta + |\alpha|^2 - \beta)})^2 - \frac{3}{4} + |\alpha|^2/\beta \right) - \left( (\beta + |\alpha| + \sqrt{(\beta + |\alpha|^2 - \beta)}) (1 + |\alpha|/\beta) \right) + \frac{1}{2} \log \left( (\beta + |\alpha|) + \sqrt{(\beta + |\alpha|^2 - \beta)} \right) \text{ for } |\alpha| \geq \frac{1}{2}(1 - \beta), \beta \geq 0$$  \hspace{1cm} (5.41)

In this case,

$$\langle \text{tr} \ U \rangle_M = \begin{cases} \frac{\alpha}{|\alpha|^2} \left( |\alpha| - \beta - \sqrt{(\beta + |\alpha|^2 - \beta)} \right) & |\alpha| \leq \frac{1}{2}(1 - \beta), 0 \leq \beta \leq 1 \\ -\frac{1}{|\alpha|^2} \left( |\alpha| - \beta - \sqrt{(\beta + |\alpha|^2 - \beta)} \right) & |\alpha| \geq \frac{1}{2}(1 - \beta), \beta \geq 0 \end{cases}$$  \hspace{1cm} (5.42)

and

$$\langle | \text{tr} \ U |^2 \rangle_M = \begin{cases} \frac{\alpha^2}{4 |\alpha|^2} \left( |\alpha| - \beta - \sqrt{(\beta + |\alpha|^2 - \beta)} \right)^2 & |\alpha| \leq \frac{1}{2}(1 - \beta), 0 \leq \beta \leq 1 \\ \frac{1}{4 |\alpha|^2} \left( |\alpha| - \beta - \sqrt{(\beta + |\alpha|^2 - \beta)} \right)^2 & |\alpha| \geq \frac{1}{2}(1 - \beta), \beta \geq 0 \end{cases}$$  \hspace{1cm} (5.43)
This result exhibits factorization in all phases. Furthermore, it exhibits \textit{spontaneous symmetry breaking} in the \textquotedblleft weak coupling phase\textquotedblright $\beta > 1$, \begin{equation}
\lim_{|\alpha| \to 0} \langle \text{tr } U \rangle_M = \begin{cases}
0 & |\alpha| \leq \frac{1}{2}(1 - \beta), 0 \leq \beta \leq 1 \\
\frac{\alpha}{\beta^2} (1 + \sqrt{1 - 1/\beta}) & |\alpha| \geq \frac{1}{2}(1 - \beta), \beta \geq 0
\end{cases}
\end{equation}

It is in such a symmetry-breaking scenario that the local $U(1)$ symmetry of the adjoint lattice gauge model (5.12) could be viewed as being eliminated and the resulting model possessing only the usual symmetries of the standard Wilson lattice gauge theory \cite{84} (which gives the latticized version of QCD). We shall examine this possibility further in Subsection 5.3.

The mixed unitary matrix model (5.36) has been studied by Makeenko and Polikarpov \cite{100} (in the context of lattice gauge theory) and they showed that at $N = \infty$ the mixed model is equivalent to the fundamental model (5.14). This follows from the factorization property of the correlators above of the mixed model. It can be verified explicitly within the weak and strong coupling expansions, and it can be proved exactly using loop equations \cite{100}. To show this, we assume for simplicity that the coupling constant $\alpha$ of the fundamental term in (5.36) is real and positive. At $N = \infty$ we use the factorization property \begin{equation} \langle |\text{tr } U|^2 \rangle_M = \langle \text{tr } U \rangle_M^2 \end{equation}

to write a differential equation for the free energy $F_M$,
\begin{equation} 2 \frac{\partial F_M}{\partial \beta} = \left( \frac{\partial F_M}{\partial \alpha} \right)^2 \end{equation}

Differentiating (5.46) with respect to $\alpha$ then yields to $O(1/N^2)$ \begin{equation} \frac{\partial}{\partial \beta} \langle \text{tr } U \rangle_M = \langle \text{tr } U \rangle_M \cdot \frac{\partial}{\partial \alpha} \langle \text{tr } U \rangle_M \end{equation}

If we substitute the ansatz \begin{equation} \langle \text{tr } U \rangle_M = \langle \text{tr } U \rangle_F \end{equation}

with the fundamental average $\langle \text{tr } U \rangle_F$ taken in the fundamental model (5.14) with the redefined coupling $\bar{\alpha}$ defined by the implicit function relation \begin{equation} \bar{\alpha} = \alpha + \beta \langle \text{tr } U \rangle_F \end{equation}

then from \begin{equation} \frac{\partial}{\partial \alpha} \langle \text{tr } U \rangle_M = \frac{d}{d\bar{\alpha}} \langle \text{tr } U \rangle_F \left[ 1 - \beta \frac{d}{d\bar{\alpha}} \langle \text{tr } U \rangle_F \right]^{-1} \end{equation}

\begin{equation} \frac{\partial}{\partial \beta} \langle \text{tr } U \rangle_M = \langle \text{tr } U \rangle_M \cdot \frac{d}{d\bar{\alpha}} \langle \text{tr } U \rangle_F \left[ 1 - \beta \frac{d}{d\bar{\alpha}} \langle \text{tr } U \rangle_F \right]^{-1} \end{equation}

it is readily seen that (5.48),(5.49) satisfies (5.47).

Thus in the mixed model (5.36), the gauge field correlators at large-$N$ depend not on the 2 variables $\alpha$ and $\beta$ separately, but are a function of their combination determined by the effective charge $\bar{\alpha}$ in (5.49). Using (5.16) this effective charge is \begin{equation} \bar{\alpha}(\alpha, \beta) = \begin{cases}
\frac{\alpha}{1 - \beta} & \alpha \leq \frac{1}{2}, 0 \leq \beta \leq 1 \\
\frac{1}{2} \left( \alpha + \beta + \sqrt{(\beta + \alpha)^2 - \beta} \right) & \frac{\alpha}{1 - \beta} \geq \frac{1}{2}, \beta \geq 0
\end{cases} \end{equation}

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The first order phase transition in the model occurs at the points where the specific heat changes its sign (and becomes infinite). These are the points where the right-hand sides of (5.50) vanish, i.e.

$$
\beta_c = \left( \frac{d}{d\tilde{\alpha}} \langle \text{tr } U \rangle_F \right)^{-1} , \quad \alpha_c = \tilde{\alpha} - \beta_c \langle \text{tr } U \rangle_F
$$

(5.52)

Finally, the string tension $\sigma_M$, which is defined by the area law behaviour of the Wilson loops

$$
\langle \text{tr } U \rangle_M \equiv e^{-\sigma_M A}
$$

(5.53)

can easily be found from (5.16) and (5.49) to be

$$
\sigma_M(\alpha, \beta) = \begin{cases} 
\frac{1}{A} \log \left( \frac{1 - \beta}{\alpha} \right) , & \frac{\alpha}{1 - \beta} \leq \frac{1}{2} , \ 0 \leq \beta \leq 1 \\
\frac{1}{A} \log \left( \frac{2 (\alpha + \beta + \sqrt{(\beta + \alpha)^2 - \beta} - \beta)}{(2 (\alpha + \beta + \sqrt{(\beta + \alpha)^2 - \beta} - 1)} \right) , & \frac{\alpha}{1 - \beta} \geq \frac{1}{2} , \ \beta \geq 0
\end{cases}
$$

(5.54)

Notice that for finite-$N$ one generally expects a perimeter law for the adjoint Wilson loops $\langle |\text{tr } U|^2 \rangle_A$. However, the above analysis shows that at $N = \infty$ they satisfy the same area law as the fundamental Wilson loops $\langle \text{tr } U \rangle_F$. The perimeter law at finite-$N$ thus enters at $\mathcal{O}(1/N^2)$. We also note from (5.31) that while the string tension in the adjoint unitary matrix model in the weak coupling phase is finite and corresponds to normal quark confinement, that in the strong coupling phase is infinite and is associated with local confinement (i.e. the absence of quarks in that phase).

### 5.3 Mean Field Theory and Confinement at $N = \infty$

We now carry out a detailed analysis of the phase structure at $N = \infty$ of the gauge theory induced by (5.1). We first note that factorization implies that the gauge theory with adjoint action (5.12) is equivalent at $N = \infty$ to that with the fundamental representation Wilson action $S_W[U]$ provided that the coupling in (5.6) obeys the self-consistency condition $[100]$

$$
\hat{\beta} \equiv \frac{1}{g^2} = \beta_A W_F(\square; \hat{\beta})
$$

(5.55)

where $W_F(\square; \hat{\beta})$ denotes the fundamental plaquette average

$$
W_F(\square; \hat{\beta}) = \frac{\int \prod_{(x,y) \in \mathcal{L}^D} [dU(x,y)] e^{S_W[U(x,y)]} W[U; \square]}{\int \prod_{(x,y) \in \mathcal{L}^D} [dU(x,y)] e^{S_W[U(x,y)]}}
$$

(5.56)

(5.55) can be solved for $\hat{\beta}(\beta_A)$ at strong coupling by substituting in the strong coupling expansion of (5.56) $[54, 133]$

$$
W_F(\square; \hat{\beta}) = \frac{\hat{\beta}}{2} + \frac{\hat{\beta}^5}{8} \quad \text{for} \quad \hat{\beta} \ll 1
$$

(5.57)
(5.55) in the strong coupling regime always possesses the trivial solution $\tilde{\beta} = 0$. However, for $\beta_A \sim 2$ there is the nontrivial solution

$$\tilde{\beta} \propto \left( \frac{1}{2} - \frac{1}{\beta_A} \right)^{1/4} \tag{5.58}$$

which matches the weak coupling solution

$$\tilde{\beta} \sim \beta_A - 1/4 \quad \text{for} \quad \tilde{\beta} \gg 1 \tag{5.59}$$
as $\beta_A \to \infty$.

The basic observable of the induced gauge theory (5.12) is the adjoint plaquette average

$$W_A(\square; \beta_A) = \frac{\int \prod_{(x,y) \in \mathcal{L}^D} [dU(x,y)] \ e^{S_A[U(x,y)]} \left( |W[U; \square]|^2 - \frac{1}{N_f^2} \right)}{\int \prod_{(x,y) \in \mathcal{L}^D} [dU(x,y)] \ e^{S_A[U(x,y)]}} \tag{5.60}$$

Note that since the product of 2 fundamental representation link operators contains both the adjoint and scalar representations, i.e. $(U_{ij}U_{kl}^\dagger)_{A} = U_{ij}U_{kl}^\dagger - \delta_{ik}\delta_{jl}$, we subtract off the latter in (5.60). Factorization implies that at $N = \infty$ it is given by

$$W_A(\square; \beta_A) = \left( W_F(\square; \tilde{\beta}) \right)^2 = \left( \frac{\tilde{\beta}}{\beta_A} \right)^2 \tag{5.61}$$

Since the sign of the second term in (5.57) is positive, we find that the slope of (5.61) is negative for the solution (5.58) near $\beta_A = 2$. Consequently, a first order phase transition in the induced gauge theory (5.12) must occur with increasing $\beta_A$ at some critical coupling $\beta_A^c < 2$ [79, 95].

Although the integrations in (5.1) cannot be carried out explicitly, the phase diagram can be studied by employing a variational mean field analysis [54, 79, 80] to the partition function (5.1) for $N_f = 1$ and $m \gg 1$. The mean field approximation is based on a result known as Jensen’s inequality [54] which is easily derived as follows. Consider 2 partition functions $Z^{(1)}$ and $Z^{(0)}$ which are defined in terms of actions $S_1$ and $S_0$, i.e. $Z^{(i)} = \int e^{-S_i}$, with statistical averages $\langle A \rangle_i = \int A e^{-S_i} / Z^{(i)}$. For $t \in [0, 1]$, we introduce the one-parameter family of partition functions $Z^{(t)} = \int e^{-tS_0 - (1-t)S_1}$. Then a simple calculation shows that

$$\frac{d^2 \log Z^{(t)}}{dt^2} = \langle (S_1 - S_0)^2 \rangle_t - \langle S_1 - S_0 \rangle_t^2 \geq 0 \tag{5.62}$$

where we have used the fact that $\langle A^2 \rangle \geq \langle A \rangle^2$ for any statistical average. From this it follows that, as a function of $t$, $\log Z^{(t)}$ is concave up everywhere, and so it lies above all of its tangent lines. In particular,

$$\log Z^{(1)} \geq \log Z^{(0)} + \frac{d \log Z^{(t)}}{dt} \bigg|_{t=0} \tag{5.63}$$

which leads to

$$Z^{(1)} \geq Z^{(0)} e^{\langle S_0 - S_1 \rangle_0} \tag{5.64}$$

(5.64) is a special case of what is known in functional analysis as Jensen’s inequality.
Going back to our analysis, we introduce the trial partition function

\[
Z_D^{(A)} = \int \prod_{(x,y) \in \mathcal{L}^D} [dU(x,y)] e^{-\frac{N^2b_A}{2} \sum_{(x,y) \in \mathcal{L}^D} \text{tr} U(x,y)^2} \tag{5.65}
\]

which is similar in form to the unitary matrix integrations in (5.1). The partition function (5.65) is a product of one-link unitary matrix integrals which are the simplest ones with the local \(U(1)\)-gauge invariance of (5.1). It can therefore be evaluated using the results of Subsection 5.2.2 above as

\[
Z_D^{(A)} = e^{-N^2 \text{vol}(\mathcal{L}^D)DF_A(-b_A/2)} \tag{5.66}
\]

where \(\text{vol}(\mathcal{L}^D)\) is the volume of the lattice \(\mathcal{L}^D\) so that \(\text{vol}(\mathcal{L}^D)D\) is the total number of links in the lattice (there are \(2D\) nearest neighbours to each site). Then Jensen’s inequality (5.64) with \(Z^{(1)} = Z_D, Z^{(0)} = Z_D^{(A)}\) gives a bound on the partition function (5.1)

\[
Z_D \geq Z_D^{(A)} \int \prod_{x \in \mathcal{L}^D} d\bar{\psi}(x) d\bar{\psi}(x) \exp \left[ \sum_{x \in \mathcal{L}^D} \left\{ N^2m \text{tr} \bar{\psi}(x)\psi(x) - \left\langle \sum_{\ell=1}^D \left( N^2 \text{tr} (\bar{\psi}(x)\mathcal{P}_\ell^-U(x,x+\ell)\psi(x+\ell)U(x,x+\ell) + \bar{\psi}(x + \ell)\mathcal{P}_\ell^+U(x+\ell,x)\psi(x+\ell,x)) - \frac{N^2b_A}{2} \text{tr} U(x,x+\ell)^2 \right) \right\}_0 \right\} \right] \tag{5.67}
\]

where the subscript 0 denotes the normalized average with respect to the statistical ensemble (5.65). Since the argument of the exponential function in (5.67) is a sum of one-link averages, it can be written in terms of the unitary one-link integral

\[
u^2 = \langle | \text{tr} U |^2 \rangle_A(\beta = -b_A/2) = \frac{\int [dU] e^{-\frac{N^2b_A}{2} | \text{tr} U |^2} | \text{tr} U |^2}{\int [dU] e^{-\frac{N^2b_A}{2} | \text{tr} U |^2}} \tag{5.68}
\]

using the identity

\[
\langle \text{tr} \bar{\psi}(x)\mathcal{P}_\ell^\pm U(x,x+\ell)\psi(x+\ell)U(x,x+\ell) \rangle_0 = \nu^2 \text{tr} \bar{\psi}(x)\mathcal{P}_\ell^\pm\psi(x+\ell) \tag{5.69}
\]

where we have used (5.27).

The idea behind the variational mean field method is to now fix the coupling \(b_A\) by the condition that the explicitly solvable trial partition function (5.65) gives the best approximation to (5.1) in this class of models, from which it is hoped that (5.65) will have a very close behaviour to \(Z_D\). Taking the derivative with respect to \(b_A\) using (5.30), (5.68) and (5.69), we find after some algebra that the maximum value of the right-hand side of (5.67) is at the coupling

\[
b_A = \frac{\int \prod_{x \in \mathcal{L}^D} d\bar{\psi}(x) d\bar{\psi}(x) e^{S_F[\psi, \bar{\psi}; u]} \text{tr} (\bar{\psi}(0)\mathcal{P}_\ell^-\psi(0+\ell) + \bar{\psi}(0+\ell)\mathcal{P}_\ell^+\psi(0))}{\int \prod_{x \in \mathcal{L}^D} d\bar{\psi}(x) d\bar{\psi}(x) e^{S_F[\psi, \bar{\psi}; u]}} \tag{5.70}
\]

where

\[
S_F[\psi, \bar{\psi}; u] = \sum_{x \in \mathcal{L}^D} N^2 \text{tr} \left( m\bar{\psi}(x)\psi(x) - \nu^2 \sum_{\ell=1}^D \left[ \bar{\psi}(x)\mathcal{P}_\ell^-\psi(x+\ell) + \bar{\psi}(x+\ell)\mathcal{P}_\ell^+\psi(x) \right] \right) \tag{5.71}
\]

\[
84
\]
expresses the parameter $b_A$ as a function of the fermion mass $m$. Thus the variational method above amounts to substituting in a mean field value $[U(x,y)]_{ij} = u \delta_{ij}$ for every link operator except the one on the fixed distinguished link $(0,0+\ell)$, and then (5.68) gives a self-consistency condition at this link.

Notice that the large-mass expansion of (5.70) can be represented as

$$b_A = \frac{1}{m} \sum_{\Gamma_{x,\ell} \in C_D} \left( \frac{u^2}{m} \right)^{l(\Gamma_{x,\ell})} \mathrm{TR} \prod_{\ell' \in \Gamma_{x,\ell}} P_{\ell'}^\pm$$

(5.72)

where the sum is over all open contours $\Gamma_{x,\ell}$ with endpoints at $x$ and $x+\ell$. It is only for chiral fermions ($r = 0$ in (5.3)) that the mean field method gives the same large-mass expansion (5.11), since for Wilson fermions ($r = 1$ in (5.3)) backtracking paths never contribute to the sum in (5.11) because of the identity

$$\sigma \equiv P_{\ell}^\pm P_{\ell'}^\mp = r^2 - 1$$

(5.73)

and because of the unitarity of the gauge fields. In (5.72) the contribution from backtracking paths is explicitly taken into account. The parameter $b_A$ can still, however, be considered as an upper bound to the actual value which suffices to give the point of the first order phase transition in the induced gauge theory [54, 79, 80].

In the weak coupling region where $b_A > 2$ and $\frac{1}{2} \leq u \leq 1$, the integral (5.68) can be evaluated (see Subsection 5.2.2 above) and the self-consistency condition can be written as

$$2(u - u^2) = 1/b_A$$

(5.74)

In the strong-coupling phase the unitary matrix integral (5.68) vanishes. The geometric criterion for the location of the first order phase transition is that point where the $u \neq 0$ solution terminates with increasing $m$ \footnote{This requirement is equivalent to the thermodynamic criterion that the free energies of the $u = 0$ and $u \neq 0$ phases coincide.}. The large-$N$ phase transition is then associated with the freezing of the gauge field $U(x,y)$ at some mean field value $u$ while $u = 0$ in the strong coupling regime of the induced gauge theory. Notice that the condition here for the corresponding unitary one-matrix model to possess a weak coupling solution is $u > \frac{1}{2}$.

### 5.3.1 Minimization of the Mean Field Coupling

To explicitly calculate (5.70), we introduce the lattice Fourier transform

$$f(x) = \int_{-\pi}^{+\pi} \prod_{\ell=1}^{D} \frac{dp_{\ell}}{2\pi} e^{-ip \cdot x} f(p) , \quad x \in \mathcal{L}^D$$

(5.75)

where the momentum space integral in (5.75) is restricted to the first Brillouin zone of the lattice $\mathcal{L}^D$ and $p_{\ell} \in [-\pi, +\pi]$ are the Bloch momenta. For example, for chiral fermions which in the naive continuum limit ($m \rightarrow 0$) are associated generally with $2^D N_f$ flavours, the Gaussian integral in (5.70) can be evaluated by the usual lattice technique [54, 133] to give

$$b_A = \frac{2^D}{u^2} \int_{-\pi}^{+\pi} \prod_{\ell=1}^{D} \frac{dp_{\ell}}{2\pi} \left( \frac{1}{m^2} \sum_{\ell=1}^{D} \cos p_{\ell} \right)$$

$$= \frac{2^D}{Du^2} \left( 1 - \frac{m^2}{2u^4} \int_0^\infty dy \ e^{-\left( \frac{m^2}{2u^2} + D \right)y} [I_0(y)]^D \right)$$

(5.76)
where $I_0(y)$ is the modified Bessel function of order 0. Thus in this case $u^2b_A$ is a monotonically increasing function of decreasing mass $m$ which takes its maximal value $2^D/D$ at $m = 0$. Moreover, the second integral in (5.76) can be well-approximated by the leading order term in the expansion in powers of $(\frac{m^2}{2u^4} + D)$ [80] and so we can write

$$\frac{1}{b_A} \sim 2^{-D} \left( \frac{m^2}{2u^4} + D \right) u^2$$

(5.77)

The critical point $m = m_c$ can now be found from (5.74) and the geometric condition mentioned above which is

$$\frac{\partial u}{\partial m} \bigg|_{m=m_c} = \infty$$

(5.78)

For example, for $D = 4$ we find $m_c \sim \frac{13}{2}$ and $u(m_c) \sim \frac{2}{3}$. Notice that since $u^2b_A$ can always reach values larger than $\frac{1}{2}$, the equation (5.74) always has a solution. Notice also that the integral in (5.76) always converges for $0 < m^2 < \infty$ which is a consequence of the stability of the fermionic system.

