HARTREE-FOCK-BOGOLUBOV APPROXIMATION IN THE MODELS WITH GENERAL FOUR-FERMION INTERACTION

N.N. Bogolubov (Jr.)
and
A.V. Soldatov

MIRAMARE-TRIESTE
In this article the following Hamiltonian is considered:

$$H = \sum_{\langle ff' \rangle} \Omega_{\delta}(f, f') a_{f'}^+ a_f + \frac{1}{2} \sum_{\langle jfj'I \rangle} U(f_j f_{j'} f_{j'I} f'_{j'I'}) a_{j'}^+ a_j a_{j'I} a_{j'I'} ,$$

(1)

where $a_{f'}^+, a_f$ are Fermi operators of creation and annihilation obeying anticommutative relations

$$[a_f, a_{f'}^+] = a_f a_{f'}^+ + a_{f'}^+ a_f = \delta_{ff'} ,$$

$$[a_f, a_{f'}] = a_f a_{f'} + a_{f'} a_f = 0 ,$$

and

$$\delta_{ff'} = \delta(\vec{r} - \vec{r'}) \delta(s - s')$$

is interpreted as a Kronecker delta for discrete arguments or as a Dirac delta function for continuous arguments. $f = (\vec{r}, s)$ is the set of four quantum numbers (i.e. momentum components and spin projection) determining the state of free fermion:

$$\vec{r} = \left( \frac{2\pi n_1}{L}, \frac{2\pi n_2}{L}, \frac{2\pi n_3}{L} \right) , \quad s = \pm \frac{1}{2} ,$$

where $n_1, n_2, n_3$ are integers and $L^3 = V$ where $V$ is the volume of the system. It is supposed that functions $\Omega_{\delta}(f, f')$ and $U(f_j f_{j'} f_{j'I} f'_{j'I'})$ are chosen to guarantee the fulfillment of laws of conservation for the number of particles

$$[H, N] = 0 ,$$

(2)

total momentum

$$[H, P] = 0 ,$$

(3)

and total spin projection

$$[H, S_z] = 0 ,$$

(4)

where $N, P, S_z$ are operators of correspondent dynamical variables. It is assumed that

$$U(f_j f_{j'} f_{j'I} f'_{j'I'}) = U(f_{j'I} f_{j'} f_{j' I} f'_{j'I'}) .$$

Because of technical reasons we consider also more general Hamiltonian with sources:

$$H_t = \sum_{\langle ff' \rangle} \Omega(f, f', t) a_{f'}^+ a_f + \frac{1}{2} \sum_{\langle jfj'I \rangle} U(f_j f_{j'} f_{j'I} f'_{j'I'}) a_{j'}^+ a_j a_{j'I} a_{j'I'} +$$

$$+ \frac{1}{2} \sum_{\langle ff' \rangle} (\rho_{ff'}(f, f', t) a_f a_{f'} + \rho_{f'-f}(f', f) a_{f'} a_f) +$$

$$+ \frac{1}{2} \sum_{\langle ff' \rangle} (\eta_{ff'}(f, f', t) a_f a_{f'} + \eta_{f'-f}(f, f) a_{f'} a_f) +$$

$$\sum_{\langle jf \rangle} (\eta_{jf}(f, t) a_f + \eta_{j-f}(f, t) a_{f'}^+) ,$$

where

$$\Omega(f, f', t) = \Omega_0(f, f') + j(f, f', t) .$$
Sources \( j^{(m)}(f^t, f) \) are independent on time \( t \). It is assumed that the following sources are antisymmetric:

\[
  j_+(f^t, f, t) = -j_-(f^t, f, t); \quad j_-(f^t, f, t) = -j_+(f^t, f, t); \quad j^{(m)}(f^t, f) = -j^{(m)}(f^t, f) \, .
\]

In general, sources violate symmetries of the model (1) induced by conservation laws (2-4). The most important role will be played by the sources (5) which violate symmetries in the case of thermodynamic equilibrium implementing the concept of quasiequilibrium for the model (1).

Particular example of the model (1) is the generalized BCS model investigated in \( V \):

\[
  H_{\text{BCS}} = \sum_{\langle j f \rangle} \Omega(j^t, f) a_j^t a_f^t - \frac{1}{2V} \sum_{\langle j f \rangle} \lambda(f) \lambda(f^t) a_j^t a_f^t a_j^t a_f^t \, ,
\]

where \( \Omega(j^t, f) = \delta(f - f^t) T(p); \, T(p) = \frac{p^2}{2m} - \mu; \, \mu = |\mu| \) and \( \mu \) is chemical potential, \( \mu > 0 \).

It is assumed that

\[
  \lambda(f) = \text{sign}(s) |I(p)| \, ,
\]

where

\[
  \text{sign}(s) = \begin{cases} 1 & \text{if } s = \frac{1}{2} \\ -1 & \text{if } s = -\frac{1}{2} \end{cases}.
\]

It was shown in [1,3,4] that the model (6) allows asymptotically exact solutions under the limit \( V \rightarrow \infty \) and all important thermodynamic properties of the model (6) coincide under this limit with those of the reduced exactly solvable model that is called the approximating Hamiltonian. This Hamiltonian is constructed in accordance with some special rules outlined in [3,4].

Let us explain briefly the scheme of the approximating Hamiltonian method on the basis of the model (6). Let us fulfill the identity transformation of the Hamiltonian (6):

\[
  H_{\text{BCS}} = H_0(c^*, c) + H_{\text{int}}(c^*, c) \, ,
\]

where

\[
  H_0(c^*, c) = \sum_{\langle j f \rangle} \Omega(j^t, f) a_j^t a_f^t - \frac{1}{2V} \sum_{\langle j f \rangle} \lambda(f) \left( c a_{-j} a_f + c^* a_j^t a_f^t \right) + \frac{\mu^2}{2} V \, ,
\]

\[
  H_{\text{int}}(c^*, c) = -\frac{1}{2} \sum_{\langle j f \rangle} \left( \frac{\lambda(f) a_j^t a_f^t}{V} - c \right) \left( \frac{\lambda(f^t) a_{-j} a_f}{V} - c \right) \, ,
\]

and \( c \) is an arbitrary complex parameter for the present. Later the parameter \( c \) is determined by the extreme condition (absolute minimum) for the density of the free energy

\[
  \frac{\partial F(H_0(c^*, c))}{\partial c} = 0 \, ,
\]

\[
  \frac{\partial F(H_0(c^*, c))}{\partial c^*} = 0 \, ,
\]

where

\[
  F(H_0) = -\frac{1}{2V} T \sum \left( \lambda(f) a_{-j} a_f^t + a_j^t a_f \right) e^{-\beta H_0} \, ,
\]

is finite as was shown in [3,4]. Eqs. (10-11) lead to the equation of self-consistency for the parameter \( c \):

\[
  c = \frac{1}{V} \sum_{\langle j f \rangle} \lambda(f) a_{-j} a_f^t \, ,
\]

where \( a_{-j} a_f^t \) are calculated with respect to the approximating Hamiltonian (8). The modified interaction (9) does not influence on the thermodynamic properties of the model (6) under the limit \( V \rightarrow \infty \) if parameters \( c^*, c \) are chosen in a proper way (10-11). This circumstance confirms the assumption that under the limit \( V \rightarrow \infty \) operator

\[
  \sum_{\langle j f \rangle} \lambda(f) a_{-j} a_f^t
\]

can be interpreted rather as a number then as an operator. So one neglects interaction from the model (6) and substitutes successfully (8) instead of the initial Hamiltonian (6) to calculate all thermodynamic properties and equilibrium two-temperature Green functions of all orders for the model (6).

The procedure outlined above of the approximating Hamiltonian derivation is valid only for the models with separable interaction of the type (6). The aim of this study is to construct effective reasonable method of approximation for general model (1). As a whole this new method might suffer a lack of mathematical rigor especially in comparison with the standard approximating Hamiltonian method [3,4]. So the following consideration is marked with necessary formal character sometimes. At the same time it is natural to demand the coincidence of the results (at least under the limit \( V \rightarrow \infty \)) obtained by the two methods for models with separable interaction.

The first step of approximation for the model (1) comprises the following formal substitution for the product of four Fermi operators in the interaction term in the Hamiltonian (1):

\[
  a_{j_1}^t a_{j_2} a_{j_3} a_{j_4}^t \rightarrow (a_{j_1}^t a_{j_2}^t) (a_{j_3} a_{j_4}) - (a_{j_1}^t a_{j_3}) (a_{j_2}^t a_{j_4}) - (a_{j_1} a_{j_2}^t) (a_{j_3}^t a_{j_4}) + (a_{j_1} a_{j_3}) (a_{j_2} a_{j_4}^t) \, ,
\]

where

\[
  R = (a_{j_1}^t a_{j_2}) (a_{j_3} a_{j_4}) + (a_{j_1} a_{j_2}^t) (a_{j_3}^t a_{j_4}) - (a_{j_1} a_{j_2}^t) (a_{j_3} a_{j_4}^t) \, .
\]

The approximation (13) yields quadratic in Fermi operators Hamiltonian

\[
  H_0 = \sum_{\langle j f \rangle} \tilde{K}(j^t, f) a_{-j} a_f^t + \frac{1}{2} \sum_{\langle j f \rangle} \tilde{K}(j^t, f) a_j a_f^t + \frac{1}{2} \sum_{\langle j f \rangle} U(j^t f ; j f) R(j^t f ; j f) \, ,
\]

\[
  + \frac{1}{2} \sum_{\langle j f \rangle} \tilde{K}(j^t, f) a_{-j} a_f^t - \frac{1}{2} \sum_{\langle j^t f ; j f \rangle} U(j^t f ; j f) R(j^t f ; j f) \, ,
\]

where

\[
  \frac{\partial F(H_0)}{\partial R} = 0 \, ,
\]

\[
  \frac{\partial F(H_0)}{\partial R^*} = 0 \, .
\]
where

\[ \hat{K}(f', f) = \Omega_0(f', f) + \sum_{(f'f)} W(f_1 f_2 f_3 f_4) a_{f_1}^* a_{f_4} \]