The situation is the same for Kogut-Susskind fermions [86], which are associated with $2^{D/2}N_f$ continuum fermion flavours and can be obtained from the chiral fermion fields by a flavour-number reduction transformation as

$$\psi(x) = (\gamma_1)^{x_1}(\gamma_2)^{x_2} \cdots (\gamma_D)^{x_D} \chi(x)$$

(5.79)

where $x_\ell$ are the components of the lattice vector $x$. In these new spinor variables the chiral fermion action (5.2) is

$$S_F[\chi, \bar{\chi}; U] = \sum_{x \in L^D} N^2 \text{tr} \left( m \bar{\chi}(x)\chi(x) - \sum_{\ell=1}^{D} \eta_\ell(x) \left[ \bar{\chi}(x)U(x, x + \ell)\chi(x + \ell)U^\dagger(x, x + \ell) 
- \bar{\chi}(x + \ell)U^\dagger(x + \ell, x)\chi(x)U(x + \ell, x) \right] \right)$$

(5.80)

where

$$\eta_1(x) = 1 \quad; \quad \eta_\ell(x) = (-1)^{x_1+\cdots+x_{\ell-1}} \quad \text{for} \quad \ell = 2, \ldots, D \quad (5.81)$$

Since the action (5.80) is now diagonal in the spin indices, it suffices to consider only one spin component of the field $\chi(x)$ so that it can be considered as a spinless (scalar) Grassmann variable. The chiral spinors are then reproduced in the continuum limit by combining $2^D$ of the $\chi$ fields at nearest-neighbour sites of the lattice, so that this procedure manifestly reduces the number of flavours in the continuum by a factor of $2^{D/2}$. The mean field analysis for Kogut-Susskind fermions is now identical to that of chiral fermions above, the only change being that the factor of $2^D$ in (5.76) and (5.77) is now replaced by $2^{D/2}$, so that the self-consistency condition (5.74) at $D = 4$ becomes

$$2 \left( \frac{1}{u} - 1 \right) = \frac{m^2}{8u^4} + 1$$

(5.82)

This equation determines the critical values as above associated with the large-$N$ phase transition to be $m_c \sim \frac{7}{10}$ and $u(m_c) \sim \frac{1}{2}$, and therefore the large-$N$ phase transition occurs as well for the case of Kogut-Susskind fermions.
However, the situation is less clear for Wilson lattice fermions, which correspond to $N_f$ light fermions in the naive continuum limit (while the others whose masses are of the order of the inverse lattice spacing still play a role in the lattice dynamics). Then the standard lattice calculation \cite{5b,5d} of the Gaussian fermionic integral \cite{5b,5d} yields

$$b_A = \frac{2^{D/2}}{D u^2} \int_{-\pi}^{\pi} \prod_{\ell = 1}^{D} \frac{dp_\ell}{2\pi} \left( 1 - \frac{m}{2u^2} \frac{m - \sum_{\ell = 1}^{D} \cos p_\ell}{m - \sum_{\ell = 1}^{D} \cos p_\ell} \right) \right)$$  \hspace{1cm} (5.83)

As mentioned above, the nice feature of (5.83) is that its large-mass expansion coincides with that of the induced action (5.11) where the backtracking paths never contribute due to the unitarity of the gauge fields $U$. However, the price to pay for this nice property is the very complicated integral which now appears in (5.83). Nonetheless, since $b_A$ in (5.83) varies from $b_A = 0$ at $m = \infty$ to $b_A = 1$ at $m = 0$, we expect that (5.70) and (5.78) will possess a solution at some critical value $m_c$. Indeed, an approximate numerical evaluation of the integral in (5.83) \cite{80} gives $m_c \sim u(m_c) \sim \frac{1}{2}$, which resembles the situation for the case of Kogut-Susskind fermions above. However, the uncertainty in the mean field method is even larger now so that a conclusion about the existence of the large-$N$ phase transition is less certain for Wilson fermions. In any case, the large-$N$ phase transition definitely occurs in the case of Wilson fermions for $N_f \geq 2$ \cite{80}. As mentioned in the introduction, for large enough flavours $N_f$ of the fermion fields there is no asymptotic freedom for the adjoint fermion matrix model (5.1). The mean field theory of this Subsection therefore perfectly agrees with this, in that the phase transition occurs outside of the asymptotically free region which is given by \cite{80,95}

$$\frac{11}{3} - \frac{4N_f}{3} > 0$$ \hspace{1cm} (5.84)

### 5.3.2 Area Law

From the analysis of the last Subsection we can conclude that the adjoint fermion lattice gauge theory (5.1) has 2 phases separated by a first order phase transition at some $m = m_c$ above which the model is in the local confinement regime (see below) with dynamics governed by the induced single-plaquette adjoint action (5.12). For $m < m_c$ the model is in the perturbative area law phase and the local $U(1)$ symmetry is broken (in some sense) \cite{110}. The phase structure here is quite different from that of the Kazakov-Migdal model \cite{95} in several respects. For instance, in the fermionic model there is no Higgs phase or instability region due to the fermionic nature of the inducing fields (but there can be a composite Higgs phase associated with $\langle \bar{\psi} \psi \rangle \neq 0$). Furthermore, the first order phase transition is present already for a single fermion flavour, whereas in the Kazakov-Migdal model the large-$N$ first order phase transition appears only for $N_f > N_f^c \sim 30$ \cite{80}. For the Kazakov-Migdal model the continuum gauge theory is reached at the line of a second order phase transition which separates the area law and Higgs phases provided that one approaches it from the area law phase \cite{95}. For the adjoint fermion matrix model (5.1) there is no Higgs phase and the continuum gauge theory is reached as $m \to 0$. Thus in this case the naive continuum limit is in fact the true continuum limit of the lattice gauge theory.

Of course, these conclusions are only based on the mean field approximation used above, but these arguments are quite reasonable because the phase transition here occurs only for systems which are not asymptotically free. At the point of this first order large-$N$ phase
transition, the area law behaviour of the adjoint Wilson loops which is associated with normal
confinement [133] is restored just as it is for the single-plaquette adjoint action (5.12). To see
this, consider the adjoint Wilson loop

$$W_A(\Gamma) = \langle [W[U; \Gamma]]^2 \rangle - 1/N^2$$

(5.85)

We can alternatively average this with respect to the induced action (5.11) which at $N_f = \infty$
becomes the single-plaquette adjoint action (5.12). In this limit, the same factorization
formula (5.61) holds at $N = \infty$ with the elementary plaquette $\square$ replaced by the more general
loop $\Gamma$ and $\bar{\beta}(\beta_A)$ given by (5.55). Since $\bar{\beta} = 0$ for $\beta_A < \beta_A^c$, from (5.61) it follows that the
adjoint Wilson loop (5.85) vanishes there, unless the loop folds onto itself so that the area
$A_{\text{min}}(\Gamma)$ of the minimal surface spanned by $\Gamma$ vanishes

$$W_A(\Gamma) = \delta_{0,A_{\text{min}}(\Gamma)} + \mathcal{O}(1/N^2)$$

(5.86)

This means that in this phase all loops are contractable to a point, due to unitarity, and
formally this corresponds to local confinement with an infinite string tension $\sigma_A(\beta_A) = \infty$.
Thus in this phase quarks aren’t even asymptotically free and cannot propagate even within
hadrons. On the other hand, the area law [133]

$$W_A(\Gamma) \sim e^{-\sigma_A(\beta_A)A[\Gamma]}$$

(5.87)

with string tension

$$\sigma_A(\beta_A) = 2\sigma_F(\bar{\beta}(\beta_A))$$

(5.88)

holds for $\beta_A > \beta_A^c$ when (5.55) has a non-trivial solution. In this area law phase, the dynamics
of extended objects are nontrivial. Although the local confinement condition (5.86) holds to
all orders of the large-mass expansion (5.11) [93], it is not clear if the area law behaviour is
valid for finite $N_f$.

### 5.4 Mean Field Theory of a Bosonic Lattice Gauge Theory at Large-$N$

The above analysis has suggested that the Itzykson-Zuber integral undergoes a large-$N$
first order phase transition when it is defined using Grassmann-valued matrices which transform
under the adjoint representation of the Lie group $U(N)$. It is instructive at this stage to
demonstrate the breakdown of these results in the scalar case. We shall therefore in this
Subsection briefly digress from our study of the adjoint fermion matrix model (5.1) and
discuss the analog of the phase structure in a generalized bosonic lattice field theory which
reduces to the Kazakov-Migdal model (1.49) as a limiting case. We present here a modified
version [131] of the mean field analysis of Khokhlachev and Makeenko [79] who used the
same adjoint unitary matrix integral (5.65) as the trial partition function. We consider the
$D$-dimensional Hermitian matrix model (1.49), except that we include the Wilson term (5.6)
into the action to get a model more closely resemblant to QCD. We are therefore interested
in the adjoint scalar lattice gauge theory

$$Z_{KMW} = \int \prod_{x \in \mathcal{L}^D} d\phi(x) \ e^{-N^2 \sum_{x \in \mathcal{L}^D} \operatorname{tr} V(\phi(x))} \cdot Z_U[\phi, \bar{\beta}]$$

(5.89)
where

\[
Z_U[\phi; \beta] = \int \prod_{(x,y) \in \mathcal{L}^D} [dU(x,y)] \exp \left( N^2 \sum_{\ell = 0}^{D-1} \text{tr} \phi(x) U(x, x + \ell) \phi(x + \ell) U^\dagger(x, x + \ell) \right.
\]

\[
+ \left. \frac{N^2 \beta}{2} \sum_{\square \in \mathcal{L}^D} (W[U; \square] + W[U; \square]^*) \right)
\]

(5.90)

The analysis of the matrix model (5.89) will therefore include the phase structure of the exactly solvable Kazakov-Migdal model when we set \( \beta = 0 \), and more generally the behaviour of the model for all values of \( \beta \) for which (5.89) is a highly non-trivial field theory which may have an interesting critical behaviour that is relevant to a continuum limit for QCD.

To approximate (5.89) using mean field theory, we consider instead of the adjoint model (5.65) the mixed trial partition function

\[
Z_D^{(M)} = \int \prod_{(x,y) \in \mathcal{L}^D} [dU(x,y)] \exp \left\{ N^2 \sum_{(x,y) \in \mathcal{L}^D} \left\{ \text{tr} \left( \alpha^* U(x,y) + \alpha U^\dagger(x,y) \right) \right. \right.
\]

\[
\left. \left. + \beta \left| \text{tr} U(x,y) \right|^2 \right\} \right\}
\]

(5.91)

which resembles (5.90). Again, we wish to find an optimized choice of the parameters \( \alpha \) and \( \beta \) for which the behaviour of (5.91) will closely resemble that of (5.90). Since (5.91) is also a product of one-link unitary matrix integrals, it can be evaluated using the results of Subsection 5.2.3 above from

\[
Z_D^{(M)} = e^{-N^2 \text{vol}(\mathcal{L}^D) D \mathcal{M}_\mathcal{D}[\alpha, \beta]}
\]

(5.92)

Now, however, the evaluation of the expectation values in Jensen’s inequality (5.64) with \( Z^{(1)} = Z_U \), \( Z^{(0)} = Z_D^{(M)} \) and the maximization over the parameters \( \alpha \) and \( \beta \) is a bit more involved and we shall therefore go through the analysis more carefully than in the last Subsection.

First of all, the expectation value on the right-hand side of Jensen’s inequality (5.64) for \( \beta = 0 \) is again a sum over one-link averages and the normalized average \( \langle S_0 \rangle_0 \) is easily computed using (5.42) and (5.43). To evaluate \( \langle S_{KMW} \rangle_0 \), with \( S_{KMW} \) the action in (5.90), we note first that (5.37) leads to the identity

\[
\langle \text{tr} \phi(x) U(x, x + \ell) \phi(x + \ell) U^\dagger(x, x + \ell) \rangle_0
\]

\[
= \langle \left| \text{tr} U \right|^2 \rangle_M \text{tr} \phi(x + \ell) + (1 - \langle \left| \text{tr} U \right|^2 \rangle_M) \text{tr} \phi(x) \text{tr} \phi(x + \ell)
\]

(5.93)

which holds at \( N = \infty \). Next, we need the normalized average \( \langle S_W \rangle_0 \) of the Wilson action (5.6). For this we note that the correlator \( \langle W[U; \square] \rangle_0 \) factorizes into a product of one-link averages involving \( \langle U_{ij} \rangle_M \) and \( \langle U_{k\ell}^\dagger \rangle_M \). From (5.38) it therefore follows that

\[
\langle \text{Re} W[U; \square] \rangle_0 = \left| \langle \text{tr} U \rangle_M \right|^4
\]

(5.94)

Notice that there is a total of \( \text{vol}(\mathcal{L}^D) D(D - 1)/2 \) plaquettes in the lattice \( \mathcal{L}^D \). Substituting these identities into Jensen’s inequality (5.64) then gives a lower bound on the partition function (5.90)

\[
Z_U[\phi; \beta] \geq e^{-N^2 \text{vol}(\mathcal{L}^D) D \mathcal{M}_\mathcal{D}[\phi; \alpha, \beta]}
\]

(5.95)

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where
\[
\Omega[\phi; \alpha, \beta] = -A[\phi] + F_M(\alpha, \beta) + (\beta - B[\phi])(|\operatorname{tr} U|^2)_M + \alpha^* \langle \operatorname{tr} U \rangle_M + \alpha \langle \operatorname{tr} U^\dagger \rangle_M
\]
\[
- \frac{\beta(D - 1)}{2} |\langle \operatorname{tr} U \rangle_M|^4
\]
and we have defined
\[
A[\phi] = \frac{1}{\text{vol}(\mathcal{L}^D) D} \sum_{(x,y) \in \mathcal{L}^D} \operatorname{tr} \phi(x) \operatorname{tr} \phi(y)
\]
\[
B[\phi] = \frac{1}{\text{vol}(\mathcal{L}^D) D} \sum_{(x,y) \in \mathcal{L}^D} (\operatorname{tr} \phi(x) \phi(y) - \operatorname{tr} \phi(x) \operatorname{tr} \phi(y))
\]

(5.96)

5.4.1 Minimization of the Mean Field Action

We now want to minimize the action (5.96) over all values of \(\alpha\) and \(\beta\). Since \(\Omega[\phi; \alpha, \beta]\) depends only on \(|\alpha|\), we may assume without loss of generality that the parameter \(\alpha\) is real and positive in the above. From (5.41)–(5.43) we see that \(\Omega[\phi; \alpha, \beta]\) is given explicitly by

\[
\Omega[\phi; \alpha, \beta] = \begin{cases} 
-A + (1 - B)x - 16qx^2 & , \quad \alpha \leq (1 - \beta)/2, 0 \leq \beta \leq 1 \\
-A + \frac{1}{4} + \frac{1}{2} \log(2(1 - y)) - B y^2 - q y^4 & , \quad \alpha \geq (1 - \beta)/2, \beta \geq 0
\end{cases}
\]

(5.98)

where we have introduced the constants
\[
q = \beta(D - 1)/32 , \quad x = \alpha^2/(1 - \beta)^2 , \quad y = (\beta - \alpha + \sqrt{(\alpha + \beta)^2 - \beta})/2\beta
\]

(5.99)

We see therefore that the action \(\Omega\) can be simplified to a function of a single variable in both regions where it is defined. In the first region, which we shall call region 1, it is a function of \(x \in [0, 1]\) while in region 2 it is function of \(y \in [\frac{1}{2}, 1]\).

Consider first region 1. It is easy to see that there \(\Omega\) is minimized at either \(x = 0\) or \(x = 1/4\). However, \(\Omega(x = 0) \geq \Omega(x = 1/4)\) is equivalent to \(B \leq 1 - 4q\), and so over region 1 the action (5.98) is minimized at

\[
\Omega_0[\phi; \beta] = \begin{cases} 
-A , & \quad B \leq 1 - 4q \\
-A - q + (1 - B)/4 , & \quad B \geq 1 - 4q
\end{cases}
\]

(5.100)

The behaviour of \(\Omega\) in region 2 is somewhat more complicated. For the moment we ignore the constant \(A\) that appears in (5.98) above. In this region, for each fixed value of \(y \in [\frac{1}{2}, 1]\), as a function of \(B\) and \(q\), \(\Omega\) describes a plane. The minimum \(\Omega_0\) of \(\Omega\) over this region is then the envelope function (i.e. lower bound) of the set of planes parametrized by \(y\). This implies that \(\Omega_0\) in this region is a concave down function of \(q\) and \(B\). Furthermore, since \(\frac{1}{2} \leq y \leq 1\), for each fixed \(q\) the slope of \(\Omega_0\) is bounded between \(-\frac{1}{4}\) and \(-1\) and it approaches these limits as \(B \to \pm \infty\). Since the slope of \(\Omega_0\) is 0 or \(-1/4\) in region 1, it follows that there is a unique point \(B_0(q)\) for each \(q\) below which \(\Omega\) takes its minimum in region 1 and above \(B_0(q)\) in region...
2. Since \( \Omega_0(\mathcal{B} = 1) \geq -q \) for each fixed value of \( q \), it is greater than \( \Omega_0 \) in region 1 and hence \( \mathcal{B}_0(q) \geq 1 \).

To find the explicit form of \( \Omega_0 \) in region 2, we differentiate (5.98) to get

\[
\frac{\partial \Omega}{\partial y} = \frac{1 - 4\mathcal{B}y(1 - y) - 8qy^3(1 - y)}{2(1 - y)}
\]

For \( q \) and \( \mathcal{B} \) positive, it follows from (5.101) that \( \partial_y \Omega \) can have at most one zero where it is increasing, and so \( \Omega \) can have at most one minimum for \( y \in [\frac{1}{2}, 1] \). Furthermore, since \( \partial_y \Omega \to \infty \) as \( y \to 1 \) and \( \partial_y \Omega(y = 1/2) = 1 - \mathcal{B} - q/2 \), it follows that \( \partial_y \Omega \) has a unique zero for \( y \in [\frac{1}{2}, 1] \) at which point \( \Omega \) attains its global minimum. Note that we need only consider the region \( \mathcal{B} > 1 \), since as discussed above for \( \mathcal{B} < 1 \) the minimum of \( \Omega \) lies in region 1. Thus for \( \mathcal{B} > 1 \), the minimum of \( \Omega \) is \( \Omega_0[\partial; \beta] = \Omega(y(q)) \) where \( y(q) \) is the unique solution with \( y \in [\frac{1}{2}, 1] \) of the quartic equation

\[
1 - 4\mathcal{B}y(1 - y) - 8qy^3(1 - y) = 0
\]

Although an analytic solution for \( y(q) \) is possible, it is rather complicated, and it is more instructive to instead explore a small \( q \) (strong coupling) expansion of the minimum value \( \Omega_0 \).

First, we note that for \( q = 0 \), the admissible solution of (5.102) is \( y_0 = (1 + (1 - 1/\mathcal{B})^{1/2})/2 \), so that over region 2 the minimum of \( \Omega \) for \( q = 0 \) is

\[
\Omega_0^{(0)} = \frac{1}{2} + \frac{1}{2} \log \left( \mathcal{B} \left( 1 + \sqrt{1 - 1/\mathcal{B}} \right) \right) - \frac{1}{2} \mathcal{B} \left( 1 + \sqrt{1 - 1/\mathcal{B}} \right)
\]

We now Taylor expand (5.98) in region 2 about \( y = y_0 \) using (5.102) to compute the derivatives \( \frac{d^n y(q)}{dq^n} \bigg|_{q=0} \). After some algebra, we find to first order in \( q \) that

\[
\Omega_0 = \frac{1}{2} + \frac{1}{2} \log \left( \mathcal{B} \left( 1 + \sqrt{1 - 1/\mathcal{B}} \right) \right) - \frac{1}{2} \mathcal{B} \left( 1 + \sqrt{1 - 1/\mathcal{B}} \right)
\]

\[
-\frac{q}{16} \left( 1 + \sqrt{1 - 1/\mathcal{B}} \right)^4 + \mathcal{O}(q^2)
\]

Note that (5.104) approaches the value \(-q + \frac{1}{2} \log(-2\mathcal{B}) - \mathcal{B} + 3/4 \) as \( \mathcal{B} \) becomes large, which justifies the assertion made earlier that (5.101) as a function of \( q \) tends to \(-1 \) as \( \mathcal{B} \to \infty \) and that \( \Omega \) eventually gets smaller than the minimum (5.100) from the previous region at some value \( \mathcal{B}_0(q) > 1 \). The actual value of \( \mathcal{B}_0(q) \) is defined by the intersection of (5.104) with the function \(-q + (1 - \mathcal{B})/4 \), but notice that we can also determine it as

\[
\mathcal{B}_0(q) = \min_{y \in [\frac{1}{2}, 1]} \frac{q(1 - y^4) - \frac{1}{2} \log(2(1 - y))}{y^2 - 1/4}
\]

which defines the intersection of the envelope function for the collection of lines in region 2 in (5.98) with the minimum value of \( \Omega \) in region 1 in (5.100).

It can be shown by a numerical calculation [131] that for small values of \( q \) a good approximation to \( \mathcal{B}_0(q) \) is

\[
\mathcal{B}_0(q) \approx 1 + (1.32) \cdot y^{2/3} \quad \text{for} \quad q \ll 1
\]
For large values of $q$ (weak coupling), the minimizing value of $y$ in (5.105) is very close to 1, so that $B_0(q)$ tends asymptotically to

$$B_0(q) \approx 2(1 + \log 4q)/3 \quad \text{for} \quad q \gg 1 \quad (5.107)$$

Then we can write the minimum of the action $\Omega$ over both regions 1 and 2 as

$$\Omega_0[\phi; B] = \left\{ \begin{array}{ll}
-A, & B \leq 1 - 4q \\
-A - q + (1 - B)/4, & 1 - 4q \leq B \leq B_0(q) \\
-A + \frac{1}{2} + \frac{1}{2} \log(B(1 + \sqrt{1 - 1/B})) - \frac{1}{16}q(1 + \sqrt{1 - 1/B})^4, & B > B_0(q) \\
+O(q^2) &
\end{array} \right. \quad (5.108)$$

### 5.4.2 Observables and Critical Behaviour

We shall now use the action (5.108) as an approximation in the partition function (5.90) and examine the phase structure of the induced bosonic field theory (5.89). We further assume now that the field $\phi(x)$ is frozen at a mean value $\bar{\phi}$ at each site $x$ of the lattice $\mathcal{L}^D$. This is justified if we assume that the lattice is large enough so that the expectation values $\langle f(\phi(x)) \rangle$ in (5.89) are approximately the same for each site $x$. For definiteness, we shall also now take $V$ to be the Gaussian potential

$$V(\phi) = M^2 \phi^2 \quad (5.109)$$

for the scalar field. Thus we effectively want to study the properties of the Hermitian one-matrix model

$$Z_{KMW}^{(1)} = \int d\phi \, e^{-N^2 M^2 \frac{1}{2} \operatorname{tr} \phi^2 - N^2 D\Omega_0[\phi; B]} \quad (5.110)$$

where from (5.97) the functionals $A[\phi]$ and $B[\phi]$ appearing in (5.108) in the mean-field approximation are

$$A[\phi] = \langle \operatorname{tr} \phi \rangle^2, \quad B[\phi] = \operatorname{tr} \phi^2 - \langle \operatorname{tr} \phi \rangle^2 \quad (5.111)$$

The Hermitian one-matrix model (5.110) therefore possesses the reflection symmetry $\phi \to -\bar{\phi}$ so that $\langle \operatorname{tr} \phi \rangle = 0$. This suggests that we could approximate (5.111) by $A[\phi] = 0$ and $B[\phi] = \operatorname{tr} \phi^2$ without significantly changing the behaviour of the model. This approximation (i.e. that $\langle \operatorname{tr} \phi^2 \rangle \ll \langle \operatorname{tr} \phi^2 \rangle$) is justified in the large-$N$ limit of the matrix model (5.110) because of factorization. With this observation, $\Omega_0$ is now a function of $B = \operatorname{tr} \phi^2$ with parameter $q$ which we will denote by $\Omega_q(B)$.

Recall that for large $B$, $\Omega_q$ is asymptotically equal to $3/4 - B + \frac{1}{2} \log 2B - q$. It follows that for $M^2 < D$ the partition function (5.110) diverges and the action there becomes infinite as $\operatorname{tr} \phi^2 \to \infty$. This corresponds to an unphysical situation where $\langle \operatorname{tr} \phi^2 \rangle = \infty$. We must therefore require that $M^2 \geq D$ in which case all correlators of the matrix model (5.110) should be well-defined. To analyse the behaviour of this matrix model, we consider the case $D = 0$ first where the partition function is a pure Gaussian matrix model. Rescaling the field
variables in (5.110) as $\phi \to M\phi$, we see that the $M$-dependence of the partition function is $Z_{K,M,W}^{(1)}(D = 0) \sim M^{-N^2}$. From (5.110) it follows that

$$
\left\langle \text{tr } \phi^2 \right\rangle (D = 0) = -\frac{1}{N^2} \frac{\partial \log Z_{K,M,W}^{(1)}(D = 0)}{\partial M^2} = \frac{1}{2M^2}
$$

and that the root mean square fluctuation about this average value is

$$
\left[ \left\langle \left( \text{tr } \phi^2 - \left\langle \text{tr } \phi^2 \right\rangle (D = 0) \right) \right\rangle^2 \right]^{1/2} = \left[ \frac{1}{N^4} \frac{\partial^2 \log Z_{K,M,W}^{(1)}(D = 0)}{\partial (M^2)^2} \right]^{1/2} = \frac{1}{NM^2\sqrt{2}}
$$

This shows that in the large-$N$ limit the configurations with $\text{tr } \phi^2$ away from its mean value are largely suppressed.