\[ \hat{K}_I(f', f) = \frac{1}{2} \sum_{(f'f)} W(f_1 f_2 f_3 f_4) a_{f_1}^* a_{f_4} \]

\[ \hat{K}_L(f', f) = \frac{1}{2} \sum_{(f'f)} W(f_1 f_2 f_3 f_4) a_{f_1} a_{f_4} \]

and function

\[ W(f_1 f_2 f_3 f_4) = U(f_1 f_2 f_3 f_4) - U(f_1 f_2 f_3 f_4) \]

possesses properties of antisymmetry:

\[ W(f_1 f_2 f_3 f_4) = -W(f_1 f_3 f_2 f_4) \]

\[ W(f_1 f_2 f_3 f_4) = -W(f_1 f_3 f_2 f_4) \]

renorms \((a_{f_1} a_{f_2}), (a_{f_1}^* a_{f_2}^*), (a_{f_1} a_{f_2}^*)\) can be fixed by the conditions of self-consistency by analogy with condition (12). These conditions are consequence of extreme conditions for the function of free energy for the Hamiltonian (14). Note that the approximation (13) looks like a procedure of partial calculation of equilibrium averages for the product of Fermi operators by means of the Wick theorem. The approximation (13) can be named Hartree-Fock-Bogoliubov approximation because it joins ideas ascending equally to the Bogoliubov's approximating Hamiltonian approach and to the principle of self-consistency inherited from the Hartree-Fock way of approximation [5].

In principle the Hamiltonian (14) can be diagonalized by the canonical Bogoliubov-\(u-v\) transformation for Fermi operators but in general this approach may encounter significant technical difficulties. So it seems reasonable to reformulate the problem in terms of more flexible formalism of two-time temperature dependent Green functions having in mind later possible approximate calculations. One could derive equations for the Green functions directly for the approximating Hamiltonian (14). But it seems more useful to choose more general approach and to try to construct dynamical approximation of Hartree-Fock-Bogoliubov type which allows to investigate not only equilibrium but also nonequilibrium properties of the model (1).

Consider equation of motion for some dynamical variable \(A\) independent on time explicitly providing its evolution is determined by the Hamiltonian \(H\) with time-dependent sources (we put later \(\hbar = 1\)).

\[ i \hbar \frac{d}{dt} A(t) = \{A(t), H(t)\} . \]  \hspace{1cm} (17)

Then average both sides of Eq. (17) with some time-independent density operator \(\hat{D}\):

\[ \langle A(t) \rangle = \text{Sp} \langle \hat{A}(t) \hat{D} \rangle , \]

\[ i \hbar \frac{d}{dt} \langle A(t) \rangle = \langle [A(t), H(t)] \rangle . \]  \hspace{1cm} (18)

The right side of Eq. (18) contains commutator term

\[ \langle [A(t), \frac{1}{2} \sum_{(f'f)} U(f_1 f_2 f_3 f_4) a_{f_1}^* a_{f_4} (t) a_{f_2} (t) a_{f_3} (t) a_{f_4}^* (t)] \rangle \]  \hspace{1cm} (19)

where time dependence of all operators \(a_{f_1}^* (t), a_{f_2} (t)\) is determined by the Hamiltonian \(H\) with sources. Usually dynamical variable \(A\) is a product of finite number of Fermi operators \(a_{f_1}, a_{f_2}\). So under the commutation (19) correlation functions of higher orders than the initial correlation function \(\langle A(t) \rangle\) with respect to these operators would arise in the right side of Eq. (18). To unhink the chain of evolution equations for these functions and to derive closed system of equations for some set of correlation functions one has to split correlation functions of higher orders and to express them in terms of the lower ones. We elaborate here a different approach. Instead of writing down a chain of equations for correlation functions starting from some function \(\langle A(t) \rangle\) and inventing each time unique procedure of splitting to unhink the chain we introduce once and forever some universal global approximation already in commutator term (19). The approximation ensures implicit splitting of correlation functions in each chain of equations built for a given dynamical variable \(A\) under the condition that \(A\) is a product or a sum of products of finite number of Fermi operators \(a_{f_1}, a_{f_2}\). The proposed approximation results in formal substitution:

\[ a_{f_1}^* (t) a_{f_2} (t) a_{f_3} (t) a_{f_4}^* (t) = \]

\[ \langle a_{f_1}^* (t) a_{f_2} (t) a_{f_3} (t) a_{f_4}^* (t) \rangle - \langle a_{f_1}^* (t) a_{f_2} (t) a_{f_3} (t) a_{f_4}^* (t) \rangle + \]

\[ - \langle a_{f_1}^* (t) a_{f_2} (t) a_{f_3} (t) a_{f_4}^* (t) \rangle + \langle a_{f_1}^* (t) a_{f_2} (t) a_{f_3} (t) a_{f_4}^* (t) \rangle . \]  \hspace{1cm} (20)

The approximation (20) is similar to the one (13) in the equilibrium case. All operators \(a_{f_1}, a_{f_2}\) in Eq. (20) are interpreted to be operators in the Heisenberg representation with the Hamiltonian \(H\),

\[ a_{f_1} (t) = U^{-1} (t) a_{f_1}^* (t) U(t) , \]

\[ a_{f_2} (t) = U^{-1} (t) a_{f_2} (t) U(t) , \]

\[ \langle a_{f_1} (t), a_{f_2} (t) \rangle = \text{Sp} U(t) \langle a_{f_1}^* (t), a_{f_2}^* (t) \rangle U(t) \]

and so on, where the evolution operator \(U(t)\) satisfies the equation

\[ i \hbar \frac{d}{dt} U(t) = H(t) . \]

So all averages in Eq. (20) are interpreted as exact averages and the evolution of them is determined by the Hamiltonian \(H\). Taking into account symmetry properties (15-16) the approximate evolution equation resulting after substitution of (20) to exact Eq. (18) can be written as

\[ i \hbar \frac{d}{dt} \langle A(t) \rangle = \langle [A(t), Q_{eff} (t)] \rangle , \]

where the operator form \(Q_{eff} (t)\) is defined as

\[ Q_{eff} (t) = \sum_{(f', f)} K(f', f) a_{f_1}^* (t) a_{f_2} (t) + \]

\[ + \frac{1}{2} \sum_{(f', f)} (\hat{K}_I(f', f) a_{f_1} (t) a_{f_2} (t) + \hat{K}_L(f', f) a_{f_1}^* (t) a_{f_2} (t) + \]

\[ + \sum_{(f', f)} |\mu_f |^2 (t) a_{f_1} (t) a_{f_2} (t) + \mu_\mu (f, f') a_{f_1}^* (t) a_{f_2} (t) \]  \hspace{1cm} (22)
\[ K(f', f, t) = \Omega_{0}(f, f') + j_{0}(f', f, t) + \sum_{(j, k)} W(f_{j} f_{f}, f_{k} f_{k}')(a_{j}^{*}(t) a_{k}(t)) \]

\[ K_{+}(f', f, t) = j_{1}(f', f, t) + j_{0}^{+}(f, f') + \sum_{(j, k)} U(f_{j} f_{f}, f_{k} f_{k}')(a_{j}(t) a_{k}^{*}(t)) \]

\[ K_{-}(f', f, t) = j_{-}(f', f, t) + j_{0}^{-}(f, f') + \sum_{(j, k)} U(f_{j} f_{f}, f_{k} f_{k}')(a_{j}^{*}(t) a_{k}(t)) \]

Let us stress especially that the form \( Q_{ij}(t) \) can be considered as neither approximating Hamiltonian nor, generally speaking, some effective Hamiltonian at all. This form is introduced only to make derivation of approximate equations of the type (22) for various dynamical variables \( A \) more compact and clear. It follows from Eq. (21) that starting from some dynamical variable \( A \) which is a product of finite number of Fermi operators one can derive a chain of equations closed with respect to some set of correlation functions including the initial function \( \langle A(t) \rangle \). It is guaranteed because the form (22) contains only linear and quadratic dependence on Fermi operators.

Now let us write successively the approximate equation (21) for the following combinations of Fermi operators:

\[ \frac{d}{dt}(a_{j}^{*}(t)) = -\sum_{(j')} \left\{ K(f, f', t)(a_{j'}(t)) + K_{+}(f, f', t)(a_{j'}^{*}(t)) \right\} + \sum_{(j')} j_{+}(f', f, t)(a_{j'}^{*}(t) a_{j'}(t)) + \sum_{(j')} j_{-}(f', f, t)(a_{j'}(t) a_{j'}^{*}(t)) \]

\[ \frac{d}{dt}(a_{j}(t)) = \sum_{(j')} \left\{ K(f, f', t)(a_{j'}(t)) + K_{+}(f, f', t)(a_{j'}^{*}(t)) \right\} - \sum_{(j')} j_{+}(f', f, t)(a_{j'}^{*}(t) a_{j'}(t)) + \sum_{(j')} j_{-}(f', f, t)(a_{j'}(t) a_{j'}^{*}(t)) + \eta_{+}(f, t) + \eta_{-}(f, t) \]

\[ \frac{d}{dt}(a_{j}^{*}(t) a_{j}(t)) = \sum_{(j')} \left\{ K(f, f', t)(a_{j'}(t)) + K_{+}(f, f', t)(a_{j'}^{*}(t)) \right\} - \sum_{(j')} j_{+}(f', f, t)(a_{j'}^{*}(t) a_{j'}(t)) + \sum_{(j')} j_{-}(f', f, t)(a_{j'}(t) a_{j'}^{*}(t)) + \eta_{+}(f, t) + \eta_{-}(f, t) \]