In particular, the function $f(\text{tr } \phi^2) = M^2 \mathcal{B} + D\Omega_q(\mathcal{B})$ which is the action in (5.110) will be very close to its value at $\mathcal{B} \equiv \left\langle \text{tr } \phi^2 \right\rangle$. Thus, at $N = \infty$ we can replace $f$ by its first order Taylor series expansion about $\mathcal{B}$, and the partition function (5.110) can be written as

$$
\lim_{N \to \infty} Z_{K,M,W}^{(1)} = \int d\phi \ e^{-N^2 f(\mathcal{B})} \text{tr } \phi^2
$$

up to an irrelevant constant. This is simply a Gaussian Hermitian matrix integral, and so using (5.112) we have

$$
\mathcal{B} \equiv \left\langle \text{tr } \phi^2 \right\rangle = 1/2 f'(\mathcal{B})
$$

The correlator $\langle \text{tr } \phi^2 \rangle$, which displays the local behaviour of the scalar fields in the matrix model (5.89), is therefore determined as the solution of the self-consistency condition (5.115) which explicitly reads

$$
D\Omega_q'(\mathcal{B}) = 1/2 \mathcal{B} - M^2
$$

Thus for each value of $q$ we can determine the value of $\mathcal{B} = \langle \text{tr } \phi^2 \rangle$ for each value of $M^2$ by finding the point $\mathcal{B}$ where the 2 functions in (5.116) intersect.

From (5.108) we see that there are 3 different regions of behaviour to consider. For $\mathcal{B} < 1 - 4q$, we have $\Omega_q'(\mathcal{B}) = 0$, while for $1 - 4q < \mathcal{B} < B_0(q)$ we find $\Omega_q'(\mathcal{B}) = -1/4$. In the final region, we recall that $\Omega_q(\mathcal{B}) = \Omega(y(B, q), B, q)$ where $y(B, q)$ is the unique solution to $\partial_q \Omega(y, B, q) = 0$ with $y \in [1/2, 1]$ (i.e. of (5.102)). Thus in the third region in (5.108) we have

$$
\Omega_q'(\mathcal{B}) = \frac{\partial}{\partial B} \Omega(y(B, q), B, q) = -[y(B, q)]^2
$$

and so $-D/4 \leq D\Omega_q'(\mathcal{B}) \leq -D$ in this region. Furthermore, $D\Omega_q'(\mathcal{B})$ asymptotically approaches $D/2 - D$ for large $\mathcal{B}$, and, since $\Omega_q(\mathcal{B})$ is concave down everywhere, $\Omega_q'(\mathcal{B})$ decreases from some value less than $-1/4$ at $\mathcal{B} = B_0(q)$ towards $-1$ as $\mathcal{B} \to \infty$. The value of $\Omega_q(\mathcal{B})$ in the third region can be calculated numerically for any $q$ using (5.117) and (5.102) \cite{[131]}. The function $1/2 \mathcal{B} - M^2$ will intersect $D\Omega_q'(\mathcal{B})$ in the first region if the equation $1/2 \mathcal{B} - M^2 = 0$ has a solution with $\mathcal{B} < 1 - 4q$. This implies that $\langle \text{tr } \phi^2 \rangle = 1/2 M^2$ for $q < 1/4(1 - 1/2 M^2)$. Next, (5.116) has a solution in the second region if the equation $1/2 \mathcal{B} - M^2 = -D/4$ has a solution with $1 - 4q < \mathcal{B} < B_0(q)$. In the physical region, $1/2 \mathcal{B} - M^2 < 1/2 - D < -D/4$ at $\mathcal{B} = 1$ and for $D \geq 1$, and so such a solution always exists and we find $\langle \text{tr } \phi^2 \rangle = 1/2 (M^2 - D/4)$ for $q > 1/4(1 - 1/(M^2 - D/4))$. Finally, we note that the third region never plays
a role in determining the behaviour of $B$ with respect to $M^2$ and $q$, because the restriction $M^2 > D$ ensures that the functions in (5.116) always intersect in one of the first 2 regions. Recalling the definition of $q$ in (5.99), we can summarize the behaviour of the scalar field correlator as

$$
\langle \text{tr} \phi^2 \rangle = \begin{cases} 
\frac{1}{2M^2}, & \beta < \frac{8}{D-1} \left( 1 - \frac{1}{2M^2} \right) \\
\frac{1}{2(M^2 - D/4)}, & \beta > \frac{8}{D-1} \left( 1 - \frac{1}{2(M^2 - D/4)} \right) 
\end{cases} \tag{5.118}
$$

for $M^2 > D$. For $M^2 \leq D$ we are in the forbidden region of the theory where $\langle \text{tr} \phi^2 \rangle$ diverges. Notice that there is a small region of ambiguity between the 2 regions in (5.118) where the scalar correlator is not determined. This is most likely due to the fact that the action $\Omega_q(B)$ is not smooth at $B = 1 - 4q$, so that the assumption that it can be replaced by its Taylor series expansion to first order is questionable. The Hermitian one-matrix model (5.110) may have a smooth transition between $B = 1/2M^2$ and $B = 1/2(M^2 - D/4)$ in the vicinity of this ambiguous region, but there is a reasonable chance that the transition there is some sort of singular phase transition, and that there is a critical line somewhere in this region.

Finally, we would like to determine the correlator $\langle \text{Re} W[U; \Box] \rangle$ which describes the local behaviour of the gauge fields in the model (5.89). The average of this quantity over all plaquettes is given by

$$
\langle \text{Re} W[U; \Box] \rangle = \frac{2}{N^2 \text{vol}(\mathcal{L}^D) D(D-1)} \frac{\partial Z^{[1]}_{K, MW}}{\partial \beta} \tag{5.119}
$$

From (5.108), we easily find that

$$
\langle \text{Re} W[U; \Box] \rangle = \begin{cases} 
0, & \tilde{\beta} < \frac{8}{D-1} \left( 1 - \frac{1}{2M^2} \right) \\
1/8, & \tilde{\beta} > \frac{8}{D-1} \left( 1 - \frac{1}{2(M^2 - D/4)} \right) 
\end{cases} \tag{5.120}
$$

Thus, setting $\tilde{\beta} = 0$ in the above to recover the Kazakov-Migdal model, we see that the mean field theory results seem to suggest the existence of a phase transition in the Itzykson-Zuber integral $I[\phi(x), \phi(y)]$. Although this seems to contradict the exact solutions of the Gaussian Kazakov-Migdal model which are based on the Itzykson-Zuber formula, the latter formula is valid only with the \textit{a priori} assumption that there is no phase transition in this unitary matrix integral separating 2 regions of different behaviour for different sizes of the components of $\phi$. Nevertheless, we do recover here the previous exact results of Gross [64] and the mean field theory results of Khokhlachev and Makeenko [79] which both showed that there is a barrier in the Gaussian model at $M^2 = D$ (below which lies an unstable Higgs phase due to an unlimited Bose-Einstein condensation of the scalar fields) and that there is no phase transition as $M^2$ is varied in the physical region $M^2 > D$. Thus the Gaussian Kazakov-Migdal model has no continuum limit.

What is particularly interesting in the above extended mean field theory analysis is the appearance of a possible phase transition between 2 regions as $\tilde{\beta}$ is increased for any given value of $M^2$. Furthermore, the critical curve actually intersects the $\tilde{\beta} = 0$ axis at some $M^2 > 0$, although this intersection point lies in the unphysical $M^2 < D$ region. If this intersection
occurred at some value $M^2 > D$ then the corresponding phase transition could be exactly what is needed to define the continuum limit of the Kazakov-Migdal model. There is therefore some hope that a similar model may have the correct behaviour to define a continuum limit. Of course, there is less of an accuracy in the mean field theory results here than in the fermionic case [95]. It could be that the non-trivial phase structure observed above is merely an artifact of the phase transitions that occur in the trial partition function of the mixed unitary matrix model. However, it has been argued by Khokhlovchv and Makeenko [79] that such mean field approaches are reasonably accurate for the description of phase transitions. Moreover, the mean field value (5.118) for the scalar correlator at $\beta = 0$ is very close to the exact result [47, 64] (see Subsection 1.1.3 with $D / 4$).

$$\langle \text{tr} \phi^2 \rangle_{K\text{M}} = \frac{1}{2\sqrt{M^4 - 1}}$$ (5.121)

In addition, because of the local phase invariance $U \rightarrow e^{i\theta}U$ of the Kazakov-Migdal model, the mean-field value of 0 for the Wilson loop expectation value in the first phase in (5.120) is correct. On the other hand, at weak-coupling $\beta \rightarrow \infty$ the Wilson action (5.6) is minimized when all the unitary matrices there are set equal to the identity matrix, so one expects a value of 2 for the Wilson loop expectation value in the second phase in (5.120). The discrepancy in (5.120) is due to the fact that the mean field approximation is a bit too crude for the calculation of gauge field correlators. To properly compute these correlators, one must turn to the method of loop equations for the matrix model [47, 48]. This will be the topic of the next Section for the fermionic case.

6 Loop Equations and Observables in Higher Dimensional Adjoint Fermion Matrix Models

We now turn to a more precise investigation of the characteristics of the adjoint fermion matrix model (5.1). To get a more exact understanding of the behaviour of its observables, we must once again resort to the Makeenko-Migdal method of loop equations. The loop equations for the matrix model (5.1) are the only way here to study the precise behaviour of observables of this model, and to obtain the exact characteristics of the critical behaviour of the induced gauge theory. Furthermore, by investigating the observables associated with extended objects in the model (the open-loop averages) one can check to see if, while the gravitational part of the system is continuous, the matter fields become critical at a given fixed point [96].

6.1 Loop Equations for Extended Averages

To generate the observables of the adjoint fermion matrix model (5.1), we must generate all the correlators of the two-matrix model of Section 4 at each site and link. In this case the pair correlators of the gauge fields can be evaluated from the one-link expectation values (4.23), since at large-$N$ the expectation value of any quantity such as $U_{ij}U_{kl}^*$ factorizes into one-link averages. Because of the higher-dimensionality of the model now there are, however, numerous other gauge correlators that can be formed along curves in the lattice. Gauge
invariance implies that any such observable must be some sort of combination of the Wilson line operators (5.7). However, the space of gauge-invariant operators must be further reduced because of the local $U(1)$-gauge symmetry (4.4) at a given link. In particular, as mentioned before, this extra symmetry excludes the conventional Wilson line operator (5.7) since then

$$\langle W[U; C] \rangle = 0$$  \hspace{1cm} (6.1)$$

The non-vanishing observables at $N = \infty$ must contain the same number of $U$ and $U^\dagger$ operators at each link. The complete set of gauge correlators of the model (5.1) are therefore generated by the operator products

$$W[U; C_1, \ldots, C_k] = W[U; C_1] \cdots W[U; C_k]$$  \hspace{1cm} (6.2)$$

where

$$C_1 + \ldots + C_k = 0$$  \hspace{1cm} (6.3)$$

The simplest examples of such gauge-invariant operators are the adjoint Wilson loop

$$W_A[U; C] = W[U; C, -C] - 1/N^2 = |W[U; C]|^2 - 1/N^2$$  \hspace{1cm} (6.4)$$

and for $D > 1$ the filled Wilson loop

$$W[U; \Sigma] = W[U; \partial \Sigma] \prod_{\square \in \Sigma} W[U; \square]^*$$  \hspace{1cm} (6.5)$$

where $\Sigma$ is a 2-dimensional surface in the lattice $L^D$ with boundary the loop $\partial \Sigma$. The filled Wilson loops are particularly important if we wish to interpret the induced gauge theory as QCD. If this is to be so then there must be an observable such as the Wilson loop which for $N_f < \infty$ obeys an area law in the confining phase [133]. Since the adjoint Wilson loop generally only obeys a perimeter law, the filled Wilson loop (6.5) is the only reasonable candidate for such an observable [85]. Furthermore, in the continuum limit it depends only on $\partial \Sigma$ and reduces to the conventional Wilson loop [85]. It is in this way that the adjoint fermion model (5.1) may induce QCD in the continuum limit even though in QCD the extra local $U(1)$-gauge invariance is absent. Moreover, the filled Wilson loop observables in the Kazakov-Migdal model can be represented as sums over fluctuating 2-dimensional surfaces in such a way that surfaces with contractable boundaries play the role of the physical states (gluons) while uncontractable boundaries describe world-sheet vortices [85]. The filled Wilson loops are therefore important for constructing string theories in dimension $D > 1$ from the induced gauge theory model. As discussed in the Section 4, however, there is no direct way to evaluate these gauge field observables from the Schwinger-Dyson equations for (5.1).

It is possible that the critical behaviours of observables in the induced gauge theory could be seen from the criticality of the inducing matter fields. To explore the possible criticality of the matter fields of the matrix model (5.1), we investigate now the observables which are associated with extended objects in the model. The quantum equations of motion for the open-loop fermion observables (and in particular the lattice Dirac equation) relate closed adjoint Wilson loops to open ones with fermions at the ends. As usual, the loop equations are formally solved at $N = \infty$. However, for the $D$-dimensional model (5.1) it turns out that in addition they are only amenable to explicit solution in the strong coupling regime of the
theory where the local confinement condition (5.86) holds. To see this, consider as an example
the open Wilson line correlator

\[
\delta_{ij} H^{\mu\nu}(z; C_{x,y}) = \left\langle \text{tr} \psi_i^\mu(x) \left( P \prod_{(x',y') \in C_{x,y}} U(x',y') \right) \frac{1}{z - \bar{\psi}_j(y)} \psi_j^\nu(y) \rightangle \nonumber
\]

\[
\times P \prod_{(x',y') \in C_{x,y}} U^\dagger(x',y') \right\rangle \quad (6.6)
\]

where \( \mu \) and \( \nu \) are spin indices and \( C_{x,y} \) is a contour connecting the sites \( x \) and \( y \). The
delta-function on the left-hand side of (6.6) arises because the action (5.2) is diagonal in the
flavour indices. It therefore suffices in the following to consider only a single fermion flavour,
\( N_f = 1 \). The leading \( 1/z \) term in the asymptotic expansion of (6.6) is the open Wilson line
with fermions at its ends

\[
H^{\mu\nu}(C_{x,y}) = \left\langle \text{tr} \psi^\mu(x) \left( P \prod_{(x',y') \in C_{x,y}} U(x',y') \right) \bar{\psi}^\nu(y) \right. \nonumber
\]

\[
\times P \prod_{(x',y') \in C_{x,y}} U^\dagger(x',y') \left. \right\rangle \quad (6.7)
\]

At \( N = \infty \) the multi-link correlator (6.7) can be factored into a product of one-link correlators

\[
H^\mu_\ell = \langle \text{tr} \psi^\mu(x) U(x, x + \ell) \bar{\psi}^\nu(x + \ell) U^\dagger(x, x + \ell) \rangle \quad (6.8)
\]

Since the Wilson line in (6.7) is in the same (adjoint) representation as the fermion fields,
we can use the standard large-mass expansion of lattice gauge theory [54, 133] to represent
(6.7) as a sum over lattice loops

\[
H^{\mu\nu}(C_{x,y}) = \sum_{\Gamma_{y,x} \in \mathcal{C}^D} \frac{W_A(C_{x,y} \circ \Gamma_{y,x})}{n! \Gamma_{y,x} + 1} \left( P \prod_{\ell \in \Gamma_{y,x}} \mathcal{P}^\mu_\ell \right)^{\mu\nu} \quad (6.9)
\]

where the sum is over all paths \( \Gamma_{y,x} \) which when joined to \( C_{x,y} \) result in a closed contour.
In the strong coupling regime the paths \( \Gamma_{y,x} \) coincide with \( C_{x,y} \) with opposite orientation
modulo backtrackings which form a 1-dimensional tree graph. Thus in this case the problem of
calculating the path amplitude (6.9) is reduced to the combinatorial problem of summing
over 1-dimensional tree graphs embedded in a \( D \)-dimensional space. In the phase with normal
area law, non-trivial loops contribute to the sum in (6.9) and lead to rather complicated
unitary matrix integrals in what follows. It seems that only in the local confinement phase,
where the dynamics of extended objects are trivial, can one write down a complete set of
Schwinger-Dyson equations for the local objects. In this Section we shall be concerned with
the contributions from the tree-like, or polymer, graphs in (6.9).

To explicitly see how the strong-coupling assumption formally simplifies the evaluation of
 correlators, consider a general expectation value of the form

\[
H^\mu_\ell(O) = \langle \text{tr} O U(x, x + \ell) \psi^\mu(x + \ell) U^\dagger(x, x + \ell) \rangle \quad (6.10)
\]
where the operator $O$ is independent of the operators $U(x, x + \ell)$, $U^\dagger(x, x + \ell)$, $\bar{\psi}(x + \ell)$ and $\bar{\psi}(x + \ell)$ (e.g., see (6.8)). If we expand the exponent in (5.1) in a power series in all of the coupling constants of the potential except the mass $m$ of the fermion field, then the calculation of (6.10) reduces to the evaluation of Wick contractions among $\bar{\psi}(y)$ and $\psi(y)$ and an integration over all $U(y, y + \ell)$. The $U(N)$ integration can be carried out independently on each link because there is no kinetic term for the gauge fields. Thus to evaluate (6.10) it suffices to consider the one-link correlator

$$
\langle \text{tr} \, T^a U \chi^\mu U^\dagger \rangle_{1L} \equiv \frac{\int [dU] \int d\bar{\chi} \, d\chi \, e^{N^2 \text{tr} \, (V(\chi) - \bar{\psi} \gamma^\mu U^\dagger \gamma^\nu - \sigma \gamma^\mu U^\dagger \gamma^\nu \psi \chi)} \text{tr} \, T^a U \chi^\mu U^\dagger}{\int [dU] \int d\bar{\chi} \, d\chi \, e^{N^2 \text{tr} \, (V(\chi) - \bar{\psi} \gamma^\mu U^\dagger \gamma^\nu - \sigma \gamma^\mu U^\dagger \gamma^\nu \psi \chi)} \text{tr} \, T^a \bar{\psi} \gamma^\mu \psi} 
$$

(6.11)
since the remaining part of the large-mass expansion (see (6.9)) decouples due to the properties of the unitary matrix integrals in the local confinement phase and large-$N$ factorization. In (6.11) the average is with respect to $U$ and $\chi$ only with $\psi$ playing the role of an external field.

The integration in the large-mass expansion of (6.11) can be carried out over $\chi$ and $\chi$ using the Wick rules, after which the unitary matrix integration is rather simplified, because all the $U$ matrices cancel with their conjugates $U^\dagger$, and leaves a functional of only $\text{tr} \, T^a \bar{\psi} \gamma^\mu (\bar{\psi} \gamma^\nu \psi)^n$ [101]. This is because the potential depends only on $\bar{\psi} \psi$ so that terms such as $\text{tr} \, T^a \bar{\psi} \gamma^\mu (\bar{\psi} \gamma^\nu \psi)^n$ do not appear when evaluating (6.11) using the above rules. The one-link correlator (6.11) at $N = \infty$ therefore has the form

$$
\langle \text{tr} \, T^a U \chi^\mu U^\dagger \rangle_{1L} = \sum_{\nu=1}^{s} (\mathcal{P}_\ell^+)^{\mu\nu} \, \text{tr} \, T^a \psi^\nu F(\bar{\psi} \psi) 
$$

$$
= \sum_{\nu=1}^{s} (\mathcal{P}_\ell^+)^{\mu\nu} \, \sum_{n=1}^{\infty} F_n \, \text{tr} \, T^n \bar{\psi} \gamma^\mu (\bar{\psi} \psi)^n 
$$

(6.12)

where $s = 2^{[D/2]}$ is the number of spin components of the fermion fields and

$$
F(z) = \sum_{n=1}^{\infty} F_n z^n 
$$

(6.13)
is some analytic function\textsuperscript{12}. Thus the general correlator (6.10) has the form

$$
H^\mu_\ell(\mathcal{O}) = \sum_{\nu=1}^{s} (\mathcal{P}_\ell^+)^{\mu\nu} \left\langle \text{tr} \, \mathcal{O} F(\bar{\psi}(x) \psi(x)) \psi^\nu(x) \right\rangle 
$$

(6.14)

The coefficients $F_n$ for (6.11) will be different than those for the full lattice correlator (6.10), but the analytic function $F(z)$ in (6.14) is universal in that it is independent of the explicit form of the operator $\mathcal{O}$. The function (6.13) depends on the potential (5.5) and in general it

\textsuperscript{12}In the Kazakov-Migdal model, this function appears as the large-$N$ limit of the logarithmic derivative of the Itzykson-Zuber integral in the saddle point equation (1.56),

$$
F_i(\phi) \equiv \frac{1}{N} \sum_j C_{ij}(\phi) \phi_j 
$$
is not possible to determine its exact form. As will be discussed later on, one must substitute for it an appropriate ansatz.

The loop equation for the open-loop average (6.6) follows from the invariance of the integration measure over \( \tilde{\psi}(x) \) in (6.6) under the infinitesimal field shifts

\[
\tilde{\psi}(x) \to \tilde{\psi}(x) + e^\alpha(x) \mathbf{T}^\alpha
\]  

(6.15)
in analogy with (4.39). In analogy with (4.38), we consider the loop average

\[
\left\langle \text{tr} \, T^\alpha \left( P \prod_{(x',y') \in C_{x,y}} U(x', y') \right) \frac{1}{z - \tilde{\psi}(y)\tilde{\psi}(y)} \tilde{\psi}(x) \left( P \prod_{(x',y') \in C_{x,y}} U^\dagger(x', y') \right) \right\rangle \equiv 0
\]  

(6.16)

which vanishes because of the gauge invariance of (5.1). Working out the resulting correlators in the usual way by performing the shift (6.15) of \( \tilde{\psi}(x) \), using the invariance of the integration measure and calculating the derivatives \( \frac{\partial}{\partial \tilde{\psi}(x)} \), it is straightforward to show that this leads to the Dirac open-loop equation for \( \psi(x) \)

\[
\int_{\mathcal{C}} \frac{d\lambda}{2\pi i} \frac{V'(\lambda)}{z - \lambda} \mathcal{H}(\lambda; C_{x,y}) - \sum_{\ell=1}^{D} \left( P_\ell^+ \mathcal{H}(z; C_{x+\ell,x} \circ C_{x,y}) + P_\ell^- \mathcal{H}(z; C_{x-\ell,x} \circ C_{x,y}) \right)
\]

\[= \delta_{x,y} \left\langle \text{tr} \left( P \prod_{(x',y') \in C_{x,y}} U(x', y') \right) \frac{1}{z - \tilde{\psi}(y)\tilde{\psi}(y)} \text{tr} \frac{1}{z - \tilde{\psi}(y)\tilde{\psi}(y)} \right\rangle \times P \prod_{(x',y') \in C_{x,y}} U^\dagger(x', y') \]  

(6.17)

where the path \( C_{x+\ell,x} \circ C_{x,y} \) is obtained by attaching the link \( \langle x, x \pm \ell \rangle \) to the loop \( C_{x,y} \) at the endpoint \( x \), and an implicit matrix multiplication over spin indices is assumed in (6.17). The left-hand side of the loop equation (6.17) results from the variation of the action in (5.1) while the right-hand side represents the commutator term resulting from the variation of the integrand in (6.16).

The right-hand side of (6.17) is non-zero only for closed contours and at large-\( N \), when factorization holds, it becomes

\[\delta_{x,y} W_A(C_{x,x}) \omega(z)^2 + \mathcal{O}(1/N^2) = \delta_{x,y} \delta_0 A_{\min}(C_{x,x}) \omega(z)^2 + \mathcal{O}(1/N^2)\]  

(6.18)
in the strong coupling regime where all closed contours \( C_{x,x} \) are contractable. Here

\[\omega(z) = \left\langle \text{tr} \frac{1}{z - \tilde{\psi}(x)\tilde{\psi}(x)} \right\rangle \]  

(6.19)

and thus at \( N = \infty \) the right-hand side of the Dirac open-loop equation involves only the generators of the \( \tilde{\psi}\psi \)-moments. This is the primary simplification at strong coupling, since otherwise the Dirac equation (6.17) involves non-trivial adjoint Wilson loops which are not readily determined.

We should point out here that there is one more loop equation that is obeyed by the open-loop averages. These are the Schwinger-Dyson equations (and in particular the lattice Yang-Mills equation) which express solely the invariance of the Haar measure over \( U(x, x + \ell) \)
under arbitrary changes of variables. In particular, the Haar measure \(dU(x, x+\ell)\) is invariant under the infinitesimal shift

\[
U(x, x+\ell) \rightarrow (1 + i\epsilon(x, x+\ell))U(x, x+\ell)
\]

of the gauge field \(U(x, x+\ell)\) at the link \(\langle x, x+\ell\rangle\), where \(\epsilon(x, x+\ell)\) is an infinitesimal Hermitian matrix. For example, consider the open-loop correlator

\[
\mathcal{G}(z, w; C_{x,y}) = \left\langle \text{tr} \frac{1}{z - \bar{\psi}(x)\psi(x)} \left( P \prod_{\langle x', y' \rangle \in C_{x,y}} U(x', y') \right) \frac{1}{w - \bar{\psi}(y)\psi(y)} \right. \\
\times \left. P \prod_{\langle x', y' \rangle \in C_{x,y}} U^\dagger(x', y') \right\rangle
\]

(6.21)
The loop equation for (6.21) then follows from the identity

\[
0 \equiv \left\langle \text{tr} \frac{1}{z - \bar{\psi}(x)\psi(x)} \left( P \prod_{\langle x', y' \rangle \in C_{x,y}} U(x', y') \right) T^a P \prod_{\langle x', y' \rangle \in C_{x,y}} U^\dagger(x', y') \\
\times \text{tr} T^b \left( P \prod_{\langle x', y' \rangle \in C_{x,y}} U(x', y') \right) \frac{1}{w - \bar{\psi}(y)\psi(y)} P \prod_{\langle x', y' \rangle \in C_{x,y}} U^\dagger(x', y') \right\rangle
\]

(6.22)
which vanishes due to the gauge invariance of (5.1). Performing the shift (6.20) in (6.22) of \(U(z, z+\ell)\) at some link \(\langle z, z+\ell\rangle \in C_{x,y}\) and using the invariance of the Haar measure leads to a set of equations which is not closed because it introduces higher-order correlators of the gauge fields (i.e., those of the form \(\langle UUU^\dagger U^\dagger\rangle\)). Thus the Yang-Mills open-loop equation expresses the one-link correlators of products such as \(UUU^\dagger U^\dagger\) in terms of the pair correlators of the gauge fields \(\frac{1}{N} C_{ij} = \langle U_{ij} U_{ji}^\dagger \rangle\) [96]. As mentioned before, however, there is no direct way in the adjoint fermion matrix models to generate the pair correlators for the gauge fields from the equations of motion of the model.

### 6.2 Loop Equations for One-link Correlators

The extended loop averages such as (6.6) can be computed at \(N = \infty\) by solving the loop equations (6.17) directly (see below) or by first computing the corresponding one-link correlators and then convoluting them together using appropriate variations of the pair correlators for the gauge fields. In the Kazakov-Migdal model this convolution can be carried out straightforwardly [47] for the same reasons discussed in Subsection 4.1 (see eq. (4.25)). For example, the adjoint Wilson loop in the Kazakov-Migdal model at \(N = \infty\) can be computed as

\[
W_A^H(\Gamma) = \int \prod_{j=1}^{\ell(\Gamma)} d\alpha_j \rho(\alpha_j) C(\alpha_1, \alpha_2) C(\alpha_2, \alpha_3) \cdots C(\alpha_{\ell(\Gamma)-1}, \alpha_{\ell(\Gamma)}) \langle \alpha_{\ell(\Gamma)}, \alpha_1 \rangle
\]

(6.23)
and similarly the analogs of the extended loop averages such as (6.21) can be explicitly calculated by substituting the gauge field correlation function at each link \(\ell\) by \(C(\alpha_\ell, \alpha_{\ell+1})\) and integrating over the \(\alpha_\ell\)'s. Although it is not clear how this latter method will carry through in the adjoint fermion model, we can at least formally solve the one-link problem
from a complete set of Schwinger-Dyson equations in the hope of being able to study the critical behaviour of the model in terms of the one-link averages.

The complete set of observables of the matrix model (5.1) at $N = \infty$ and in the local confinement phase are generated by the even-even one-link correlator

$$
\mathcal{G}(z, w) = \left\langle \text{tr} \frac{1}{z - \bar{\psi}(x)\psi(x)} U(x, x + \ell) \frac{1}{w - \bar{\psi}(x + \ell)\psi(x + \ell)} U^\dagger(x, x + \ell) \right\rangle, \quad (6.24)
$$

the odd-odd one-link correlator

$$
\mathcal{H}_\ell^{\mu\nu}(z, w) = \left\langle \text{tr} \psi^\mu(x) \frac{1}{z - \bar{\psi}(x)\psi(x)} U(x, x + \ell) \frac{1}{w - \bar{\psi}(x + \ell)\psi(x + \ell)} \bar{\psi}^\nu(x + \ell) U^\dagger(x, x + \ell) \right\rangle, \quad (6.25)
$$

and the generators (6.19) of the $\bar{\psi}(x)\psi(x)$-moments at the same site $x \in \mathcal{L}^D$. From the symmetry $U \to U^\dagger$ of the Haar measure and the fact that the potential $V$ in (5.2) is the same at each lattice site it follows that the functions (6.19), (6.24) and (6.25) obey the usual symmetries

$$
\mathcal{G}(z, w) = \mathcal{G}(w, z) \quad , \quad \mathcal{H}_\ell^{\mu\nu}(z, w) = \mathcal{H}_\ell^{\mu\nu}(w, z) \quad (6.26)
$$

for this symmetric case. The generating function $\mathcal{G}(z, w)$ therefore has the same asymptotic expansions as in (4.15) but now with $\mathcal{G}_n(z) = \tilde{\mathcal{G}}_n(z)$. Furthermore, these generators are all independent of the lattice position $x$ because (5.1) is invariant under lattice translations and rotations of the lattice by $\pi$.

The loop equations can be derived just as in Section 4 by considering the identical identities there with $\psi = \psi^\mu(x)$, $\chi = \psi^\mu(x + \ell)$, etc. For instance, consider the identity

$$
0 = \frac{N^{-2}}{Z_D} \sum_{i,j} \int \prod_{(x, y) \in \mathcal{L}^D} \left[ dU(x, y) \right] \int \prod_{x \in \mathcal{L}^D} d\psi(x) \, d\bar{\psi}(x) \frac{\partial}{\partial \psi^\mu(x)_{ij}} e^{S_P[\bar{\psi}, \psi; U]} \times \left( \frac{1}{z - \bar{\psi}(x)\psi(x)} U(x, y) \frac{1}{w - \bar{\psi}(y)\psi(y)} \bar{\psi}^\nu(y) U(y, x) \right)_{ij} \\
= \left\langle \text{tr} \frac{1}{z - \bar{\psi}(x)\psi(x)} \frac{\psi^\mu(x)}{z - \bar{\psi}(x)\psi(x)} U(x, y) \frac{\bar{\psi}^\nu(y) U(y, x)}{w - \bar{\psi}(y)\psi(y)} \right\rangle \\
- \left\langle \text{tr} \frac{V^\nu[\bar{\psi}(x)\psi(x)]}{z - \bar{\psi}(x)\psi(x)} \frac{\psi^\mu(x)}{z - \bar{\psi}(x)\psi(x)} U(x, y) \frac{1}{w - \bar{\psi}(y)\psi(y)} \bar{\psi}^\nu(y) U(y, x) \right\rangle \\
- \sum_{\lambda = 1}^{\mathcal{L}^D} \sum_{\lambda = 1}^{\mathcal{L}^D} \left\{ \left( \mathcal{P}_{(x, u)}^{-} \right)^{\mu\lambda} \left\langle \text{tr} U(x, u) \bar{\psi}^\lambda(u) U(u, x) \frac{1}{z - \bar{\psi}(x)\psi(x)} U(x, y) \right\rangle \right. \\
\times \left. \frac{1}{w - \bar{\psi}(y)\psi(y)} \bar{\psi}^\nu(y) U(y, x) \right\} + \left( \mathcal{P}_{(u, x)}^{+} \right)^{\mu\lambda} \left\langle \text{tr} U(u, x) \bar{\psi}^\lambda(u) U(x, x) \right\rangle \\
\times \frac{1}{z - \bar{\psi}(x)\psi(x)} U(x, y) \times \frac{1}{w - \bar{\psi}(y)\psi(y)} \bar{\psi}^\nu(y) U(y, x) \right\} \quad (6.27)
$$

where $y = x + \ell$. The first correlation function in (6.27) can be factored at large-$N$ into the 2 terms $\omega(z)\mathcal{H}_\ell^{\mu\nu}(z, w)$. The third term contains link operators which connect the site.
fermions to all neighbouring points. One of these connects \( x \) to \( y = x + \ell \) and thus contributes a term proportional to \(-\omega(z) + w G(z, w)\). In this same term, there are also \(2D-1\) links which connect \( x \) to other sites. For these we can use the fact that the quantity \( U(u, x)\psi^\lambda(u)U^\dagger(u, x)\) inside the expectation value bracket can be replaced by the analytic function \( F(\psi(x)\psi(x))\) at \( N = \infty\) as discussed in the last Subsection (c.f. eq. (6.14)).

With this input, we arrive at the analog of the loop equation (4.33) for the \( D \)-dimensional matrix model (5.1),

\[
\omega(z) H^{\mu\nu}_\ell (z, w) - \frac{1}{s} (P^{\mu\nu}_\ell) (w G(z, w) - \omega(z)) = \oint_{C} \frac{d\lambda}{2\pi i} \frac{\mathcal{V}(\lambda) \lambda}{z - \lambda} H^{\mu\nu}_\ell (\lambda, w) = 0
\]  

(6.28)

Multiplying (6.28) by \((P^{\mu\nu}_\ell)^{\nu\sigma}\) and summing over the spin indices, we can rewrite it as

\[
\omega(z) \dot{H}(z, w) - \sigma(w G(z, w) - \omega(z)) = \oint_{C} \frac{d\lambda}{2\pi i} \frac{\mathcal{V}(\lambda) \lambda}{z - \lambda} \dot{H}(\lambda, w) = 0
\]  

(6.29)

where the effective potential \(\mathcal{V}(z)\) is defined through

\[
\mathcal{V}(z) \equiv V'(z) - 2\sigma(2D - 1) F(z)
\]  

(6.30)

and we have introduced

\[
\dot{H}(z, w) \equiv \text{TR} P^\pm_\ell H_\ell(z, w)
\]  

(6.31)

which has the asymptotic expansion

\[
\dot{H}(z, w) = \sum_{n=0}^{\infty} \frac{\dot{H}_n(z)}{w^{n+1}}
\]  

(6.32)

and from the symmetries (6.26) and the definition (6.31) it follows that it has the symmetry property

\[
\dot{H}(z, w) = \dot{H}(w, z)
\]  

(6.33)

In a similar fashion, we can write the analog of the loop equation (4.29) for the \( D \)-dimensional matrix model (5.1) as

\[
(s + 1 - z\omega(z)) G(z, w) + \dot{H}(z, w) + \oint_{C} \frac{d\lambda}{2\pi i} \frac{\mathcal{V}(\lambda) \lambda}{z - \lambda} G(\lambda, w) = 0
\]  

(6.34)

The analogs here of the other 2 loop equations (4.31) and (4.35) are identical to the above loop equations because of the symmetries (6.26) and (6.33). For \( D = \frac{1}{2} \) \((s = 1)\) and chiral fermions \((\sigma = -1)\), the loop equations (6.34) and (6.29) coincide, respectively, with the loop equations (4.29) and (4.33) of the fermionic two-matrix model (4.1) with \( V = \bar{V} \). The analogs of the asymptotic equations (4.41)–(4.43) of the two-matrix model in the present case are respectively then

\[
\dot{H}_n(z) = (z\omega(z) - (s + 1)) G_n(z) - \oint_{C} \frac{d\lambda}{2\pi i} \frac{\mathcal{V}(\lambda) \lambda}{z - \lambda} G_n(\lambda)
\]  

(6.35)

\[
\sigma G_{n+1}(z) = \omega(z) \dot{H}_n(z) - \oint_{C} \frac{d\lambda}{2\pi i} \frac{\mathcal{V}(\lambda) \lambda}{z - \lambda} \dot{H}_n(\lambda)
\]  

(6.36)

and

\[
\sigma z\omega(z) = \sigma - \oint_{C} \frac{d\lambda}{2\pi i} \frac{\mathcal{V}(\lambda) \lambda}{z - \lambda} \dot{H}(z, \lambda)
\]  

(6.37)
6.3 The Ansatz

The equations (6.35)–(6.37) formally determine the one-loop correlator $\omega(z)$ in terms of the effective potential (6.30), rather than the given potential in (5.2). In general this effective potential is not known and must be assumed to take a particular form at the onset. To determine $\mathcal{V}(z)$ formally from the above equations, and hence find $F(z)$, we note that from (6.14), (6.25), (6.31) and (6.32) we have the equation

$$\hat{\mathcal{H}}_0(z) = -\sigma \oint_{C} \frac{d\lambda}{2\pi i} \frac{F(\lambda)\lambda}{z - \lambda} \omega(\lambda)$$

(6.38)

(6.38) can be rewritten using (6.35) as

$$(s + 1 - z\omega(z))\omega(z) + \oint_{C} \frac{d\lambda}{2\pi i} \frac{\mathcal{V}(\lambda)\lambda}{z - \lambda} \omega(\lambda) - \sigma \oint_{C} \frac{d\lambda}{2\pi i} \frac{F(\lambda)\lambda}{z - \lambda} \omega(\lambda) = 0$$

(6.39)

which resembles the large-$N$ loop equation (3.18) of the fermionic one-matrix model. From this we can conclude that the continuous part of $\omega(z)$ across its singularities leads to the saddle point equation

$$2(\omega(\alpha + \epsilon_\perp) + \omega(\alpha - \epsilon_\perp)) = (s + 1)/\alpha - \mathcal{V}'(\alpha) + \sigma F(\alpha)$$

(6.40)

where $\alpha$ lies on a branch cut of $\omega(z)$. Notice that for Kogut-Susskind fermions where $s = 1$ and $\sigma = -1$ [80, 86], the above equations coincide with those of the fermionic two-matrix model (4.1) with $V \equiv \mathcal{V}$. Notice also that the above equations are drastically simplified for Wilson fermions which have $\sigma = 0$. As mentioned before, this simplifying feature for Wilson fermions is a consequence of the fact that if (5.73) vanishes then backtracking paths never contribute to the representations of the correlators as sums over lattice paths.

Although the equations above formally determine the effective potential $\mathcal{V}(z)$, they are still rather complicated because in general $\mathcal{V}(z)$ can be non-polynomial and have singularities in the complex plane away from the branch cut and pole singularities of $\omega(z)$. The problem simplifies, however, if we assume that $\mathcal{V}(z)$ is analytic on the entire complex plane except possibly at infinity,

$$\mathcal{V}(z) = \sum_{n=1}^{\infty} \frac{g_n}{n} z^n$$

(6.41)

so that the asymptotic equation (6.37) becomes

$$\sigma z\omega(z) = \sigma - \sum_{n=1}^{\infty} g_n \hat{\mathcal{H}}_{n-1}(z)$$

(6.42)

and similarly the other contour integrals simplify to forms analogous to those in Subsection 4.1. Then the equations (6.30), (6.40) and (6.42) unambiguously fix the functions $\mathcal{V}(z), F(z)$ and $\omega(z)$ for a given potential $V(z)$. (6.41) is considered as an ansatz for $\mathcal{V}(z)$ to be used in solving the loop equations. Once $\mathcal{V}(z)$ and $F(z)$ are then explicitly determined, the original potential $V(z)$ of the adjoint fermion matrix model (5.1) can be determined from (6.30). Because the terms involving $F(z)$ disappear in certain instances (e.g. $D = \frac{1}{2}$) it is natural to regard (6.41) as the potential of the proper two-matrix model to which the $D$-dimensional matrix model has been effectively simplified. As for the two-matrix model, however, the equation determining $\omega(z)$ will be a $(2K)$-th order polynomial equation when (6.41) is a
polynomial of degree $K$, and moreover these equations will all involve the set of unknown coefficients of the ansatz (6.41).

Similarly, the loop equations (6.17) for the extended loop correlators are solved by substituting in an appropriate ansatz for $\mathcal{H}^{\mu\nu}(z;C_{x,y})$. The proper ansatz that one should take can be deduced from the large-mass expansion (6.9). From the definitions (6.6) and (6.10) and the relation (6.14) it follows that the Dirac equation (6.17) can be solved in the strong coupling regime using the ansatz

$$\mathcal{H}^{\mu\nu}(z;C_{x,y}) = \left(\frac{F(z)}{z}\right)^{L(C_{x,y})} \omega(z) \left(P \prod_{\ell \in C_{x,y}} \mathcal{P}_\ell^\pm\right)^{\mu\nu}$$

(6.43)

where $L(C_{x,y})$ is the algebraic length of the path $C_{x,y}$ (i.e. the lattice length after eliminating all backtrackings in $C_{x,y}$). The spin factors in (6.43) are needed to cancel the projection operators appearing in (6.17). One must now substitute an ansatz for $F(z)$, and when this is substituted into the loop equation (6.17) we will obtain an equation both for $L(C_{x,y}) \neq 0$ when $x \neq y$ in (6.17) and also for $L(C_{x,x}) = 0$ when $x = y$ and all closed contours are contractable. The exact solutions of the loop equations for a given potential then represent the solution to the combinatorial problem of summing over 1-dimensional tree graphs embedded in a $D$-dimensional space.

### 6.4 The Gaussian Model

For the Gaussian potential $V(z) = mz$, we substitute into the loop equations the ansatz

$$\mathcal{V}(z) = \Pi^{-1} z$$

(6.44)

The asymptotic equation (6.42) then implies

$$\sigma z \omega(z) = \sigma - \Pi^{-1} \hat{\mathcal{H}}_0(z)$$

(6.45)

where from (6.35)

$$\hat{\mathcal{H}}_0(z) = (z\omega(z) - (s + 1))\omega(z) + \Pi^{-1} z \omega(z) - \Pi^{-1}$$

(6.46)

These 2 equations combine to give as usual a quadratic equation for the one-loop correlator $\omega(z)$, and the relations (6.30) and (6.40) imply that

$$\Pi = \frac{2}{\sqrt{m^2 - 4\sigma(2D-1) + m}}$$

(6.47)

and

$$F(z) = \Pi z = \frac{2z}{\sqrt{m^2 - 4\sigma(2D-1) + m}}$$

(6.48)

The one-loop correlator is thus

$$\omega(z) = \frac{1}{2} \left(\frac{s + 1}{z} - \mu + \frac{1}{z} \sqrt{\mu z^2 - 2(s - 1)\mu z + (s + 1)^2}\right)$$

(6.49)
where

$$
\mu = \Pi^{-1} + \sigma \Pi = \frac{(D - 1)m + D \sqrt{m^2 - 4\sigma(2D - 1)}}{2D - 1}
$$

(6.50)

Notice that only for Kogut-Susskind fermions, which are effectively spinless Grassmann variables [86], does the usual fermion chiral symmetry become an invariance of the model. Finally, combining the loop equations (6.34) and (6.29) determines the even-even and odd-odd one-link correlators as

$$
\mathcal{G}(z, w) = \frac{\sigma w(\omega(z) + \omega(w)) - \sigma - \Pi^{-1} \omega(w)(\omega(z) + \Pi^{-1} z)}{\omega(z)(2(s + 1) - (\mu + \Pi^{-1}) z - \Pi^{-1} z^2) + \Pi^{-1}(s + 1) z - \Pi^{-1} z^2 - \sigma w + \mu}
$$

(6.51)

$$
\hat{\mathcal{H}}(z, w) = (\Pi^{-1} z + z\omega(z) - (s + 1))\mathcal{G}(z, w) - \Pi^{-1} \omega(w)
$$

(6.52)

It is easy to verify that these correlators are symmetric and non-singular for any $z$ and $w$.

The constant $\Pi$ above and the one-loop correlator (6.49) could also have been found by substituting the ansatz

$$
\mathcal{H}^{\mu\nu}(z; C_{x,y}) = \Pi^{L(C_{x,y})} \omega(z) \left( P \prod_{\ell \in C_{x,y}} P_\ell^\pm \right)^{\mu\nu}
$$

(6.53)

into the Dirac equation (6.17) which in the case at hand reads

$$
m\mathcal{H}(z; C_{x,y}) - \sum_{\ell=1}^D \left( P_\ell^+ \mathcal{H}(z; C_{x+\ell,x} \circ C_{x,y}) + P_\ell^- \mathcal{H}(z; C_{x-\ell,x} \circ C_{x,y}) \right)
$$

$$
= \delta_{x,y} \delta_{0,A_{\text{min}}(C_{x,y})} |\omega(z)|^2
$$

(6.54)

Setting $x \neq y$ in (6.54) then yields (6.47) and the $x = y$ equation for contractable loops yields the quadratic equation above for $\omega(z)$. Notice that for Kogut-Susskind fermions, (6.49) coincides with the solution of the Gaussian model of the Subsection 4.2. For Wilson fermions the above solutions simplify since when the backtracking parameter $\sigma$ vanishes we have

$$
\Pi^L = 1/m^L
$$

(6.55)

and the solution therefore coincides with the leading-order of the large-mass expansion. This is not surprising, since as discussed before for Wilson fermions backtracking paths never contribute and thus the leading term of the large-mass expansion yields the exact result for the adjoint fermion matrix model (5.1) with Wilson fermions.

The $\frac{1}{s^2}$ coefficient in the asymptotic expansion of the one-loop correlator (6.49) is the expectation value

$$
\xi = \langle \text{tr} \ \tilde{\psi}(x)\psi(x) \rangle = \frac{s}{\mu} = \frac{(2D - 1)s}{(D - 1)m + D \sqrt{m^2 - 4\sigma(2D - 1)}}
$$

(6.56)

This correlator coincides with that obtained for lattice QCD with fundamental representation fermions when the coefficient of the Wilson action (5.6) vanishes [82]. Indeed, there is an analogy based on the Dirac equation between adjoint and fundamental fermions in the phase with local confinement for the Gaussian potential [80]. The fact that (6.56) is the same for adjoint and fundamental fermions follows from the lattice loop representation of the
propagators of the theory, i.e. in the case of the fundamental representation the exact same combinatorial problem of summing over tree diagrams emerges. In particular, as discussed in [82], the non-vanishing of (6.56) implies that the adjoint fermion matrix model (5.1) in this way furnishes an interesting example of a quantum field theory with spontaneous chiral symmetry breaking and associated non-trivial fermion chiral condensate leading to a possible composite Higgs phase of the matrix model. This coincidence is no longer valid, however, for the higher order moments of these 2 fermion matrix models.

Notice that the above strong coupling solutions of the Gaussian model are non-singular for any $m^2 > 0$, contrary to the Kazakov-Migdal model [47, 64, 79, 93, 95] (see Subsection 5.4 above), and so the solutions in the fermionic case are stable everywhere. However, the matrix model (5.1) undergoes a first order large-$N$ phase transition beyond which the strong coupling solutions above are no longer applicable. The critical point of this phase transition is quite regular from the point of view of the large-mass expansion and the strong coupling solution is insensitive to the large-$N$ phase transition which occurs inside the region where the large-mass expansion converges. In the phase with normal area law the ansatz of Subsection 6.3 above is no longer valid and the solution of the loop equations in this regime will be quite different. The weak coupling phase where the dynamics of extended objects are non-trivial is very similar to the standard Wilson lattice gauge theory [133]. We remark that one could also study the $1/N$-expansion of the $D$-dimensional model (5.1) in the same way as in the previous Sections. For instance, the next-to-leading order in $1/N$ contribution to the Dirac equation (6.17) for the Gaussian model leads to the adjoint Wilson loop [109]

$$W_A(\Gamma) = \delta_{0, L(\Gamma)} + [\Pi^{L(\Gamma)} (1 - \delta_{0, L(\Gamma)}) - 1]/N^2$$

(6.57)

The leading term here is the usual contribution from the backtrack loop which corresponds to an infinite string tension. The $1/N^2$ correction describes the perimeter law for pointlike quarks propagating along the adjoint double loop. The Schrödinger wave equation is then effectively obtained by summing over all of these closed contours. Thus the higher order terms in the $1/N$-expansion of the adjoint fermion matrix model (5.1) provide information relevant to the physical matter content of the gauge theory.