\[ \frac{d}{dt}(a_{j}(t) a_{j}^{*}(t)) = \sum_{(j')} \left\{ K(f, f', t)(a_{j'}(t)) + K_{+}(f, f', t)(a_{j'}^{*}(t)) \right\} + \sum_{(j')} j_{+}(f', f, t)(a_{j'}^{*}(t) a_{j'}(t)) + \sum_{(j')} j_{-}(f', f, t)(a_{j'}(t) a_{j'}^{*}(t)) + \eta_{+}(f, t) - \eta_{-}(f, t) a_{j}(t) a_{j}^{*}(t) \]

\[ \frac{d}{dt}(a_{j}^{*}(t) a_{j}^{*}(t)) = -\sum_{(j')} \left\{ K(f, f', t)(a_{j'}^{*}(t)) a_{j'}^{*}(t) + K_{-}(f, f', t)(a_{j'}^{*}(t)) a_{j'}^{*}(t) \right\} + \sum_{(j')} j_{-}(f', f, t)(a_{j'}^{*}(t) a_{j'}^{*}(t)) - \sum_{(j')} j_{+}(f', f, t)(a_{j'}^{*}(t) a_{j'}^{*}(t)) + \eta_{+}(f, t) a_{j}^{*}(t) a_{j}^{*}(t) - \eta_{-}(f, t) a_{j}^{*}(t) a_{j}^{*}(t) \]
\begin{align*}
K_+(f', f; 0) &= j_+^{(0)}(f', f) + \sum_{(f'; f; 0)} U(f'; f; 0)f'(a^\dagger_f a_f) \\
K_-(f', f; 0) &= j_-^{(0)}(f', f) + \sum_{(f'; f; 0)} U(f'; f) f(a_f a^\dagger_f)
\end{align*}

and \((a_f(t)a_f(\tau))\), \((a^\dagger_f(t)a^\dagger_f(\tau))\) are commutative retarded or advanced two-time temperature Green functions defined as

\begin{align*}
\langle [A(t)|B(\tau)] \rangle^{(t-r)} &= \theta(t - \tau) \langle [A(t), B(\tau)] \rangle \quad (31) \\
\langle [A(t)|B(\tau)] \rangle^{(t-r)} &= -\theta(\tau - t) \langle [A(t), B(\tau)] \rangle \quad (32)
\end{align*}

\[ \theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases} \]

The system of Eqs. (29-30) can be simplified if we take into account selecting rules for the correlation functions and Green functions. The Hamiltonian (28) is invariant under canonical transformation

\[ a_f^\dagger \rightarrow a_f, \quad a_f \rightarrow -a_f \]

So only these Green and correlation functions differ from zero which contain even number of Fermi operators. Then sources \( j_+^{(0)}(f', f), j_-^{(0)}(f', f) \) can be chosen in such a manner that only the law of particle number conservation (2) and corresponding symmetry are violated but other symmetries connected with laws of momentum and spin projection conservation (3-4) take place. It can be guaranteed if to put

\[ j_+^{(0)}(f', f) = \nu \lambda(f) \delta(f + f'), \quad j_-^{(0)}(f', f) = -\nu \lambda(f) \delta(f + f'), \]

where \( \nu \) is a formal small parameter and \( \lambda(f) \) is, for example, antisymmetric function (7). Sources (33) have been used for the introduction of quasimomentum in order to investigate the phenomenon of superconductivity in fermion system [1,2]. Conservation laws (3-4) induce the following selecting rules for correlation functions

\[ \langle a_f(t) \rangle = \langle a^\dagger_f(t) \rangle = 0 \quad (34) \]

\[ \langle a^\dagger_f(t) a^\dagger_f(t') \rangle = \langle a^\dagger_f(t) a_f(t') \rangle \delta(f + f') \quad (35) \]

\[ \langle a_f(t) a_f(t') \rangle = \langle a_f(t) a_f(t') \rangle \delta(f + f') \quad (36) \]

\[ \langle a^\dagger_f(t) a_f(t') \rangle = \delta(f - f') \langle a^\dagger_f(t) a_f(t') \rangle \quad (37) \]

The same rules take place for pair Green functions. Rules (34-37) lead to the system of equations:

\[ \frac{d}{dt} \langle [a^\dagger_f(t)|a_f(\tau)] \rangle = -\left\{ K^{(0)}(f, f; 0) \langle [a^\dagger_f(t)|a_f(\tau)] \rangle \right. \]

\[ + K^{(0)}(f, -f; 0) \langle [a_f(t)|a^\dagger_f(\tau)] \rangle \right\} + \delta(t - \tau) (2a^\dagger_f a_f - 1) \quad (38) \]

\[ \frac{d}{dt} \langle [a_f(t)|a^\dagger_f(\tau)] \rangle = \left\{ K^{(0)}(-f, -f; 0) \langle [a^\dagger_f(t)|a_f(\tau)] \rangle \right. \]

\[ + K^{(0)}(f, -f; 0) \langle [a^\dagger_f(t)|a_f(\tau)] \rangle \right\} + 2\delta(t - \tau) (a^\dagger_f a_f) \quad (39) \]

where

\[ K^{(0)}(f, f; 0) = (j_+^{(0)}(f', f)) + \sum_{(f; 0)} U(f, f; 0)f'(a^\dagger_f a_f) \]

\[ K^{(0)}(f', f) = j_+^{(0)}(f', f) + \sum_{(f'; 0)} U(f'; f, 0)f(a_f a^\dagger_f) \]

\[ K^{(0)}(f', f) = \Omega_0 (f, f) + \sum_{(f'; f; 0)} W(f', f; 0)f(a^\dagger_f a_f) \]

Equilibrium Green functions depend only on the difference of time arguments \( t - \tau \) so it is possible to implement energetic E-representation for them:

\[ \langle [A(t)|B(\tau)] \rangle^{(t-r)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle [A(t)|B(\tau)] \rangle^{(t-r)} e^{i\omega t} dt \quad (40) \]

Then

\[ E(\langle a^\dagger_f a_f \rangle) = -\left\{ K^{(0)}(f, f; 0) \langle [a^\dagger_f a_f \rangle \right) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle [A(t)|B(\tau)] \rangle^{(t-r)} e^{i\omega t} dt \quad (41) \]

\[ E(\langle a_f a_f \rangle) = \left\{ K^{(0)}(-f, -f; 0) \langle [a_f a_f \rangle \right) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle [A(t)|B(\tau)] \rangle^{(t-r)} e^{i\omega t} dt \quad (42) \]

Note that only one approximation (20) was involved to derive Eqs. (41-42). These equations contain anomalous averages \( \langle a^\dagger_f a_f \rangle \) and anomalous Green functions \( \langle [a^\dagger_f a_f \rangle \) because the Hamiltonian (28) contains sources (33) violating the conservation law (2). These anomalous averages must be interpreted as quasimoments with usual rules of handling with them [1,2]. Eqs. (41-42) contain following coefficients in the right side:

\[ \sum_{f}\langle W(f; f; f) \rangle \quad (43) \]

\[ \sum_{f} \langle U(f; f; f) \rangle \quad (44) \]

\[ \sum_{f} \langle U(f; f; f) \rangle \quad (45) \]

\[ \sum_{f} \langle U(f; f; f) \rangle \quad (46) \]

where the dependence on \( f \) is not factorable in general. So although the spectral theorem for the Green functions allows to derive formally closed system of integral equations with respect to averages

\[ \langle a^\dagger_f a_f \rangle, \quad \langle a_f a_f \rangle, \quad \langle a^\dagger_f a^\dagger_f \rangle \]

it is impossible to solve these equations as a rule. At the same time the system (41-42) gives a possibility of further approximations in coefficients (43-46) to obtain approximate solutions for the Green functions, correlation functions and spectrum of quasiparticle excitations. To understand what the character of these approximations ought to be let us
analyze traditional BCS model \((6)\) within frames of the proposed method. For the BCS model

\[
U(f_1 f_2; f_1' f_2') = -\frac{1}{T} M(f_1) M(f_2) \delta(f_1 + f_2) \delta(f_1' + f_2').
\]

\[
W(f_1 f_2; f_1' f_2') = -\frac{2}{T} M(f_1) M(f_2) \delta(f_1 + f_2) \delta(f_1' - f_2').
\]

\[
\kappa^{(0)}(f', f) = T(p) \delta(f - f') \text{ if } V \to \infty.
\]

\[
\kappa^{(0)}(f', f) = -\delta(f + f') \lambda(f') \left[\frac{1}{2} \sum_{j \in \Omega} M(f_j)(a_1^j a_2^j) + \nu\right] = -\delta(f + f') \lambda(f') \delta(\nu + \nu).
\]

\[
\kappa^{(0)}(f', f) = -\delta(f - f') \lambda(f) \left[\frac{1}{2} \sum_{j \in \Omega} M(f_j)(a_1^j + a_2^j) + \nu\right] = -\delta(f + f') \lambda(f) |e + \nu|.
\]

So Eqs. (44-42) take the form:

\[
(E + T(p))\langle a_1^j a_2^j | \nu | a_1^j a_2^j \rangle_E = \frac{i}{2\pi} (2a_1^j a_2^j + \nu).
\]

\[
(E - T(p))\langle a_1^j a_2^j | \nu | a_1^j a_2^j \rangle_E = \frac{i}{2\pi} (2a_1^j a_2^j - \nu).
\]

Eqs. (47-48) are similar in structure to the equations for the commutative retarded or advanced Green functions written for the approximating Hamiltonian \((8)\). So results of approximation \((20)\) coincide with the ones obtained previously by the approximating Hamiltonian method \([1,3,4]\). So here we outline only final results derived from Eqs. (47-48).

\[
\langle a_1^j a_2^j | \nu | a_1^j a_2^j \rangle_E = \frac{1}{2\pi} \left(\frac{T(p)}{E(p)} \frac{\lambda(f)}{2}\right).
\]

where \(E(p)\) is a spectrum of quasiparticle excitations

\[
E(p) = \sqrt{T^2(p) + c^2 |2g(f)|^2}
\]

and the parameter \(c\) is defined by the solution of the following transcendental equation:

\[
\nu \left(1 - \frac{1}{(2\pi)^3} \int dp |f(p)|^2 \frac{T(p)}{E(p)} \frac{\lambda(f)}{2}\right) = 0.
\]

which always has trivial solution \(c = 0\). Nontrivial solution is eligible only if \(\lambda > \nu\), where the critical value \(\lambda\) is determined by the relation

\[
1 = \frac{1}{(2\pi)^3} \int dp |f(p)|^2 \frac{T(p)}{E(p)} \frac{\lambda(f)}{2}.
\]