7 Supersymmetric Matrix Models

In this final Section we shall present yet another application of the fermionic matrix model formalism to the representation of the polymer phase of bosonic string theory in target space dimensions $D > 1$ [97]. We shall be ultimately interested in the next level of complexity of the matrix models we have considered thus far, namely the combination of fermionic and bosonic matrix degrees of freedom into a supersymmetric matrix model. We will primarily discuss models which are defined in terms of matrix superfields defined in zero-dimensional target and supersurface spaces. These models are therefore a version of the $D = 1$ Marinari-Parisi superstring models [103] which were the first proposed supersymmetric extensions of the matrix models describing strings and discretized random surfaces. These models differ from the Gilbert-Perry Hermitian supermatrix models which are related to the Penner matrix

\[\text{[13]}A\text{rectangular adjoint Wilson loop, with space and time lengths } L \text{ and } T, \text{respectively, can be interpreted in the limit } T \rightarrow \infty \text{as the energy of a pair of mesons with separation } L \text{[133].}\]
models [2, 60]. As always, in these simple \( D = 0 \) cases a Hermitian fermionic matrix formalism is avoided because nilpotency always implies that \( \text{tr} \, \psi^2 = 0 \). The higher-dimensional supermatrix models will be briefly described at the end of this Section.

From a combinatorial point of view, the supersymmetry of the matrix model renders the partition function trivial. However, the matrix propagators in \( D = 0 \) correspond to a sum over polymer trees which describe a particular combinatorial problem that we shall describe here in detail. Thus the supersymmetric matrix model contains a dimensional reduction, because of the cancellation of bosonic and fermionic matrix field loops due to supersymmetry, so that for \( D = 0 \) it is very similar to a vector model in that it counts random polymers rather than random surfaces. Indeed, the Marinari-Parisi supersymmetric matrix models in \( D = 1 \) were originally introduced to describe the same random surface theories as the \( D = 0 \) Hermitian matrix models. Adding supersymmetry to the purely fermionic (or bosonic) theory makes it more rigid and reduces the number of degrees of freedom. As we will discuss, this is observed in an extreme way, whereby only a very small subclass of graphs corresponding to branched polymers survives when one examines the diagrammatic expansions of non-supersymmetric observables. In the \( D = 0 \) cases, these models are motivated by the notion of ‘folding’, an important concept in polymer physics. We can consider a statistical model of randomly branching polymer chains which are made up of \( n \) independent constituents, and which may therefore fold onto themselves. The entropy of such a system is obtained by counting the number of inequivalent ways of folding the chain. The combinatorial problem of enumerating all compact foldings is equivalent to another geometrical problem, the meander problem [39, 90], i.e. the problem of enumerating the configurations of a closed road crossing an infinite river through \( n \) bridges.

### 7.1 Meander Numbers and Random Matrix Models for the Meander Problem

Consider an infinite straight line (river). A meander of order \( n \) is defined as a closed self-avoiding connected loop (road) which intersects the line through \( 2n \) points (bridges). It can be viewed as a compact folding configuration of a closed chain of \( 2n \) constituents (in one-to-one correspondence with the \( 2n \) bridges) by putting a hinge on each section of the road between \( 2 \) bridges. The principal meander number \( \mathcal{M}_n \) is defined as the number of topologically inequivalent meanders of order \( n \). It also therefore describes the number of different foldings of a closed strip of \( 2n \) stamps or of a closed polymer chain. One can also consider a generalized version of this problem. The multi-component meander numbers \( \mathcal{M}^{(k)}_n \) are defined as the number of topologically inequivalent meanders of order \( n \) with \( k \) connected components, i.e. made up of \( k \) closed connected non-intersecting but possibly interlocking bridges which cross the river through a total of \( 2n \) bridges (\( k \) loops of the road). Clearly \( \mathcal{M}_n \equiv \mathcal{M}^{(1)}_n \). The results of a computer enumeration of the meander numbers up to \( n = 12 \) have been presented in [39, 90].

To obtain a matrix model representation of meanders [39], we note that the enumeration of (planar) meanders is very close to that of 4-valent (genus \( h = 0 \)) fat-graphs constructed from 2 self-avoiding loops (1 black \( B \), 1 white \( W \)) intersecting each other at simple nodes. The \( B \)-loop corresponds to the river (closed at infinity) while the \( W \)-loop corresponds to the road. Such a fat-graph is called a black and white graph. Since the river now becomes a loop,
this replaces the order of the bridges by a cyclic order and identifies the regions above and below the river. Thus $\mathcal{M}_n$ is now given by the meander number for the inequivalent black and white graphs with $2n$ intersections times $4n$ (2 for the up-down symmetry, $2n$ for the cyclic symmetry) weighted by the symmetry factor $|G(\mathcal{F})|$ for each fat-graph $\mathcal{F}$. The same connection holds between $\mathcal{M}_n^{(k)}$ and the black and white graphs where the white loop has $k$ connected components.

The black and white graphs can be generated by the $N \times N$ Hermitian multi-matrix integral

$$Z(m, n, c; N) = \int \prod_{b=1}^{n} dB_b \prod_{a=1}^{m} dW_a \ e^{-N^2 \ |P[B, W]|} \tag{7.1}$$

where the action is

$$P[B, W] = \sum_{b=1}^{n} \frac{B_b^2}{2} + \sum_{a=1}^{m} \frac{W_a^2}{2} - \frac{c}{2} \sum_{a,b} B_b W_a B_b W_a \tag{7.2}$$

Here $a, b$ are “colour” indices and $c$ is the coupling constant for the quartic interactions between the fields $B_b$ and $W_a$. The perturbative expansion of $\log Z$ in powers of $c$ gives a series in which the term of order $v$ can be readily evaluated as a Gaussian multi-matrix integral. The Feynman rules lead to the propagators

$$\langle [B_b]_{ij} [B_{b'}]_{k'l'} \rangle_{c=0} = \frac{\delta_{i,j} \delta_{k,l}}{N} \delta_{bb'}, \quad \langle [W_a]_{ij} [W_{a'}]_{k'l'} \rangle_{c=0} = \frac{\delta_{i,j} \delta_{k,l}}{N} \delta_{aa'} \tag{7.3}$$

and the only vertices are 4-valent ones which have alternating black and white edges and which therefore describe simple intersections of black and white loops. The normalized averages in (7.3) are Gaussian ones obtained from the partition function (7.1) at $c = 0$. Thus the perturbative expansion of $\log Z$ can be obtained as the sum over 4-valent fat-graphs whose $v$ vertices have to be connected by means of the 2 types of edges (7.3) which have to alternate around each vertex. This gives an exact realization of the black and white graphs except that any number of loops in each colour is allowed.

With these Feynman rules, a graph of genus $b$ with $v$ vertices and $b$ (respectively $w$) black (white) loops receives a weight $N^{2-2b} c^v m^w$. We can reduce the number of black loops $b$ to 1 by the so-called ‘replica trick’ [39] which is similar to the replica symmetry of the $O(N)$ vector model which picks out the $O(N^0)$ component corresponding to a self-avoiding random walk. Here we let the number $n$ of black matrices $B$ approach zero and retain only the contribution of order $n^0 = 1$ in $n$. This leads to the generating function

$$E_H(m, c; N) \equiv \lim_{n \to 0} \frac{1}{n} \log Z(m, n, c; N) = \sum_{\mathcal{F}} \frac{N^{2-2b} c^v m^w}{|G(\mathcal{F})|} \tag{7.4}$$

where the sum is over all black and white connected graphs $\mathcal{F}$ with one $B$ loop. The genus 0 contribution to (7.4) then gives a relation to the meander numbers,

$$E_H^{(0)}(m, c) \equiv \lim_{N \to \infty} \frac{1}{N^2} E(m, c; N) = \sum_{p=1}^{\infty} \frac{c^{2p}}{4p} \sum_{k=1}^{p} \mathcal{M}_p^{(k)} m^k \tag{7.5}$$

where we have used the aforementioned relation between the number of black and white graphs and the multi-component meander numbers. Alternatively, the generating function
(7.4) can be obtained directly from the Hermitian matrix model [97, 99]

\[
\mathcal{E}_H(m, c; N) = \frac{2}{N^2} \int \prod_{a=1}^{m} dW_a \quad e^{-\frac{\alpha^2}{2} \sum_{a=1}^{m} \text{tr} \, W_a^2} \times \log \left( \int d\phi \; e^{-\frac{\alpha^2}{2} \sum_{a=1}^{m} \text{tr} \, \phi W_a \phi W_a} \right)
\]

where the logarithm leaves only one closed loop of the Hermitian matrix model. The \(\phi\)-integral in (7.6) is a version of the so-called curvature matrix models considered in [31] which provide explicit examples of the polymer structures in \(D > 1\) string theory that we discussed at the end of Subsection 1.1.3.

We can carry out the exact Gaussian integration over the \(\phi\) matrices in (7.6) to get

\[
\mathcal{E}_H(m, c; N) = -\frac{N}{2} \int \prod_{a=1}^{m} dW_a \; \text{tr} \; \log \left( I \otimes I - c \sum_{a=1}^{m} W_a^T \otimes W_a \right) e^{-\frac{\alpha^2}{2} \sum_{a=1}^{m} \text{tr} \, W_a^2} \]

\[
= N \sum_{p=1}^{\infty} \frac{e^p}{2p} \left\langle \sum_{a=1}^{m} \text{tr} \left( W_a^T \otimes W_a \right)^p \right\rangle_H \]

\[
= N^2 \sum_{p=1}^{\infty} \frac{e^p}{2p} \sum_{1 \leq a_1, \ldots, a_p \leq m} \left\langle \left( \text{tr} \, W_{a_1} \cdots W_{a_p} \right)^2 \right\rangle_H \]

where we have used the Hermiticity of the \(W\) matrices, \(W^T = W^*\), and the normalized average \(\left\langle \cdot \right\rangle_H\) over the \(W\) matrices in (7.7) is with respect to the (free field) Gaussian \(W\) measure in (7.6). At \(N = \infty\), factorization implies \(\left\langle \left( \text{tr} \, \Pi_i W_{a_i} \right)^2 \right\rangle_H = \left\langle \text{tr} \, \Pi_i W_{a_i} \right\rangle_H^2\), and reflection symmetry \(W \to -W\) of the Gaussian measure in (7.6) implies that the only nonvanishing contributions in (7.7) are for \(p = 2n\) even. Comparing this with (7.5) leads to a representation of the meander numbers of order \(n\) with \(k\) connected components in terms of Gaussian averages of the \(W\)-fields,

\[
\sum_{k=1}^{n} \mathcal{M}_n^{(k)} m^k = \sum_{1 \leq a_1, \ldots, a_{2n} \leq m} \left( \lim_{N \to \infty} \left\langle \left( \text{tr} \, \Pi_{i=1}^{2n} W_{a_i} \right)^2 \right\rangle_H \right) \]

The left-hand side of (7.8) is a polynomial of degree \(n\) in \(m\) with vanishing constant coefficient, and it is therefore completely determined by the first \(n\) values \(m = 1, \ldots, n\). Thus the right-hand side of (7.8) evaluated for \(m = 1, \ldots, n\) determines completely the coefficients \(\mathcal{M}_n^{(k)}\). Notice that the Gaussian moments there are given explicitly by making Wick pairing contractions to get

\[
\left\langle \left( \text{tr} \, W_{a_1} W_{a_2} \cdots W_{a_{2n-1}} W_{a_{2n}} \right)^2 \right\rangle_H = \delta_{a_1 a_2} \delta_{a_3 a_4} \cdots \delta_{a_{2n-1} a_{2n}} + \mathcal{P}[a_1, a_2, \ldots, a_{2n}]
\]

where \(\mathcal{P}\) contains the sum of delta-functions over all permutations of the indices \(a_i\). Thus a Hermitian random matrix model can be used to generate a convenient and practical way of calculating meander numbers.

The ordered but cyclic-symmetric sequence of indices \((a_1, a_2, \ldots, a_{2n-1}, a_{2n})\) in (7.8) is called a ‘word’ constructed of \(m\) letters. The average on the right-hand side of (7.8) is called the ‘meaning’ of the word. The meander problem is therefore equivalent to the problem of summing over all words with a Gaussian meaning. The principal meander numbers \(\mathcal{M}_n\) can
be obtained from (7.8) using an analog of the replica trick which suppresses higher loops of the field $W$, i.e. they are determined as the linear terms of the expansion of (7.8) in $m$,

$$
\mathcal{M}_n = \lim_{m \to 0} \frac{1}{m} \sum_{1 \leq a_1, \ldots, a_{2n} \leq m} \left( \lim_{N \to \infty} \left\langle \left( \text{tr} \prod_{i=1}^{2n} W_{a_i} \right)_H \right\rangle \right)^2
$$

(7.10)

For $m = 1$, the Gaussian moments are (see (1.32))

$$
\lim_{N \to \infty} \left\langle \text{tr} W^{2n} \right\rangle_H = \frac{(2n)!}{(n+1)!n!} = C_n
$$

(7.11)

which is known as the Catalan number of order $n$. The quantity $C_n$ is the number of parenthesings of words of $n + 1$ letters with $n$ opening and $n$ closing parentheses. From (7.8) it then follows that

$$
\sum_{k=1}^{n} \mathcal{M}_n^{(k)} = C_n^2
$$

(7.12)

which is just the first Di Francesco-Golinelli-Guitter meander sum rule [39].

It is also possible to define higher-genus meander numbers $\mathcal{M}_n^{(k)}(h)$ which correspond to the higher-genus fat-graphs of the matrix model. Here the indexation is now by the number of intersections (or bridges), i.e. $\mathcal{M}_n^{(k)}(0) = \mathcal{M}_n^{(k)}$. The generating function for these meanders is then given as the genus expansion

$$
\sum_{h=0}^{\infty} \sum_{k=1}^{\infty} \mathcal{M}_n^{(k)}(h) m^k N^{-2h} = \sum_{1 \leq a_1, \ldots, a_p \leq m} \left\langle \left( \text{tr} \prod_{i=1}^{p} W_{a_i} \right)^2 \right\rangle_H
$$

(7.13)

and the sum rule (7.12) becomes [39]

$$
\sum_{h=0}^{\infty} \sum_{k=1}^{\infty} \mathcal{M}_n^{(k)}(h) m^k = m(m+2)(m+4) \cdots (m+2p-2)
$$

(7.14)

For example, a genus 1 meander is made of a collection of $p$ loops intersecting the river only once (since the right-hand side of (7.14) is a polynomial of degree $p$ with leading coefficient 1).

The meander numbers can alternatively be represented as a Gaussian average over complex matrices [97, 99]. The generating function is

$$
\mathcal{E}_C(m, c; N) = \frac{1}{N^2} \left\langle \log \left( \int d\phi_1 \, d\phi_2 \, e^{-\Sigma_C[\phi, W]} \right) \right\rangle_C
$$

(7.15)

where

$$
\left\langle Q(W) \right\rangle_C \equiv \frac{1}{N} \int \prod_{a=1}^{m} d[W_a]_{ij} \, d[W_a^\dagger]_{ij} \, Q(W) \, e^{-N^2 \sum_{a} \text{tr} W_a^\dagger W_a}
$$

(7.16)

with $N$ a normalization constant, and the action is

$$
\Sigma_C[\phi, W] = \frac{N^2}{2} \text{tr} \phi_1^2 + \frac{N^2}{2} \text{tr} \phi_2^2 - cN^2 \sum_{a=1}^{m} \text{tr} \phi_1 W_a^\dagger \phi_2 W_a
$$

(7.17)

Here $\phi_1$ and $\phi_2$ are $N \times N$ Hermitian matrices and $W_a$ are general $N \times N$ complex-valued matrices. Notice that in the generating function (7.6) the Hermitian matrix $\phi$ can be represented
in diagonal form by absorbing the unitary matrices $U$ of the diagonalization transformation into the Hermitian matrices by the adjoint action $W_a \rightarrow U^\dagger W_a U$ of $U(N)$ which leaves the integration measure over Hermitian matrices $W_a$ invariant. The same feature is true of the multi-matrix integral (7.15) - diagonalizing the Hermitian matrices $\phi_1$ and $\phi_2$ by unitary transformations generated by $U_1, U_2 \in U(N)$, respectively, the unitary matrices $U_1$ and $U_2$ can be absorbed by the $U(N) \otimes U(N)$ adjoint transformation $W_a \rightarrow U_2 W_a U_1, W_a^\dagger \rightarrow U_1 W_a^\dagger U_2$ which leaves the integration measure in (7.15) unchanged because it is over general complex matrices $W_a$.

Consider in addition the generating function

$$\mathcal{M}_C(m, c; N) = c \left\langle \left( \frac{\int d\phi_1 d\phi_2 e^{-\Sigma_C[\phi, W]} \text{tr} \phi_1 W_{a_0}^\dagger \phi_2 W_{a_0}}{\int d\phi_1 d\phi_2 e^{-\Sigma_C[\phi, W]}} \right) \right\rangle_c$$

(7.18)

which is averaged in terms of a single component $W_{a_0}$. Differentiating (7.15) with respect to $c$ and noting that all $W_a$ in the resulting expression are weighted the same way, we get a relation

$$c \frac{\partial}{\partial c} \mathcal{E}_C(m, c; N) = m \mathcal{M}_C(m, c; N)$$

(7.19)

between these 2 generating functions.

To see how the complex matrix model (7.15) generates meander numbers, we explicitly calculate the Gaussian integrals over $\phi_1$ and $\phi_2$ in (7.15) as before using

$$\int d\phi_1 d\phi_2 e^{-\Sigma_C[\phi, W]} = \int d\phi_2 e^{-\frac{\alpha^2}{2} \text{tr} \phi_2^2 + \frac{\beta^2 \alpha^2}{2} \sum_{a,b=1}^m \text{tr} (\phi_2 W_a^\dagger \phi_2 W_b^\dagger W_a^\dagger W_b^\dagger)}$$

$$= \text{det}^{-1/2} \left[ I \otimes I - \beta^2 \sum_{a,b=1}^m W_a^\dagger (W_b^\dagger)^T \right]$$

(7.20)

Using (7.19) we get

$$\lim_{N \rightarrow \infty} \mathcal{M}_C(m, c; N) = \sum_{n=1}^{\infty} c^{2n} \sum_{k=1}^{n} \mathcal{M}_n^{(k)} m^{k-1}$$

(7.21)

with

$$\sum_{k=1}^{n} \mathcal{M}_n^{(k)} m^{k-1} = \sum_{1 \leq a_1, \ldots, a_{2n} \leq m} \left( \lim_{N \rightarrow \infty} \left\langle \text{tr} W_{a_0} W_{a_2}^\dagger \cdots W_{a_{2n-1}}^\dagger W_{a_{2n}}^\dagger \right\rangle_c \right)^2$$

(7.22)

The planar limit of the Gaussian average in (7.22) coincides with that of (7.9) where the former is obtained from Wick contractions among $W$ and $W^\dagger$. In particular, the analog of the $m = 1$ formula (7.11) for complex matrices is (see Subsection 1.2.2)

$$\lim_{N \rightarrow \infty} \left\langle \text{tr} (W^\dagger W)^n \right\rangle_c = C_n$$

(7.23)

which leads again to the sum rule (7.12).

7.2 Arch Configurations and Fermionic Matrix Models

A general meander of order $n$ with an arbitrary number of connected components is uniquely specified by its upper half (above the river) and its lower half (below the river). Both halves
form systems of \( n \) non-intersecting arches connecting \( 2n \) bridges by pairs. Each of these systems are called arch configurations. Thus any meander is a superposition of 2 arch configurations. Any 2 arches are either disjoint or included, one into the other. The number of arch configurations of order \( n \), linking \( n \) bridges, is the Catalan number \( C_n \). Any arch configuration is completed by reflecting with respect to the river. This gives a one-to-one correspondence between the \( n \)-component meanders of order \( n \) and the arch configurations of order \( n \), so that

\[
\mathcal{M}_n^{[n]} = C_n
\] (7.24)

More generally, any multi-component meander of order \( n \) is obtained by superimposing any 2 arch configurations of order \( n \), one above the river and one below, and connecting them through the \( 2n \) bridges. This fact implies immediately the sum rule (7.12), which therefore expresses combinatorially the total number of multi-component meanders of order \( n \) as the total number of pairs of (top and bottom) arch configurations of order \( n \).

The construction of meanders is equivalent to 2 types of moves on arch configurations [39]. The first move, which we shall denote by an operator \( I_1 \), is to pick any exterior arch configuration of a meander and cut it, and then pull the 2 edges of the cut across the river (left part of exterior arch to the left, right part to the right). Then paste them around the lower ‘rainbow’. This increases the rainbow configuration by 1 arch (by definition a rainbow configuration of order \( n \) has 1 arch of each depth between 1 and \( n \) and the number of bridges by 2. This yields a meander of order \( n+1 \) with the same number \( k \) of connected components, and so from each meander \( \mathcal{M} \) of order \( n \) with \( k \) components we can construct \( E(\mathcal{M}) \) distinct meanders of order \( n+1 \) with \( k \) components, where \( E(\mathcal{M}) \) is the number of exterior arches of \( \mathcal{M} \). The second move \( I_2 \) is to take a meander of order \( n \) with \( k-1 \) connected components, add an extra circular loop around it (which increases the lower rainbow configuration of order \( n \) by 1 arch) and then add 2 bridges. This yields an order \( n+1 \) meander with \( k \) connected components which cannot be obtained by the move \( I_1 \) above (since it has only 1 exterior arch, whereas \( I_1 \) yields at least 2 exterior arches). Conversely, it is possible to show that any meander of order \( n \) with \( k \) connected components can be obtained in this way [39].