So the condition

\[
\frac{1}{(2\pi)^3} \int dp \left|f(p)\right|^2 \frac{T(p)}{E(p)} \frac{\lambda(f)}{2} > 1
\]

is a condition of existence for nontrivial solution of Eq. (49). When solving Eqs. (47-48) the following formulation of the spectral theorem was used:

\[
(1 - e^{-i\Delta})\langle a_1^j a_2^j | \nu | a_1^j a_2^j \rangle_E = \langle a_1^j a_2^j | e^{-i\Delta} | a_1^j a_2^j \rangle_E - \langle a_1^j a_2^j | e^{-i\Delta} | a_1^j a_2^j \rangle_E.
\]

\[
\langle a_1^j a_2^j | e^{-i\Delta} | a_1^j a_2^j \rangle_E = \langle a_1^j a_2^j | e^{-i\Delta} | a_1^j a_2^j \rangle_E.
\]

\[
\langle a_1^j a_2^j | e^{-i\Delta} | a_1^j a_2^j \rangle_E = \langle a_1^j a_2^j | e^{-i\Delta} | a_1^j a_2^j \rangle_E.
\]

We also put \(\nu = 0\) in final results according to the theory of quasiparticles. Now the criterion of selection between trivial and nontrivial solutions in the region \(\lambda > \nu\) must be worked out. Striving to stay within frames of the proposed formalism we avoid to involve auxiliary conditions such that the true solution must provide the absolute minimum for the free energy of the system and so on. Instead of that we show that if nontrivial solution exists, i.e. if condition (50) fulfills, the trivial solution has to be neglected because it contradicts to spectral properties of the dynamical system. To prove this statement let us consider equation for Green function \(\langle a_1^j a_2^j | a_1^j a_2^j \rangle\) in the case when \(\lambda > \nu\) and only trivial solution is chosen to be true solution. To derive the equation we vary Eq. (25) with respect to the source \(j, -g, a, \tau\) as was done before when deriving Eqs. (29-30) and then take into account selecting rules induced by laws of conservation (3-4) and use representation (40). The result is:

\[
\langle E - 2T(p) \rangle (a_1^j a_2^j | a_1^j a_2^j \rangle_E = \lambda f_1 \left(\langle a_1^j a_2^j | a_1^j a_2^j \rangle \right) \left(\langle a_1^j a_2^j | a_1^j a_2^j \rangle \right) - \frac{\lambda f_1}{2\pi} \left(\langle a_1^j a_2^j | a_1^j a_2^j \rangle \right) E - \frac{\lambda f_1}{2\pi} \left(\langle a_1^j a_2^j | \nu | a_1^j a_2^j \rangle \right) E.
\]

where it was denoted that

\[
C_\lambda(E) = \sum_{j \in \Omega} \frac{\lambda f_1}{2\pi} \left(\langle a_1^j a_2^j | \nu | a_1^j a_2^j \rangle \right) E.
\]

It follows from Eqs. (53-54) that

\[
C_\lambda(E) = \frac{1}{2\pi} \frac{2\lambda f_1}{2\pi V (2T(p) - E)} \frac{T(p)}{E(p)} \frac{\lambda(f)}{2}.
\]
Eqs. (53, 55) show that the deviation of the magnitude \(\langle a_{-m} a_{m}^* \rangle_k\) from its value in the absence of interaction \(\lambda(f) = 0\) tends to zero while \(V \rightarrow \infty\) if we chose trivial solution \(c = 0\). Now let us introduce dynamical variable

\[
A = \frac{1}{\sqrt{V}} \sum_{j\downarrow} \lambda(f) a_{-m_{j}} a_{m_{j}}^*
\]

It follows from Eq.(54) that

\[
\langle\langle A | A^{*}\rangle \rangle_k = \sum_q \lambda(q) C_q(E)
\]

and from Eq.(55) one has

\[
\langle\langle A | A^{*}\rangle \rangle_k = -\frac{i}{2\pi} \frac{2F(E)}{1 - F(E)}
\]

where

\[
F(E) = \frac{2}{(2\pi)^{1/2}} \int |\langle p|\rangle^2 \frac{\partial H(\delta q)}{\partial \delta q} d\rho^2.
\]

Taking into account spectral representation (51-52) we have

\[
\langle\langle A | A^{*}\rangle \rangle_k = -\frac{i}{2\pi} \int \frac{d\omega \mathcal{J}_{AA^{*}}(\omega) 1 - e^{-\frac{\omega}{E}}}{E - \omega}
\]

where \(\mathcal{J}_{AA^{*}}(\omega)\) is correspondent spectral density. Then

\[
\frac{1}{1 - F(E)} = 1 - \frac{1}{2} \int_{-\infty}^{+\infty} d\omega \mathcal{J}_{AA^{*}}(\omega) 1 - e^{-\frac{\omega}{E}}
\]

(57)

Variable \(E\) can take values on the imaginary axis

\(E = iz\).