An important idea for the combinatorics of the meander problem is the notion of the signature of arch configurations [39]. This is defined recursively for an arch configuration \( \mathcal{A} \) with respect to the moves \( I_1 \) and \( I_2 \) above, starting from the empty arch,

\[
\text{sig}(\emptyset) = 1 \quad ; \quad \text{sig}(I_1 \mathcal{A}) = (-1)^{|\mathcal{A}|} \text{sig}(\mathcal{A}) \quad , \quad \text{sig}(I_2 \mathcal{A}) = (-1)^{|\mathcal{A}|+1} \text{sig}(\mathcal{A})
\] (7.25)

where \( |\mathcal{A}| \) is the number of arches in \( \mathcal{A} \). The relevant combinatorial quantity for the meander problem is the number

\[
s(n) \equiv \sum_{\mathcal{A}_n} \text{sig}(\mathcal{A}_n)
\] (7.26)

where the sum is over all arch configurations \( \mathcal{A}_n \) of order \( n \). Given a meander \( \mathcal{M} \) with upper and lower arch configurations \( \mathcal{A}^{(u)} \) and \( \mathcal{A}^{(d)} \), respectively, we define its signature as

\[
\text{sig}(\mathcal{M}) = \text{sig}(\mathcal{A}^{(u)}) \cdot \text{sig}(\mathcal{A}^{(d)})
\] (7.27)

For a meander \( \mathcal{M}_n^{(k)} \) of order \( n \) with \( k \) connected components, it follows from this definition that [39]

\[
\text{sig}(\mathcal{M}_n^{(k)}) = (-1)^{k+n}
\] (7.28)
We now turn to a matrix model representation of these notions. We can modify the complex matrix model (7.15) by considering Grassmann-valued matrices instead of complex-valued ones. The generating function is

$$E_F(m, c; N) = \frac{1}{N^2} \left\langle \log \left( \int d\phi_1 \, d\phi_2 \, e^{-\Sigma_F[\phi, \bar{\psi}]} \right) \right\rangle_F$$

(7.29)

with fermionic Gaussian average

$$\left\langle Q[\psi, \bar{\psi}] \right\rangle_F = \frac{1}{N} \int \prod_{a=1}^{m} d\psi_a \, d\bar{\psi}_a \, Q[\psi, \bar{\psi}] \, e^{N^2 \sum_a \text{tr} \, \bar{\psi}_a \psi_a}$$

(7.30)

and action

$$\Sigma_F[\phi, \psi, \bar{\psi}] = \frac{N^2}{2} \, \text{tr} \, \phi_1^2 + \frac{N^2}{2} \, \text{tr} \, \phi_2^2 - cN^2 \sum_{a=1}^{m} \text{tr} \, \phi_1 \bar{\psi}_a \phi_2 \psi_a$$

(7.31)

where $\phi_1$ and $\phi_2$ are as before $N \times N$ Hermitian matrices, and $\bar{\psi}_a$ and $\psi_a$ are independent $N \times N$ fermionic Grassmann-valued matrices. Most of the formalism which identifies the complex matrix model representation of meanders that we discussed in Subsection 7.1 above now carries through in exactly the same way for the generating function (7.29) with the replacements $W_a^1 \rightarrow \bar{\psi}_a$, $W_a \rightarrow \psi_a$ there. There are, however, 2 important differences in the fermionic case. First of all, we recall that the Feynman rules associate a factor of $-1$ to each loop of the fermion fields. This modifies the meander generating function (7.21) to

$$\lim_{N \rightarrow \infty} M_F(m, c; N) = \lim_{N \rightarrow \infty} \cF \left\langle \frac{\int d\phi_1 \, d\phi_2 \, e^{-\Sigma_F[\phi, \bar{\psi}]} \, \text{tr} \, \bar{\phi}_1 \bar{\psi}_a \bar{\phi}_2 \psi_a}{\int d\phi_1 \, d\phi_2 \, e^{-\Sigma_F[\phi, \bar{\psi}]}} \right\rangle_F$$

(7.32)

$$= \sum_{n=1}^{\infty} c^{2n} \sum_{k=1}^{n} M_n^{(k)} (-m)^{k-1}$$

with

$$\sum_{k=1}^{n} M_n^{(k)} (-m)^{k-1}$$

$$= \sum_{1 \leq a_2, \ldots, a_{2n} \leq m} \lim_{N \rightarrow \infty} \left\langle \text{tr} \, \psi_{a_0} \bar{\psi}_{a_2} \cdots \psi_{a_{2n-1}} \bar{\psi}_{a_{2n}} \right\rangle_F \left\langle \text{tr} \, \bar{\psi}_{a_{2n}} \psi_{a_{2n-1}} \cdots \bar{\psi}_{a_2} \psi_{a_0} \right\rangle_F$$

(7.33)

where we have kept track of the order of the matrices as they appear from the generating function (7.32) because the signs are essential for Grassmann-valued matrices. As we shall see below, the factors of $(-1)^{k-1}$ are associated with the signatures of the arch configurations.

The second distinguishing difference from the bosonic case is the $m = 1$, $N = \infty$ Gaussian average, which from the generating function (3.36) is easily found to be

$$\lim_{N \rightarrow \infty} \left\langle \text{tr} \, (\bar{\psi} \psi)^n \right\rangle_F = \begin{cases} 0, & \text{for } n = 2p \text{ even} \\ C_p, & \text{for } n = 2p + 1 \text{ odd} \end{cases}$$

(7.34)

Using (7.33) this implies that the $m = 1$ sum rule (7.12) is now modified to

$$\sum_{k=1}^{n} (-1)^{k-1} M_n^{(k)} = \begin{cases} 0, & \text{for } n = 2p \text{ even} \\ C_p^2, & \text{for } n = 2p + 1 \text{ odd} \end{cases}$$

(7.35)
which is just the second Di Francesco-Golinelli-Guitter meander sum rule [39]. Notice, in particular, that the left-hand side of (7.35) is given by the combinatorical quantity
\[ \sum_{k=1}^{n} (-1)^{k-1} M_n^{(k)} = (-1)^{n-1} \sum_{A_n^{(u)}} \sum_{A_n^{(d)}} \text{sig}(A_n^{(u)}) \text{sig}(A_n^{(d)}) = (-1)^{n-1} s(n)^2 \] (7.36)
which combined with (7.35) yields \( s(2p) = 0, s(2p+1) = (-1)^{p+1} C_p \) for \( p = 1, 2, \ldots \) (with the initial values \( s(0) = 1, s(1) = -1 \)). Thus the adjoint fermion matrix models provide a natural representation of the notion of the signature of arch configurations that we discussed above.

### 7.3 Principal Meander Numbers and Supersymmetric Matrix Models

As a final example of a matrix model representation of the meander problem, we shall now combine the models discussed in the previous 2 Subsections and consider a general complex matrix model with both bosonic and fermionic matrices, i.e. we combine the bosonic representation (7.21),(7.22) of meanders with the fermionic one (7.32),(7.33). To this end, we consider a general \( m = m_b + m_f \) component matrix field \( W_a \) with \( m_b \) bosonic (complex) components \( \phi_a \) and \( m_f \) fermionic (Grassmann) components \( \psi_a \),
\[ W_a = (\phi_1, \ldots, \phi_{m_b}, \psi_1, \ldots, \psi_{m_f}) , \quad \bar{W}_a = (\phi_1, \ldots, \phi_{m_b}, \bar{\psi}_1, \ldots, \bar{\psi}_{m_f}) \] (7.37)
The generating function is defined by
\[ \mathcal{E}_S^{(m_b, m_f)}(c, N) = \frac{1}{N^2} \left\langle \log \left( \int d\phi_1 d\phi_2 e^{-\Sigma_S^{(m_b, m_f)}[\phi, W, \bar{W}]} \right) \right\rangle_S \] (7.38)
with Gaussian average
\[ \left\langle Q[W, \bar{W}] \right\rangle_S = \frac{1}{\mathcal{N}} \int dW_a d\bar{W}_a Q[W, \bar{W}] e^{-N^2 \sum_{a=1}^{m} \text{tr} \bar{W}_a W_a} \] (7.39)
and action
\[ \Sigma_S^{(m_b, m_f)}[\phi, W, \bar{W}] = \frac{N^2}{2} \text{tr} \phi_1^2 + \frac{N^2}{2} \text{tr} \phi_2^2 - c N^2 \sum_{a=1}^{m} \text{tr} \phi_1 W_a \phi_2 W_a \] (7.40)
where the integration measure is
\[ dW_a d\bar{W}_a \equiv d\psi_a d\bar{\psi}_a \prod_{i,j} d[\tilde{\phi}_a]_{ij} d[\tilde{\phi}_a^i]_{ij} \] (7.41)
Since fermion loops are always accompanied by a minus sign, it follows that the generating function (7.38) is related to the meander numbers by
\[ \lim_{N \to \infty} \mathcal{E}_S^{(m_b, m_f)}(c, N) = \sum_{n=1}^{\infty} \frac{c^{2n}}{2n} \sum_{k=1}^{n} M_n^{(k)} (m_b - m_f)^k \] (7.42)

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To generalize the relations of the previous Subsections between the meander problem and the problem of summing over all words with Gaussian meaning, we again introduce another generating function

$$\mathcal{M}_S(m_b, m_f, c; N) = c \left\langle \int \frac{d\phi_1 \ d\phi_2 \ e^{-\Sigma_S^{(m_b, m_f)}[\phi, W, \bar{W}]} \ tr \phi_1 \bar{W}_{a_0} \phi_2 W_{a_0}}{\int d\phi_1 \ d\phi_2 \ e^{-\Sigma_S^{(m_b, m_f)}[\phi, W, \bar{W}]}} \right\rangle_S$$

(7.43)

which satisfies

$$c \frac{\partial}{\partial c} \mathcal{E}_S^{(m_b, m_f)}(c, N) = (m_b - m_f) \mathcal{M}_S(m_b, m_f, c; N)$$

(7.44)

From (7.42) and (7.44) it follows that (7.43) is related to the meander numbers by

$$\lim_{N \to \infty} \mathcal{M}_S(m_b, m_f, c; N) = \sum_{n=1}^{\infty} c^{2n} \sum_{k=1}^{m} \mathcal{M}_n^{(k)}(m_b - m_f)^{k-1}$$

(7.45)

The Gaussian integrations above can be computed as before to get

$$\int d\phi_1 \ d\phi_2 \ e^{-\Sigma_S^{(m_b, m_f)}[\phi, W, \bar{W}]} = \int d\phi_2 \ e^{-\frac{\phi_2^2}{2} + \frac{c^2}{2} \sum_{\alpha, \beta=1}^{m} \tr (\bar{W}_{a_\alpha} \phi_2 W_{a_\alpha} \bar{W}_{a_\beta} \phi_2 W_{a_\beta})}$$

$$= \int d\phi_2 \ e^{-\frac{\phi_2^2}{2} + \frac{c^2}{2} \sum_{\alpha, \beta=1}^{m} \tr (\phi_2 \bar{W}_{a_\alpha} \phi_2 \bar{W}_{a_\beta})}$$

$$= \det^{-1/2} \left[ I \otimes I - c^2 \sum_{\alpha, \beta=1}^{m} \sigma_{a_\alpha} W_{a_\alpha} \bar{W}_{a_\beta} \otimes (W_{a_\beta} \bar{W}_{a_\alpha})^T \right]$$

(7.46)

where \(\sigma_{a}\) is the signature factor (or Klein number) of the component \(W_{a}\), defined as +1 for the bosonic components and −1 for the fermionic ones.

Expanding the determinant in (7.46) as before on the right-hand side of (7.38) in powers of \(c^2\), we arrive at the representation

$$\lim_{N \to \infty} \mathcal{E}_S^{(m_b, m_f)}(c, N)$$

$$= \sum_{n=1}^{\infty} \frac{c^{2n}}{2n} \sum_{a_1, \ldots, a_{2n-1}, a_{2n}=1}^{m} \sigma_{a_1} \sigma_{a_2} \cdots \sigma_{a_{2n-1}}$$

$$\times \lim_{N \to \infty} \left\langle \tr W_{a_1} \bar{W}_{a_2} \cdots \bar{W}_{a_{2n-1}} W_{a_{2n}} \right\rangle_S \left\langle \tr W_{a_{2n}} \bar{W}_{a_{2n-1}} \cdots \bar{W}_{a_2} W_{a_1} \right\rangle_S$$

(7.47)

$$= \sum_{n=1}^{\infty} \frac{c^{2n}}{2n} \sum_{a_1, \ldots, a_{2n-1}, a_{2n}=1}^{m} \lim_{N \to \infty} \left\langle \tr W_{a_1} \bar{W}_{a_2} \cdots \bar{W}_{a_{2n-1}} W_{a_{2n}} \right\rangle_S$$

$$\times \left\langle \tr W_{a_{2n}} \bar{W}_{a_{2n-1}} \cdots \bar{W}_{a_2} W_{a_1} \right\rangle_S$$

where the signs of the fermionic components in the first equality in (7.47) have been absorbed by reordering the components in the second Gaussian correlator there. Comparing this result with the above expressions for the meander generating functions we find

$$\sum_{k=1}^{m} \mathcal{M}_n^{(k)}(m_b - m_f)^{k-1}$$

$$= \sum_{a_2, \ldots, a_{2n}=1}^{m} \lim_{N \to \infty} \left\langle \tr W_{a_0} \bar{W}_{a_2} \cdots \bar{W}_{a_{2n-1}} \bar{W}_{a_{2n}} \right\rangle_S$$

$$\times \left\langle \tr \bar{W}_{a_{2n}} W_{a_{2n-1}} \cdots \bar{W}_{a_2} W_{a_0} \right\rangle_S$$

(7.48)
Notice that the order of the matrices in (7.48) is chosen to absorb the signature factors for the fermionic components.

We are especially interested in the situation above where the number of bosonic and fermionic matrix fields coincide, \( m_b = m_f \). In that case, in addition to the usual gauge and charge-conjugation invariances of the complex and fermionic matrix models, the model (7.38) is invariant under the supersymmetry (boson-fermion) transformation \( \phi_a \leftrightarrow \psi_a, \phi_a^\dagger \leftrightarrow \bar{\psi}_a \). We can use this supersymmetry to kill the loops of the \( W \)-fields which represents an alternative to the replica trick [97, 99]. For brevity of notation, we restrict our attention to the case \( m = 2 \) (\( m_b = m_f = 1 \)), i.e. we consider a 2-component field \( W \_a \) whose first component is a bosonic, complex-valued matrix and whose second component is a fermionic matrix,

\[
W_a = (\phi, \psi) , \quad \bar{W}_a = (\phi^\dagger, \bar{\psi})
\]

Since the propagators for the bosonic and fermionic matrices in (7.49) coincide (see Section 1), the simple \( D = 0 \) supersymmetry here reduces to just rotations between the bosonic and fermionic components. The proper infinitesimal supersymmetry transformation is

\[
\delta_\epsilon \phi = \bar{\epsilon} \psi , \quad \delta_\epsilon \psi = -\epsilon \phi \quad ; \quad \delta_\epsilon \phi^\dagger = \bar{\psi} \epsilon , \quad \delta_\epsilon \bar{\psi} = -\phi^\dagger \epsilon
\]

where \( \epsilon \) and \( \bar{\epsilon} \) are infinitesimal anticommuting parameters. We shall briefly discuss more general supersymmetry transformations at the end of this Section.

The key property of the supersymmetry is that the contributions from the bosonic and fermion loops are mutually cancelled for any potential which is constructed symmetrically from the superfields (7.49), and which is therefore supersymmetric. In the generating function \( \mathcal{E}_S^{[1,1]}(c, N) \) defined as the supersymmetric one-matrix integral in (7.38), the potential there is the simplest Gaussian superpotential

\[
\mathcal{W}_{\text{Gauss}}[\phi, \psi, \bar{\psi}] = \bar{W} \bar{W} \equiv \sum_{a=1,2} \bar{W}_a W_a = \bar{\phi}^\dagger \phi + \bar{\psi} \psi
\]

which reproduces the propagators given in Section 1. Notice that it is even invariant under the rotation (7.50) when the parameters \( \epsilon \) and \( \bar{\epsilon} \) are fermionic \( N \times N \) matrices. This large degree of (super)symmetry means that the supersymmetric generating function \( \mathcal{E}_S^{[1,1]}(c, N) \) is identically zero, due to the mutual cancellation between the loops of the bosonic and fermionic matrix fields. Instead, one should use the generating function (7.43) which for the supersymmetric matrix model above can be represented as

\[
\mathcal{M}_S(c; N) \equiv \mathcal{M}_S(1,1; c; N) = \left\langle \frac{\text{tr} \, \bar{\phi} \phi}{\phi_1 \phi_2} \log \left( \int d\phi_1 \, d\phi_2 \, e^{-\Sigma_s^{[1,1]}[\phi, W, \bar{W}]} \right) \right\rangle_S
\]

where the action is explicitly given by

\[
\Sigma_s^{[1,1]}[\phi, W, \bar{W}] = \frac{N^2}{2} \text{tr} \, \phi_1^2 + \frac{N^2}{2} \text{tr} \, \phi_2^2 - c N^2 \text{tr} \, \phi_1 \phi_2 \bar{\phi} \bar{\phi} - c N^2 \text{tr} \, \phi_1 \bar{\psi} \phi_2 \bar{\psi}
\]

That (7.52) is equivalent to (7.43) (with \( a_0 = 1 \)) in this supersymmetric case follows from an integration by parts in the identity

\[
0 \equiv \left\langle \left\{ N^{-1} \frac{\partial}{\partial \phi_{ij}} \log \left( \int d\phi_1 \, d\phi_2 \, e^{-\Sigma_s^{[1,1]}[\phi, W, \bar{W}]} \right) \right\} \right\rangle_S ,
\]

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summing over all $i = k, j = \ell$, and using the fact that $\mathcal{E}^{(1,1)}_S(c, N) \equiv 0$.

Because the bosonic and fermionic generating functions (7.21), (7.22) and (7.32), (7.33) alternate in sign relative to one another, only the $k = 1$ terms there survive for the supersymmetric model. Thus the multi-component meanders vanish and from (7.45), (7.48) we get the representation

$$\mathcal{M}_n = \sum_{a_2, \ldots, a_{2n} = 1, 2} \lim_{N \to \infty} \left\langle \text{tr} \, \mathcal{W}_{a_2} \cdots \mathcal{W}_{a_{2n-1}} \mathcal{W}_{a_{2n}} \right\rangle_S \left\langle \text{tr} \, \mathcal{W}_{a_{2n}} \mathcal{W}_{a_{2n-1}} \cdots \mathcal{W}_{a_2} \phi \right\rangle_S^{(7.55)}$$

for the principal meander numbers. Again we keep care here of the order of matrices as the signs are crucial for the fermionic components of the $W$-field. Alternatively, replacing $\phi$ by $\psi$ and $\bar{\phi}$ by $\bar{\psi}$ in (7.52), (7.54) ($a_0 = 2$ in (7.43)), we also have

$$-\mathcal{M}_n = \sum_{a_2, \ldots, a_{2n} = 1, 2} \lim_{N \to \infty} \left\langle \text{tr} \, \mathcal{W}_{a_2} \cdots \mathcal{W}_{a_{2n-1}} \mathcal{W}_{a_{2n}} \right\rangle_S \left\langle \text{tr} \, \mathcal{W}_{a_{2n}} \mathcal{W}_{a_{2n-1}} \cdots \mathcal{W}_{a_2} \psi \right\rangle_S^{(7.56)}$$

Thus the supersymmetric matrix model provides a representation of the principal meander numbers which looks much more natural than the one before based on the replica trick. It is hoped that the large supersymmetry of the problem will make it simpler to solve the $m = 2$ supersymmetric matrix model than a pure bosonic or fermionic one at arbitrary $m$, for example by solving it using Ward identities associated with the supersymmetry (i.e. loop equations associated with the transformations (7.50)). For explicit results of the calculations of the principal meander numbers up to $n = 4$ based on the supersymmetric matrix model equations (7.55) and (7.56), as well as a comparison with those based on the purely bosonic matrix model equations (7.22), see [99].

In [97, 99] it is also shown how to represent the meander problem in terms of unitary matrix models, and ultimately its connection with the Kazakov-Migdal model on a $D$-dimensional lattice. The words are the same for both the meander problem and the Kazakov-Migdal model, the only difference residing in the meaning of non-vanishing words which is unity in the case of averages over unitary matrices in Haar measure. This relation could give a hint on how to solve the meander problem. Its connection with higher-dimensional matrix models, and in particular the combinatorial problem of summing over all closed loops of a given length in a $D$-dimensional embedding space with all possible backtrackings (or foldings) included of which we found the Gaussian fermionic representation in Section 6 above, is an interesting representation of the polymer phase of higher-dimensional string theories.

### 7.4 Loop Equations

We shall now proceed to discuss the evaluation of observables in supersymmetric matrix models. Again, we shall be primarily interested with the novel model of the last Subsection describing the meander problem, and we shall start first from the generic complex matrix model (7.38) and then later on impose the restriction to supersymmetry. However, the method for solving the corresponding loop equations in these cases, developed recently in
is an elaborate technique for dealing with general supersymmetric matrix models, not just the one that we are dealing with thus far. The observables in the cases at hand are words. To deal with these appropriately, we must modify somewhat extensively the generating functions considered in our earlier matrix theories. The observables of the matrix model are now generated by introducing non-commutative sources which reduce the calculation to computing averages in a Boltzmannian Fock space. This method has been applied recently to the loop equations of bosonic matrix models in [27, 51, 52, 63]. To this end, we introduce a set of non-commuting variables \( \hat{u}_a, \hat{u}_a^\dagger \), \( a = 1, \ldots, m \), which generate the Cuntz algebra

\[
\hat{u}_a \hat{u}_b^\dagger = \delta_{ab}
\]  

and regard them as creation and annihilation operators on some representation Hilbert space. The normalized vacuum state \( |\Omega\rangle \), \( \langle\Omega|\Omega\rangle = 1 \), satisfies

\[
\hat{u}_a |\Omega\rangle = \langle\Omega|\hat{u}_a^\dagger = 0
\]

and the completeness relation is

\[
\sum_{a=1}^{m} \hat{u}_a^\dagger \hat{u}_a = 1 - |\Omega\rangle \langle\Omega|
\]

where \( 1 \) is the identity operator on the given Hilbert space.

The non-commutative variables are introduced above to define the generating functions for words,

\[
G(z; \hat{u}) = \left\langle \left( \text{tr} \frac{1}{z - \sum_{a,b=1}^{m} \hat{u}_a \hat{u}_b W_a W_b} \right) \right\rangle_S
\]

\[
= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z^{n+1}} \sum_{a_1, \ldots, a_{2n}=1}^{m} \hat{u}_{a_1} \hat{u}_{a_2} \cdots \hat{u}_{a_{2n-1}} \hat{u}_{a_{2n}} \left\langle \text{tr} W_{a_1} W_{a_2} \cdots W_{a_{2n-1}} W_{a_{2n}} \right\rangle_S
\]

\[
\tilde{G}(z; \hat{u}) = \left\langle \left( \text{tr} \frac{1}{z - \sum_{a,b=1}^{m} \hat{u}_a \hat{u}_b \tilde{W}_a \tilde{W}_b} \right) \right\rangle_S
\]

\[
= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z^{n+1}} \sum_{a_1, \ldots, a_{2n}=1}^{m} \hat{u}_{a_1} \hat{u}_{a_2} \cdots \hat{u}_{a_{2n-1}} \hat{u}_{a_{2n}} \left\langle \text{tr} \tilde{W}_{a_1} \tilde{W}_{a_2} \cdots \tilde{W}_{a_{2n-1}} \tilde{W}_{a_{2n}} \right\rangle_S
\]

where the general complex matrices \( W_a, \tilde{W}_a \) are defined in (7.37). These 2 generating functions are not independent because of the cyclic symmetry of the traces. Rearranging the components appropriately, we have

\[
\tilde{G}(z; \hat{u}_a) = G(z; \sqrt{\alpha_a} \hat{u}_a)
\]

so that the components of \( \hat{u}_a \) associated with bosonic components are unchanged while those associated with fermionic components are multiplied by a factor of \( i \).

The Schwinger-Dyson equations for these generating functions follow in the usual way by shifting the matrix integration variables

\[
\tilde{W}_{2n} \rightarrow \tilde{W}_{2n} + \epsilon \tilde{W}_{2n}^m, \quad W_1 \rightarrow W_1 + \epsilon W_1^m
\]

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in (7.39). The shifts (7.63) lead, respectively, to the standard set of recurrence relations at $N = \infty$

$$
\begin{align*}
&\left\langle \text{tr } W_{a_1} W_{a_2} \cdots W_{a_{2n-1}} W_{a_{2n}} \right\rangle_S \\
= & \sum_{k=0}^{n-1} \delta_{a_{2n},a_{2k+1}} \left\langle \text{tr } W_{a_1} W_{a_2} \cdots W_{a_{2k}} \right\rangle_S \left\langle \text{tr } W_{a_{2k+2}} \cdots W_{a_{2n-1}} \right\rangle_S \\
= & \sum_{k=1}^{n} \delta_{a_1,a_{2k}} \left\langle \text{tr } W_{a_2} W_{a_3} \cdots W_{a_{2k-1}} \right\rangle_S \langle \text{tr } W_{a_{2k+1}} \cdots W_{a_{2n}} \rangle_S
\end{align*}
$$
\tag{7.64}

(7.65)

for the words with Gaussian meaning. Multiplying these equations by $1/z^{n+1}$ and summing over all $n \in \mathbb{Z}^+$ using (7.60),(7.61) and the cyclic symmetry of the trace results in the loop equations

$$
1 - z \mathcal{G}(z; \hat{u}) = \sum_{a=1}^{m} \hat{u}_a \mathcal{G}(z; \hat{u}) = \sum_{a=1}^{m} \mathcal{G}(z; \hat{u}) \hat{u}_a \mathcal{G}(z; \hat{u}) \hat{u}_a
$$
\tag{7.66}

$$
1 - z \mathcal{G}^{\bar{c}}(z; \hat{u}) = \sum_{a=1}^{m} \sigma_a \mathcal{G}^{\bar{c}}(z; \hat{u}) \hat{u}_a \mathcal{G}(z; \hat{u}) \hat{u}_a = \sum_{a=1}^{m} \sigma_a \hat{u}_a \mathcal{G}(z; \hat{u}) \hat{u}_a \mathcal{G}(z; \hat{u}) \hat{u}_a
$$
\tag{7.67}

Notice that, because of the relation (7.62), these 2 loop equations are not independent and cannot be obtained from one another by the substitution $\hat{u}_a \rightarrow \sqrt{\sigma_a} \hat{u}_a$. For a combinatorial interpretation of these loop equations which ties in directly with the ideas of the meander problem [39], see [99].

The loop equations (7.66) and (7.67) generalize those of the Gaussian adjoint fermion one-matrix model. They are easily solved for the pure fermionic case $(m_b = 0, m_f = m)$ when the variables $\hat{u}_a$ above are commutative sources (i.e. ordinary Euclidean vectors $\hat{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m$). In that case, we can add (7.66) and (7.67) together to get the pair of simple equations

$$
\mathcal{G}(z; \hat{u}) = 2/z - \mathcal{G}^{\bar{c}}(z; \hat{u}) , \quad 1 - z \mathcal{G}^{\bar{c}}(z; \hat{u}) = \hat{u}^2 z \mathcal{G}(z; \hat{u})^2 - 2 \hat{u}^2 \mathcal{G}^{\bar{c}}(z; \hat{u})
$$
\tag{7.68}

which have solutions

$$
\mathcal{G}(z; \hat{u}) = \frac{1}{z} - \frac{1}{2\hat{u}^2} + \frac{1}{2\hat{u}^2z} \sqrt{z^2 + 4\hat{u}^4} , \quad \mathcal{G}^{\bar{c}}(z; \hat{u}) = \frac{1}{z} + \frac{1}{2\hat{u}^2} - \frac{1}{2\hat{u}^2z} \sqrt{z^2 + 4\hat{u}^4}
$$
\tag{7.69}

The solution for $\mathcal{G}$ coincides with the one-loop correlator (3.36) of the Gaussian fermionic one-matrix model with $t \equiv 1/\hat{u}^2$, while the large-$z$ expansion of $\mathcal{G}$ in (7.69) yields the moments (7.34). Similarly, for a purely bosonic model $(m_b = m, m_f = 0)$ it is possible to show that the above results reproduce the Wigner semi-circle distribution for the Gaussian Hermitian (or complex) one-matrix model [99].

The formal solution to the loop equations in the general case can be found by rewriting (7.66) and (7.67) as

$$
\mathcal{G}(z; \hat{u}) = \frac{1}{z + z \sum_{a} \hat{u}_a \mathcal{G}(z; \hat{u}) \hat{u}_a} , \quad \mathcal{G}^{\bar{c}}(z; \hat{u}) = \frac{1}{z + z \sum_{a} \sqrt{\sigma_a} \hat{u}_a \mathcal{G}(z; \hat{u}) \sqrt{\sigma_a} \hat{u}_a}
$$
\tag{7.70}

As shown in [99], iterations of these equations lead to a formal solution for the generating functions of words as continued fractions which represents another form of the solutions for
Gaussian matrix models. Alternatively, using the Cuntz algebra (7.57) we can write the loop equations (7.66) and (7.67) as

\[(1 - z \mathcal{G}(z; \hat{u})) \hat{u}_a = z \mathcal{G}(z; \hat{u}) \hat{u}_a \mathcal{G}(z; \hat{u}) , \quad \hat{u}_a (1 - z \mathcal{G}(z; \hat{u}^\dagger)) = z \mathcal{G}(z; \hat{u}^\dagger) \hat{u}_a \mathcal{G}(z; \hat{u}^\dagger) \]  \hspace{1cm} (7.71)

\[(1 - z \mathcal{G}(z; \hat{u})) \hat{u}_a^\dagger = z \sigma_a \mathcal{G}(z; \hat{u}) \hat{u}_a \mathcal{G}(z; \hat{u}) , \quad \hat{u}_a (1 - z \mathcal{G}(z; \hat{u}^\dagger)) = z \sigma_a \mathcal{G}(z; \hat{u}^\dagger) \hat{u}_a^\dagger \mathcal{G}(z; \hat{u}^\dagger) \]  \hspace{1cm} (7.72)

where the left-hand sides of (7.71) and (7.72) are explicitly

\[\hat{u}_a (z \mathcal{G}(z; \hat{u}^\dagger) - 1) \]

\[= \sum_{n=1}^{\infty} \frac{1}{z^n} \sum_{a_2, \ldots, a_{2n} = 1} \hat{u}^\dagger_{a_2} \cdots \hat{u}^\dagger_{a_{2n-1}} \hat{u}_{a_{2n}} \left\langle \text{tr} W_{a_2} \cdots W_{a_{2n-1}} \mathcal{W}_{a_{2n}} \right\rangle_S \]  \hspace{1cm} (7.73)

\[z \mathcal{G}(z; \hat{u}) - 1) \hat{u}_a^\dagger \]

\[= \sum_{n=1}^{\infty} \frac{1}{z^n} \sum_{a_1, a_2, \ldots, a_{2n-1} = 1} \hat{u}_{a_1} \hat{u}_{a_2} \cdots \hat{u}_{a_{2n}} \left\langle \text{tr} \mathcal{W}_{a_1} W_{a_2} \cdots W_{a_{2n-1}} W_{a_n} \right\rangle_S \]  \hspace{1cm} (7.74)

The generating function (7.43) is determined as

\[\langle \Omega | \mathcal{G}(a; \hat{u}) \mathcal{G}(z; \hat{u}^\dagger) \rangle | \Omega \rangle = \langle \Omega | \mathcal{G}(z; \hat{u}) \mathcal{G}(z; \hat{u}^\dagger) | \Omega \rangle \]

\[= c^2 + c^2 (m_b - m_f) \lim_{N \to \infty} \mathcal{M}_S(m_b, m_f, c; N) \]  \hspace{1cm} (7.75)

with \(z = 1/c\). The equation (7.75) follows from (7.57) and (7.58) which imply that the left-hand side of it correctly reproduces the contraction of indices in (7.48).

We now specialize to the supersymmetric case \(m_b = m_f = 1\). Then (7.75) does not determine the meander numbers and one should instead use (7.55) or (7.56), where there is no summation over one of the indices \(a_1\), to get the principal meander numbers. In this case we denote the components of \(\hat{u}_a\) as \(\hat{u}_a = (u, v)\). Using the completeness relation (7.59) we have

\[z^2 \mathcal{G}(z; \hat{u}) u^\dagger u \mathcal{G}(z; \hat{u}^\dagger) = -z^2 \mathcal{G}(z; \hat{u}) v^\dagger v \mathcal{G}(z; \hat{u}^\dagger) + z^2 \mathcal{G}(z; \hat{u}) \mathcal{G}(z; \hat{u}^\dagger) - |\Omega \rangle \langle \Omega | \]

and because of the supersymmetry we further have from (7.75)

\[z^2 \langle \Omega | \mathcal{G}(z; \hat{u}) \mathcal{G}(z; \hat{u}^\dagger) | \Omega \rangle = 1 \]  \hspace{1cm} (7.77)

Thus, taking the vacuum expectation value of the expression (7.76), the supersymmetry of the matrix model implies that (7.73) and (7.74) yield

\[\lim_{N \to \infty} \mathcal{M}_S(1/z; N) = z^2 \langle \Omega | \mathcal{G}(z; \hat{u}) u^\dagger u \mathcal{G}(z; \hat{u}^\dagger) | \Omega \rangle = -z^2 \langle \Omega | \mathcal{G}(z; \hat{u}) v^\dagger v \mathcal{G}(z; \hat{u}^\dagger) | \Omega \rangle \]  \hspace{1cm} (7.78)

or alternatively

\[\lim_{N \to \infty} \mathcal{M}_S(1/z; N) = z^2 \langle \Omega | \mathcal{G}(z; \hat{u}) u^\dagger u \mathcal{G}(z; \hat{u}^\dagger) | \Omega \rangle = -z^2 \langle \Omega | \mathcal{G}(z; \hat{u}) v^\dagger v \mathcal{G}(z; \hat{u}^\dagger) | \Omega \rangle \]  \hspace{1cm} (7.79)

Using (7.71) and (7.72) these equations can then be written, respectively, as

\[\lim_{N \to \infty} \mathcal{M}_S(1/z; N) = \langle \Omega | \mathcal{G}(z; \hat{u}) u \mathcal{G}(z; \hat{u}) \mathcal{G}(z; \hat{u}^\dagger) u^\dagger \mathcal{G}(z; \hat{u}^\dagger) | \Omega \rangle \]

\[= \langle \Omega | \mathcal{G}(z; \hat{u}) v \mathcal{G}(z; \hat{u}) \mathcal{G}(z; \hat{u}^\dagger) v^\dagger \mathcal{G}(z; \hat{u}^\dagger) | \Omega \rangle \]  \hspace{1cm} (7.80)
\[
\lim_{N \to \infty} \mathcal{M}_s(1/z; N) = \langle \Omega | \tilde{\mathcal{G}}(z; \hat{u}) u \mathcal{G}(z; \hat{u}) \mathcal{G}(z; \hat{u}^\dagger) u^\dagger \mathcal{G}(z; \hat{u}^\dagger) | \Omega \rangle
\]

\[
= \langle \Omega | \tilde{\mathcal{G}}(z; \hat{u}) v \mathcal{G}(z; \hat{u}) \mathcal{G}(z; \hat{u}) v^\dagger \mathcal{G}(z; \hat{u}^\dagger) | \Omega \rangle
\]  

(7.81)

These latter 2 representations of the generating function for the principal meander numbers are convenient for an iterative evaluation. The standard trick \cite{27, 52} for dealing with 2 non-commutative variables is to expand the quantities in \( v \). This will lead to the calculation of the principal meander numbers order by order in \( c = 1/z \). We therefore introduce the expansions

\[
\mathcal{G}(z; \hat{u}) = \sum_{n=0}^{\infty} \mathcal{G}_n(z; \hat{u}) \quad , \quad \tilde{\mathcal{G}}(z; \hat{u}) = \sum_{n=0}^{\infty} (-1)^n \mathcal{G}_n(z; \hat{u})
\]

(7.82)

where the \( v \)-expansion of \( \tilde{\mathcal{G}} \) is an alternating series because of the relation (7.62). The coefficients \( \mathcal{G}_n(z; \hat{u}) \) in (7.82) are those terms in (7.60) that involve exactly \( 2n \) factors of \( v \). From the definition (7.49) and (7.23) it follows that

\[
\mathcal{G}_0(z; \hat{u}) = \frac{1 - \sqrt{1 - 4u^2/z}}{2u^2}
\]

(7.83)

which we note is just the Wigner semi-circle law for the distribution of the complex Gaussian moments \( \langle \text{tr} (\hat{\phi}^n \hat{\phi}) \rangle_c = C_n \). The functions \( \mathcal{G}_n(z; \hat{u}) \) for \( n \geq 1 \) can be found recursively from (7.70) which now takes the form

\[
\sum_{n=0}^{\infty} \mathcal{G}_n(z; \hat{u}) = \frac{1}{z} \left( \frac{1}{1 + \sum_{n=0}^{\infty} (-1)^n u \mathcal{G}_n(z; \hat{u}) v \mathcal{G}_0(z; \hat{u}) + \sum_{n=0}^{\infty} (-1)^n v \mathcal{G}_n(z; \hat{u}) v \mathcal{G}_0(z; \hat{u})} \right)
\]

(7.84)

At each order of the \( v \)-expansion in (7.84) we have to solve the equation

\[
\mathcal{G}_n(z; \hat{u}) = -\frac{1}{z} \left( (-1)^n \mathcal{G}_0(z; \hat{u}) u \mathcal{G}_n(z; \hat{u}) v \mathcal{G}_0(z; \hat{u}) + A_n(z; \hat{u}) \right)
\]

(7.85)

for some functions \( A_n(z; \hat{u}) \) which will be determined recursively below. The solution of (7.85) is found by iterating it to get

\[
\mathcal{G}_{2p-1}(z; \hat{u}) = -\frac{1}{z} \sum_{\ell=0}^{\infty} \left( \mathcal{G}_0(z; \hat{u}) u \right)^\ell \frac{A_{2p-1}(z; \hat{u})}{z^\ell} \left( \mathcal{G}_0(z; \hat{u}) u \right)^\ell
\]

(7.86)

\[
\mathcal{G}_{2p}(z; \hat{u}) = -\frac{1}{z} \sum_{\ell=0}^{\infty} (-1)^\ell \left( \mathcal{G}_0(z; \hat{u}) u \right)^\ell \frac{A_{2p}(z; \hat{u})}{z^\ell} \left( \mathcal{G}_0(z; \hat{u}) u \right)^\ell
\]

where \( p \geq 1 \). The 2 solutions in (7.86) can be written nicely as the contour integral

\[
\mathcal{G}_n(z; \hat{u}) = -\frac{1}{z} \oint_C \frac{d\lambda}{2\pi i} \frac{1}{\lambda - (-1)^{n+1} \mathcal{G}_0(z; \hat{u}) u \lambda \frac{A_n(z; \hat{u})}{1 - \lambda \mathcal{G}_0(z; \hat{u}) u / z} \equiv \{ A_n \}
\]

(7.87)

where the contour \( C \) encircles the origin of the complex \( \lambda \)-plane, and we have introduced a short-hand bracket notation for the quantities on the right-hand sides of (7.86) or (7.87) to simplify some of the cumbersome formulas which follow.
The functions $A_n(z;\hat{u})$ above are now computed recursively using (7.84) and (7.85). The first few lower order ones are explicitly

$$A_1 = G_0vG_0vG_0, \quad A_2 = -G_0vG_0vG_0 + z^2G_1\frac{G_1}{G_0}$$

$$A_3 = G_0vG_2vG_0 + z^2G_2\frac{G_1}{G_0}G_1 + z^2G_1\frac{G_1}{G_0}G_2 + z^3G_1\frac{G_1}{G_0}G_1\frac{G_1}{G_0}$$

(7.88)

Generally, substituting the expansion (7.82) into (7.71) (or (7.72)), equating the orders of the $v$-expansions and using the fact that

$$G_n(z;\hat{u})v^\dagger = -A_n(z;\hat{u})v^\dagger/z$$

(7.89)

which follows from (7.86) and (7.57), we see that the $A_n$ satisfy the identities

$$A_n(z;\hat{u})v^\dagger = z\sum_{k=0}^{n-1}(-1)^kG_{n-1-k}(z;\hat{u})v^g_k(z;\hat{u})$$

$$vA_n(z;\hat{u}^\dagger) = z\sum_{k=0}^{n-1}(-1)^kG_k(z;\hat{u})v^g_{n-1-k}(z;\hat{u}^\dagger)$$

(7.90)

Then, using (7.88) and the fact that $G_0$ commutes with the brackets defined by (7.87), the first few $G_n$'s are given by

$$G_1 = G_0\{vG_0v\}G_0, \quad G_2 = -G_0\{vG_0v\}G_0vG_0 + z^2G_0\{\{vG_0v\}G_0vG_0\}G_0$$

$$G_3 = -G_0\{vG_0v\}G_0\{vG_0v\}G_0vG_0G_0 + z^2G_0\{vG_0v\}G_0\{vG_0v\}G_0vG_0vG_0G_0$$

$$- z^2G_0\{\{vG_0v\}G_0\{vG_0v\}G_0\}G_0 - z^2G_0\{\{vG_0v\}G_0\{vG_0v\}G_0\}G_0vG_0vG_0G_0$$

$$+ z^3G_0\{\{vG_0v\}G_0\{vG_0v\}G_0\}G_0 + z^4G_0\{\{\{vG_0v\}G_0\{vG_0v\}G_0\}G_0\}G_0$$

$$+ z^4\{\{vG_0v\}G_0\{\{vG_0v\}G_0\}G_0\}G_0$$

(7.91)

The brackets in (7.91) must in general always pair equal numbers of open and closed parentheses, and an inner pair of brackets must always be embedded in an overall pair of outer brackets. For general $n$, $G_n(z;\hat{u})$ contains $C_n$ terms of the types in (7.91) with alternating signs up to order $z^{2n-1}$. Some rules for representing a general term can be formulated which resemble the Wick pairing of bilinear combinations of the $v$'s [99].

Most of the relations above are similar to those that one would obtain for 2 bosonic matrix fields ($m_b = m = 2, m_f = 0$), except that there are no alternating signs in the pure bosonic case. The occurrence of these minus signs at appropriate places makes the supersymmetric matrix model somewhat simpler in structure than the pure bosonic $m = 2$ case. For instance, the leading order $v^4$ terms could be cancelled for $A_2$ in (7.88) and for $G_2$ in (7.91), which follows from the general property (7.34) of fermionic Gaussian averages. For this same reason the asymptotic behaviours of the functions $G_n(z;\hat{u})$ are

$$\lim_{|z|\to\infty} G_n(z;\hat{u}) = \begin{cases} 
1/z^{2n+2} & \text{for } n \text{ odd} \\
1/z^{2n+4} & \text{for } n \geq 2 \text{ even}
\end{cases}$$

(7.92)

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Some further relations are imposed by the supersymmetry of the model. The simplest one follows from substituting the expansion (7.82) into (7.77) which leads to
\[
z^2 \sum_{n=0}^{\infty} (-1)^n \langle \Omega | G_n(z; \hat{u}) G_n(z; \hat{u}^\dagger) | \Omega \rangle = 1
\] (7.93)
where only diagonal terms contribute in (7.93) because of the definitions (7.57) and (7.58).

Finally, we evaluate the principal meander numbers \( \mathcal{M}_n \) using the above relations. Notice that (7.92) implies that a contribution to \( \mathcal{M}_n \) in (7.79) (the coefficient of \( 1/z^{2n} \)) can come at most from \( G_n \) with \( n \) odd and from \( G_{n-1} \) with \( n \) even. Substituting the expansion (7.82) into (7.79) we find
\[
\lim_{N \to \infty} \mathcal{M}_S(1/z; N) = z^2 \sum_{n=1}^{\infty} (-1)^{n-1} \langle \Omega | G_n(z; \hat{u}) v^\dagger v G_n(z; \hat{u}^\dagger) | \Omega \rangle
\]
\[
= \sum_{n=1}^{\infty} (-1)^{n-1} \langle \Omega | A_n(z; \hat{u}) v^\dagger v A_n(z; \hat{u}^\dagger) | \Omega \rangle
\] (7.94)
where in the second equality in (7.94) we have used (7.89). From this expression, we can immediately calculate the first and second principal meander numbers \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \). From (7.91) and (7.87) we have
\[
G_1(z; \hat{u}) v^\dagger = -G_0(z; \hat{u}) v G_0(z; \hat{u}) / z^2 \quad , \quad v G_1(z; \hat{u}^\dagger) = -G_0(z; \hat{u}^\dagger) v^\dagger G_0(z; \hat{u}) / z^2
\] (7.95)
so that (7.57), (7.58) and (7.83) give
\[
\lim_{N \to \infty} \mathcal{M}_S(1/z; N) = \frac{1}{z^2} \left( \sum_{k=0}^{\infty} \frac{C_k^2}{z^{2k}} \right)^2 + \sum_{n=2}^{\infty} (-1)^{n-1} \langle \Omega | G_n(z; \hat{u}) v^\dagger v G_n(z; \hat{u}^\dagger) | \Omega \rangle
\] (7.96)
The first term on the right-hand side of (7.96) (the \( n = 1 \) term in (7.94)) yields the anticipated values \( \mathcal{M}_1 = C_0^2 = 1 \), \( \mathcal{M}_2 = 2C_0^2 C_1^2 = 2 \) [39, 99]. The contributions of the next orders are controlled by the asymptotic behaviours (7.92) above.

For the terms in (7.94) up to \( n = 3 \), at most the expansion coefficients \( G_3 \) are essential, and (7.91) along with (7.83) give
\[
G_2(z; \hat{u}) v^\dagger = (vw^2 u - uv^2 w) / z^8 + \mathcal{O}(1/z^{10})
\]
\[
v G_2(z; \hat{u}^\dagger) = (-v^\dagger u^\dagger v^2 u + u^\dagger (v^2 u^\dagger v) / z^8 + \mathcal{O}(1/z^{10})
\]
\[
G_3(z; \hat{u}) v^\dagger = v^6 / z^8 + \mathcal{O}(1/z^{10}) \quad , \quad v G_3(z; \hat{u}^\dagger) = (v^\dagger)^6 / z^8 + \mathcal{O}(1/z^{10})
\] (7.97)
Substituting (7.97) into (7.96) yields the expected third principal meander number \( \mathcal{M}_3 = 8 \) [39, 99]. In general, however, the iterative procedure described above becomes rather cumbersome when trying to evaluate the vacuum expectation values in (7.96) which involve \( C_n^2 \) terms for each \( n > 1 \). While each individual term is in principle calculable, it is not as such clear if this iterative procedure could lead to recurrence relations for the meander numbers. The main difference between the non-commutative loop equations for the supersymmetric model above and those for the two-matrix model with polynomial potential [52] is that the \( v \)-expansion in the supersymmetric case does not lead to an algebraic equation determining the coefficients.
Thus, just as in the pure fermionic cases, the supersymmetric models aren’t as amenable to explicit solution as the pure bosonic cases because of the complexity of the loop equations involving fermionic degrees of freedom. As always, this complexity can lead to novel critical phenomena so that the existence of fermionic inducing fields leads to exotic types of random surface models.

### 7.5 $D = 0$ Supersymmetric Matrix Models and Branched Polymers

In this Subsection we shall elaborate on the uses of $D = 0$ supersymmetric matrix models to represent the branched polymer phase of string theory, following the approach of Ambjørn, Makeenko and Zarembo [13]. We consider a more general interaction potential defined by

$$\mathcal{W}(\bar{W} W) = \sum_{k \geq 1} \frac{g_k}{k} (\bar{W} W)^k \quad (7.98)$$

The invariance of (7.98) under the matrix supersymmetry transformations (7.50) follows from

$$\delta_{\epsilon}(\bar{W} W) = \delta_{\epsilon}(\bar{\phi}^i \phi + \bar{\psi} \psi) = \bar{\psi} e \bar{\phi} - \bar{\psi} e \phi = 0 \quad (7.99)$$

and similarly for the action of $\delta_{\epsilon}$. The supersymmetry transformations can be generated using the matrix supercharges

$$Q_{ij} = N \sum_{k=1}^{N} \left( \psi_{ik} \frac{\partial}{\partial \phi_{jk}} - \frac{\partial}{\partial \psi_{ki}} \bar{\phi}^i_{kj} \right), \quad Q_{ij} = \bar{N} \sum_{k=1}^{N} \left( \frac{\partial}{\partial \bar{\phi}^i_{ki}} \bar{\psi}_{kj} - \frac{\partial}{\partial \bar{\psi}_{ki}} \bar{\phi}^i_{kj} \right) \quad (7.100)$$

whose action on matrix fields $\Psi$ of the model give the infinitesimal supersymmetry variations,

$$\delta_{\epsilon} \Psi = [ \text{tr } Q_{ij}, \Psi ], \quad \delta_{\epsilon} \Psi = [ \text{tr } \epsilon Q_{ij}, \Psi ] \quad (7.101)$$

The anti-commutation relations between these supercharges are

$$\{ Q_{ij}, Q_{mn} \} = \{ \bar{Q}_{ij}, \bar{Q}_{mn} \} = 0 \quad (7.102)$$

$$\{ Q_{ij}, \bar{Q}_{mn} \} = -N \delta_{mn} \sum_{k=1}^{N} \left( \bar{\phi}_{mk} \frac{\partial}{\partial \phi_{jk}} + \frac{\partial}{\partial \bar{\phi}^i_{km}} \bar{\phi}^i_{kj} \right) - N \sum_{k=1}^{N} \left( \psi_{ik} \frac{\partial}{\partial \psi_{nk}} - \frac{\partial}{\partial \bar{\psi}_{ki}} \bar{\psi}_{kn} \right) \delta_{mj} \quad (7.103)$$

We now set $g_1 = -1$ in (7.98) and consider the supersymmetric matrix model with partition function

$$Z_S[g] = \int dW \ d\bar{W} \ e^{N^2 \text{ tr } \mathcal{W}(\bar{W} W)} \quad (7.104)$$

We shall be interested in the generating function for the correlators of the complex, bosonic matrices,

$$\Xi(z) = \left\langle \text{tr} \frac{1}{z - \phi^i} \right\rangle = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{zn+1} \xi_n \quad (7.105)$$

where the bosonic moments are

$$\xi_n = \left\langle \text{tr} (\bar{\phi}^i \phi)^n \right\rangle \quad (7.106)$$

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The Schwinger-Dyson equation for this generating function follows from the identity

\[
\frac{1}{N^2} \sum_{i,j} \int dW \, d\bar{W} \frac{\partial}{\partial \phi_{ij}} \left( e^{N^2 \text{tr} \, \mathcal{W}(\bar{W}W)} \frac{1}{z - \phi^\dagger \phi} \right)_{ij} = 0
\]  

(7.107)

which can be expanded out into averages to give

\[
\left\langle \text{tr} \, \mathcal{W}'(\bar{W}W) \frac{\phi^\dagger \phi}{z - \phi^\dagger \phi} \right\rangle = z \left\langle \left( \frac{1}{z - \phi^\dagger \phi} \right)^2 \right\rangle
\]  

(7.108)

In contrast to the loop equations derived previously, the equation (7.108) can actually be expressed in a closed form by exploiting the invariance of the action and integration measure in (7.104) under the supersymmetry transformations (7.50). This leads to the Ward identity

\[
\left\langle \delta_v \left( \psi - \frac{1}{w - \bar{W}W} \frac{1}{z - \phi^\dagger \phi} \right) \right\rangle = 0
\]  

(7.109)

which, after the proper contraction of matrix indices, can be expanded to give

\[
\left\langle \text{tr} \, \frac{\bar{W}W}{w - \bar{W}W} \frac{1}{z - \phi^\dagger \phi} \left( 1 + \text{tr} \, \frac{\phi^\dagger \phi}{z - \phi^\dagger \phi} \right) \right\rangle = \left\langle \text{tr} \, \frac{1}{w - \bar{W}W} \frac{\phi^\dagger \phi}{z - \phi^\dagger \phi} \right\rangle
\]  

(7.110)

As mentioned before, the supersymmetry leads to a cancellation of bosonic and fermionic loops, so that the partition function is unity and all supersymmetric correlators vanish. This can be formally proved by taking the limit \( z \to \infty \) in (7.110) and comparing the \( \mathcal{O}(1/z) \) coefficients on both sides to give

\[
\left\langle \text{tr} \, (\bar{W}W)^n \right\rangle = 0
\]  

(7.111)

for all \( n \geq 1 \). Since \( \left\langle \text{tr} \, (\bar{W}W)^n \right\rangle = \frac{1}{N^2} \frac{\partial}{\partial \mu} Z_{\mathcal{F}}[\tilde{g}] \), the partition function is independent of the potential \( \mathcal{W} \) and is formally equal to 1. However, non-trivial physical characteristics of the model reside in non-supersymmetric correlators, such as those of the bosonic matrices (7.106).

In the large-\( N \) limit, when factorization holds, the equations of motion (7.108) and (7.110) of the supersymmetric matrix model can be written succinctly as the respective integral equations

\[
\int_C \frac{d\mu}{2\pi i} \int_C \frac{d\lambda}{2\pi i} \frac{\mu \mathcal{W}'(\mu) \mathcal{Q}(\mu, \lambda)}{z - \lambda} = z \Xi(z)^2
\]  

(7.112)

\[
\int_C \frac{d\mu}{2\pi i} \frac{\mu \mathcal{Q}(\mu, z)}{w - \mu} \left( 1 + \int_C \frac{d\lambda}{2\pi i} \frac{\lambda \Xi(\lambda)}{z - \lambda} \right) = \int_C \frac{d\mu}{2\pi i} \frac{\mu \mathcal{Q}(w, \mu)}{z - \mu} \int_C \frac{d\lambda}{2\pi i} \frac{\lambda \Xi(\lambda)}{z - \lambda}
\]  

(7.113)

where we have introduced the additional generating function

\[
\mathcal{Q}(w, z) = \left\langle \text{tr} \, \frac{1}{w - \bar{W}W} \frac{1}{z - \phi^\dagger \phi} \right\rangle
\]  

(7.114)

and the closed contour \( \mathcal{C} \) above encircles all singularities of \( \mathcal{Q} \) and \( \Xi \), but not the points \( z, w \) or \( \infty \), with counterclockwise orientation. Using the usual asymptotic behaviours

\[
\mathcal{Q}(w, z) = 1/wz + \mathcal{O}(1/z^2) = \Xi(z)/w + \mathcal{O}(1/w^2) \quad , \quad \Xi(z) = 1/z + \mathcal{O}(1/z^2)
\]  

(7.115)
the contour integrations in (7.113) can be evaluated by computing the residues at \( z, w \) and \( \infty \) to yield, after some algebra, an equation determining the generating function \( Q \) in terms of \( \Xi \),

\[
Q(w, z) = \frac{w z \Xi(z)^2 - z \Xi(z) + 1}{w z (w \Xi(z) - z \Xi(z) + 1)} \quad (7.116)
\]

We now substitute (7.116) into the other integral equation (7.112) to generate an equation for the generating function \( \Xi \). The contour integrals over \( \mu \) and \( \lambda \) can be carried out by first computing the residues of the poles of \( Q(\mu, \lambda) \) in (7.116) as a function of \( \mu \), and then computing the residues at \( \lambda = z \) and \( \lambda = \infty \). After some algebra, we arrive finally at

\[
(z \Xi(z) - 1)W'(z - 1/\Xi(z)) = z \Xi(z)^2 \quad (7.117)
\]

Using the asymptotic expansion (7.105), the expression (7.117) can be equated order by order on both sides in \( 1/z \), leading to various mixed equations for the bosonic correlators (7.106). In particular, equating the leading-order \( 1/z \) coefficients of (7.117) leads to the expression

\[
\xi_1 W'(\xi_1) = 1 \quad (7.118)
\]

for the correlator \( \xi_1 = \langle \text{tr} \, \bar{\phi}^i \phi^j \rangle \). We immediately identify (7.118) with the saddle-point equation (2.109) (see also (2.45)) describing the continuum limit of a branched polymer model, that we derived from the vector model representation. The fact that we arrive at a closed equation for the bosonic propagator \( \xi_1 \) is a consequence of the cancellations between the bosonic and fermionic loops. The Feynman graphs which survive the supersymmetric cancellation in the diagrammatic expansion of the propagator are the so-called “cactus diagrams” [13] which consist of a closed loop of links with one marked vertex. These Feynman diagrams have an orientation, since the cactus loops can only be attached to the exterior of already existing loops, and thus the bosonic correlators of the supersymmetric matrix model here describe “chiral” branched polymers, which branch out only at one side of an open line. This is one of the fundamental differences between the supersymmetric matrix models and the representation of branched polymers via cactus graphs of the vector models that were studied in Section 2.

Thus in this way, the bosonic propagator generates the random polymer model

\[
\xi_1 = \sum_{\bar{P}} w(\bar{P}) e^{-\Lambda L(\bar{P})} \quad (7.119)
\]

where the sum is over all chiral branched polymers \( \bar{P} \) and

\[
w(\bar{P}) = \prod_{v \in \bar{P}} g(v) = \prod_{k \geq 2} (-g_k) \quad (7.120)
\]

are the branching weights, with the local weight factors \( g(v) = -g_k \) depending only on the order \( k \) (number of nearest neighbours) of the vertex \( v \). The supersymmetric matrix model representation of the branched polymer theory allows a detailed study of its spectral properties [13], specifically the determination of the universality classes of pure branched polymers as a function of the weight attributed to the branching at the individual nodes. For instance, the discontinuity of the generating function \( \Xi(z) \) above determines the eigenvalue distribution of the Hermitian matrix \( \bar{\phi}^i \phi^j \), and therefore the spectrum of the proper statistical model. It is also possible to study multi-matrix versions of the above supersymmetric model involving more than one matrix superfield \( W \). For example, a 2-matrix version of the above model was solved in [13] and shown to reproduce the well-known features of the Ising model on a branched polymer [16].
We have considered thus far in this Section relatively simple supersymmetric matrix models. Nonetheless, these models result in more complicated structures than the general $D$-dimensional pure fermionic or pure bosonic matrix models. The combinatorial problem of enumerating meander numbers that they describe belongs to the same generic class of problems of words as large-$N$ multi-colour QCD but is presumably simpler. It would be interesting to study these novel supersymmetric matrix models in connection with other physical applications to quantum gravity, other than those represented by a random polymer phase. For instance, the connection between supersymmetric matrix models and the meander problem could be useful for analysing statistical theories of discretized super-Riemann surfaces and superstrings. We conclude this Section with a brief discussion about the possibility of such applications.

In the above models we considered supersymmetric matrix fields on a $D = 0$ dimensional embedding space, whereby the bosonic and fermionic matrix propagators coincide. This means essentially that we don’t need introduce the usual (dimensionless) superspace coordinates $\theta$ which are canonically associated with the superspace formulation of supersymmetric field theories. As we have mentioned before, the first attempt at incorporating fermionic degrees of freedom into bosonic matrix theories was the Gilbert-Perry Hermitian supermatrix model [2, 60, 136]. However, the (Hermitian) fermion matrix fields can be integrated out and the model in this case reduces to an ordinary bosonic Hermitian one-matrix model (which is trivial when the number of bosonic and fermionic degrees of freedom are the same). Here we shall discuss a slight modification of the above supersymmetric matrix models. We assume that our matrices are superfields defined on a supersurface associated with an $N = 1$ supersymmetry (the generalization to higher-component supersymmetries is immediate). This means that we augment the zero-dimensional space of the matrix model by 2 anticommuting superspace coordinates. Using the usual interpretation of Grassmann coordinates as negative dimensions, we can think of such a model as a $D = -2$ dimensional matrix model. The planar triangulations of closed surfaces in $D = -2$ dimensions was studied by Kazakov, Kostov and Migdal in [76], and the matrix model approach was developed in [34, 81, 89].

The supersymmetric matrix model we consider here is thus defined by the partition function

$$Z_{MP} = \int d\Phi \ e^{-N^2\Sigma_{WZ}[\Phi]}$$

(7.121)

where the “action” in (7.121) is that of the standard Wess-Zumino model [132] in the superspace formulation

$$\Sigma_{WZ}[\Phi] = \int d\theta \ d\bar{\theta} \ \text{tr} \ (\overline{D}\Phi D\Phi + W(\Phi))$$

(7.122)

with $W(\Phi)$ some superpotential. Here

$$\Phi(\theta, \bar{\theta}) = \phi + \psi \bar{\theta} + \bar{\psi} \theta + F \bar{\theta}$$

(7.123)

are the usual supermatrix fields defined on a purely Grassmannian supersurface, with $\phi$ a bosonic $N \times N$ Hermitian matrix, and $\psi$ and $\bar{\psi}$ independent fermionic $N \times N$ Grassmann matrices. The bosonic $N \times N$ Hermitian matrix $F$ is an external field, $\theta$ and $\bar{\theta}$ are independent, anti-commuting Grassmann coordinates and the covariant derivatives on the supersurface are

$$D = \frac{\partial}{\partial \theta}, \quad \overline{D} = \frac{\partial}{\partial \bar{\theta}}$$

(7.124)
which generate the infinitesimal supersymmetry transformations. The integration measure in (7.121) is

\[ d\Phi = d\phi \, d\psi \, d\tilde{\psi} \, dF \]

(7.125)

and, as always, the complex-conjugation convention for the Grassmann numbers implies that \( \bar{D}^* = -D \).

The statistical model above is the \( D = 0 \) direct simplification of the Marinari-Parisi model [103] which is the standard form for a quantum field theory with Poincaré supersymmetry in a superspace formulation. It describes super-triangulations of super-surfaces with weights that are reparametrization invariant in parameter space and supersymmetry invariant in target space. More precisely, the weights here associate to each triangulation a factor proportional to the free propagator of the superfield (7.123) (the product of functions of the relative super-distances of all contiguous triangles which are therefore invariant under super-rotations) and the supersymmetric string theory (in a non-existent embedding space) is then described by summing over all possible topologies of these triangulations and integrating the corresponding weights in the superspace. For non-Gaussian super-potentials in (7.122), it can be shown [103] that there are critical points where the correlation length becomes much larger than the fundamental length of the discretization of the superstring theory. This can be worked out by first performing the Grassmann integrals over \( \theta \) and \( \tilde{\theta} \) in (7.122) to get

\[ \Sigma_{WZ}[\Phi] = \text{tr} \left( -F^2 + W'(\phi) F + \tilde{\psi} W''(\phi) \psi + \tilde{\psi} \psi W'(\phi) \right) \]

(7.126)

The integration over the auxiliary field \( F \) is thus Gaussian and can be carried out explicitly. Then the partition function (7.121) becomes [103]

\[ Z_{MP} = \int d\phi \, d\psi \, d\tilde{\psi} \, \exp \left\{ -\frac{N^2}{2} \text{tr} \left( \frac{\partial W(\phi)}{\partial \phi^*} \frac{\partial W(\phi)}{\partial \phi} - \tilde{\psi} \frac{\partial^2 W(\phi)}{\partial \phi^2} \psi - \tilde{\psi} \psi \frac{\partial^2 W(\phi)}{\partial \phi^2} \right) \right\} \]

(7.127)

Notice that when the superpotential \( W(\Phi) \) is quadratic in \( \Phi \), the action in (7.127) reduces to that considered for the meander problem above. More complicated, non-Gaussian superpotentials lead to a more complicated supersymmetry among the bosonic and fermionic components which results in a critical behaviour for the statistical system described above. In the above superspace formulation in terms of superfields, the model is readily generalized to higher-dimensional matrix fields. The matrix model (7.127) is invariant under the infinitesimal supersymmetry transformations

\[ \delta \phi = e \psi + \tilde{\psi} e \quad , \quad \delta \psi = -\frac{1}{2} e W'(\phi) \quad , \quad \delta \tilde{\psi} = -\frac{1}{2} e W'(\phi) \]

(7.128)

Again, the supersymmetric matrix model is trivial, in that the partition function is independent of the form of the superpotential \( W \). In the present case this follows from using the invariance of the model (7.127) under the simultaneous \( U(N) \) transformations \( \phi \rightarrow U \phi U^\dagger \), \( \psi \rightarrow U \psi U^\dagger \) and \( \tilde{\psi} \rightarrow U \tilde{\psi} U^\dagger \) to write the partition function in terms of the eigenvalues of the Hermitian matrix \( \phi \), and then integrating out the fermion fields [81]. From this it follows from differentiating the partition function with respect to a set of couplings \( g_k \) in the usual way that any integrated supersymmetric correlator vanishes,

\[ \left\langle \int d\theta \, d\tilde{\theta} \, \text{tr} \, \Phi^n \right\rangle = 0 \]

(7.129)

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Thus the supersymmetric matrix model free energy itself exhibits no critical behaviour. But a critical model is obtained as before by examining the correlators of the Hermitian matrix field $\phi$. The result of integrating out the fermions above shows that these correlators are determined as the Hermitian Gaussian correlators of the inverse of the map $\phi \to W(\phi)$ (i.e. the Nicolai map),

$$\langle \text{tr } \phi^n \rangle = \left\langle \text{tr } [W^{-1}(\phi)]^k \right\rangle \quad (7.130)$$

Despite the triviality of the supersymmetric partition function, such correlation function do display non-trivial critical behaviour. The diagrammatics of the matrix theory for such correlation functions can be written down and shown to coincide the sort of super-discretizations mentioned earlier. Furthermore, the (planar) scaling limit reproduces the correct string susceptibility exponent of the Liouville theory prediction for (non-unitary) $D = -2$ matter fields coupled to two-dimensional quantum gravity [81, 89].

As mentioned before, the supersymmetry of a matrix model always leads to a dimensional reduction, so that the supersymmetry of the matrix theory does not coincide with the supersymmetry of the supergravity or superstring theory [2, 30]. This fact was originally exploited by Marinari and Parisi [103] who considered time-dependent matrix fields above (so that the partition function describes a supersymmetric matrix quantum mechanics). The Hamiltonian which follows from quantizing this one-dimensional system then has the standard form of a Witten-type supersymmetric quantum mechanics [134] (in our $D = 0$ case above, the Hamiltonian vanishes and the supersymmetry charges generating the infinitesimal supersymmetry transformations of the theory coincide with the covariant derivatives (7.124)). The dimensional reduction mechanism in this $D = 1$ model comes from the stochastic quantization method associated with regarding the bosonic sector Hamiltonian as the forward Fokker-Planck Hamiltonian in the Langevin time-evolution equation [38]. However, as for the pure adjoint fermion matrix models, the supersymmetric theory is always well-defined and no ambiguities occur, in contrast to the standard $D = 0$ Hermitian matrix models which exhibit non-perturbative ambiguities, instabilities and violations of the Schwinger-Dyson equations [36, 115] (as discussed previously). For $N \to \infty$, the expectation values in the supersymmetric matrix model coincide with those of the ill-defined $D = 0, m = 2$ bosonic theory, and moreover the genus expansion is completely reproduced. Thus the dimensionally reduced Fokker-Planck supersymmetry can be taken as an alternative definition of the $D = 0$ bosonic matrix theory.

Thus, as with the adjoint fermion models that we have extensively studied throughout this Review, supersymmetric matrix models define better behaved random surface theories than the conventional bosonic matrix models and reproduce features similar to these cases. Besides this feature, it would be interesting to develop these supersymmetric models to describe non-critical superstring theories and models of dynamical supersymmetry breaking. The $D = 1$ Marinari-Parisi model provides a good example of the latter mechanism which is associated with the phase transition and the double-scaling limit, and which arises from the dimensional reduction mechanism discussed above, i.e. the supersymmetry of the matrix model is broken leaving an effectively non-supersymmetric model.

However, the main obstacle in arriving at a matrix model representation of supergravity and superstrings is precisely this dimensional reduction. The critical string susceptibility index for a superstring embedded in a $D$-dimensional space as calculated from super-Liouville
theory \cite{123} is

$$\gamma_{\text{str}}^{(s)} = \frac{1}{4} \left( D - 1 - \sqrt{(1 - D)(9 - D)} \right)$$

(7.131)

It is not known at this stage how to relate superstring theories to supersymmetric matrix models and therefore to geometrical descriptions of discretized super-surfaces. The only successful discrete approach to date to 2-dimensional supergravity coupled to minimal superconformal models are the super-eigenvalue models introduced in \cite{3}. These models reproduce the super-Virasoro algebra associated with the Neveu-Schwarz sector of the superstring theory. The idea is to use the prominent role played by the Virasoro constraints of the usual Hermitian matrix models to construct a partition function for \(N\) Grassmann even and odd variables (the “super-eigenvalues”) which obeys a set of super-Virasoro constraints. To formulate these constraints, one must introduce, in addition to the usual coupling constants \(g_k\), a set of Grassmann-valued couplings \(\xi_{k+1/2}\). Many of the well-known features of matrix models, such as the genus expansion, loop equations and loop insertion operators, have supersymmetric counterparts in this formalism. The problem of solving a super-eigenvalue model can be reformulated as a set of superloop equations obeyed by superloop correlators. The double-scaling limit was further studied in \cite{4}, and the complete iterative solution based on the moment technique of \cite{14} (see Subsection 3.4) was carried out by Plefka in \cite{120, 121}. For the connections between \(N = 1\) super-Liouville amplitudes and super-eigenvalue correlators, see \cite{137}.

There is, unfortunately, no known connection between the super-eigenvalue model and any type of super-matrix model. The idea to determine whether or not a super-Virasoro algebra can be realized in a super-matrix model, in terms of a set of differential operators in the coupling constants of a general matrix potential, is to construct matrix generators analogous to the matrix super-charges \((7.100)\) which generate, as \(N \to \infty\), the super-Virasoro algebra associated with the Ramond sector of superstring theory in a \(D = 0\) dimensional target space. This problem has been discussed somewhat by Makeenko in \cite{98}. In order for such constructions to work, though, the symmetry of the matrix model should be reduced by modifying the potential so that a larger class of Feynman graphs (beyond the tree-like, cactus diagrams) survives the perturbative expansion.

8 Conclusions

We have shown that a novel class of matrix models with fermionic degrees of freedom are formally solvable by the method of loop equations, even though they do not admit the standard Riemann-Hilbert problem \cite{25, 109, 111} which arises for the statistical distribution of eigenvalues in the more conventional matrix models. The solutions of the loop equations for fermionic matrix models are completely analogous to those of Hermitian and complex matrix models \cite{14}, and they coincide at each order of the \(\frac{1}{N}\)-expansion with those of a Hermitian matrix model with a generalized Penner interaction. However, the observables in the fermionic case are always well-defined and convergent quantities, contrary to Hermitian matrix models, and they may have an interesting interpretation as dynamically triangulated theories of random surfaces. A price to pay for this good convergence is the larger degree of complexity of the equations which completely determine the set of correlators of these models. This complexity leads to a more complicated phase structure of the models and results in a critical behaviour which is non-characteristic of the usual matrix models. It would be interesting to give the
fermionic nature of these matrix models an interpretation in terms of worldsheet discretizations in string theory. One step along these lines could involve examining the continuum limit of the fermionic models in relation to their novel Virasoro and $W$-algebra constraints to determine precisely what the double-scaling limit continuum theory is [92].

The $D > 0$ dimensional models can only be explicitly solved for Gaussian potentials, since higher degree potentials lead to large degree algebraic equations and even when their solutions are explicitly known (e.g. for a Penner-type potential) they are rather obscure and are not at all informative. It would be interesting to develop other methods to solve these models, such as an explicit formula for the Itzykson-Zuber integral (4.26) with Grassmann-valued matrices, in order to explore the critical behaviour associated with higher order potentials in these cases. This critical behaviour might provide new insights into quantum gravity in 2 and higher dimensions, and especially for the $D > 1$ dimensional models which might induce QCD. In these latter models the loop equations are even more complex since one doesn’t know right away what potential the ansatz (6.41) is associated with. Moreover, the extended loop correlators such as (6.6) can only be found from a separate set of Schwinger-Dyson equations for the extended objects. Contrary to the Kazakov-Migdal model, however, these higher-dimensional models do have a first order phase transition in the Gaussian case which is associated with the restoration of the area law. Furthermore, the fermionic nature of the inducing fields in these cases makes the adjoint fermion matrix model resemble ordinary QCD in many respects. It would be interesting though to develop some other formalism, such as a master field formulation [93, 95, 109, 111], for solving these models in the weak coupling phase where the dynamics of extended objects are non-trivial.

Insights into the nature of the solutions of the loop equations for these higher dimensional fermionic matrix models could also come from further investigations of the supersymmetric theories. It would be interesting to generalize these reduced matrix theories to other combinatorial problems, especially those relevant to superstring theory. As always the problem is the dimensional reduction which appears in these models, so that the supersymmetry of the matrix model does not represent the same supersymmetry of the superstring theory [2, 30]. However, a supersymmetric model in $D > 0$ dimensions could be relevant to a statistical theory of discretized super-Riemann surfaces. As in the case of the pure matrix theories, modified supersymmetric vector models could provide huge insights into these problems.

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