So

\[
\frac{1}{E - \omega} = \frac{-iz + \omega}{z^2 + \omega^2}
\]

and

\[
\omega(1 - e^{-i\omega}) > 0.
\]

Therefore for the real part we have

\[
\text{Re} \left( -\frac{1}{2} \int_{-\infty}^{+\infty} d\omega \mathcal{J}_{AA^{*}}(\omega) 1 - e^{-\frac{\omega}{E}} \right) = \frac{1}{2} \int_{-\infty}^{+\infty} d\omega \mathcal{J}_{AA^{*}}(\omega) \left( 1 - e^{-\frac{\omega}{E}} \right) > 0.
\]
Appendix A

Let us consider the following Hamiltonian with sources

$$H_t = H + J_t,$$

$$J_t = (\sum_j \int \! dx \, B(x) j(x, t))$$

where operators $H$ and $B(x)$ are independent on time $t$ and $j(x, t)$ contain time dependence. The symbol $(\sum f) \! dx$ implies summation over discrete and integration over continuous indexes simultaneously. The complicated index $x$ may denote a set of quantum number or (and) other ordinary indexes defining type of a particular source. It is assumed that each source $j(x, t)$ contains multiplier $e^{it\tau}$, $(\tau > 0)$, so

$$j(x, t) = 0 \quad \text{if} \quad t \rightarrow -\infty.$$  \hspace{1cm} (A.1)

The equation for the evolution operator is ($k = 1$):

$$i \frac{dU^+(t)}{dt} = HU^+(t) \quad \text{or} \quad \dot{U}^+(t) = U^+(t)J_t,$$  \hspace{1cm} (A.2)

and the boundary condition follows from Eq. (A.1):

$$e^{itU^+(0)}U^+(t) = 0 \quad \text{if} \quad t \rightarrow -\infty.$$  \hspace{1cm} (A.3)

In the Heisenberg representation one has for any time independent dynamical variable

$$A(t) = U^+(t)AU^-(t)$$

and the statistics operator $D$ is some statistical operator. Let us vary $\delta(A(t))$ with respect to $j(x, t)$. So

$$\delta(A(t)) = \langle \delta(U^+(t)AU(t)) \rangle = \langle \delta(U^+(t)|\delta J_t)U^+(t) \rangle.$$  \hspace{1cm} (A.4)

Let us define

$$\delta U^+(t) = U^+(t)\delta U(t)$$

then the equation for $\delta U^+(t)$ is (if one varies Eq. (A.2)):

$$i \frac{d\delta U^+(t)}{dt} = U^+(t)(\delta J_t)U^+(t)$$  \hspace{1cm} (A.5)

But

$$U^+(t)B(x)U(t) = B(t, x)$$

$$U^+(t)(\delta J_t)U^+(t) = \delta J(t) = (\sum_j \int \! dx \, B(t, x)j(x, t))$$

and taking into account boundary condition (A.3) leads to the formal solution of Eq. (A.5):

$$\delta U^+(t) = -U^+(t) \int_{-\infty}^{+\infty} \delta J(t_1)dt_1 = -U^+(t) \int_{-\infty}^{+\infty} \theta(t - t_1)\delta J(t_1)dt_1.$$  \hspace{1cm} (A.6)

In full analogy with above calculations one has

$$-i \frac{dU^+(t)}{dt} = U^+(t)H_t$$

with boundary condition

$$U^+(t) = 1 \quad \text{if} \quad t \rightarrow -\infty$$

and

$$-i \frac{dU^+(t)}{dt} = (\delta U^+(t))H_t + U^+(t)(\delta J_t).$$  \hspace{1cm} (A.7)

So

$$U^+(t) = \int_{-\infty}^{+\infty} \theta(t - t_1)\delta J(t_1)dt_1U^+(t).$$  \hspace{1cm} (A.8)

From Eqs. (A.1, A.6, A.7) it follows that

$$\delta(A(t)) = -\int_{-\infty}^{+\infty} \theta(t - t_1)([A(t), \delta J(t_1)]).$$

Therefore

$$\delta(A(t)) \mid_{\delta J(x_1, \tau), \tau x_1<0} = -\delta(t - \tau)\langle [A(t), B(\tau, x_1)] \rangle.$$  \hspace{1cm} (A.9)

Let us note that we did not use any special form of density operator $D$ deriving relation (A.8). For

$$D = e^{-itH}/Sp(e^{-itH})$$

the right side of (A.8) becomes proportional to usual two-time temperature retarded Green function $\langle \delta J \rangle$. The result (A.9) for the advanced Green functions can be derived in the same manner. It is only necessary to assume that each source contains multiplier $e^{-it\tau}$ and to impose correspondent boundary conditions for $t \rightarrow +\infty$:

$$U^+(t) = 1 \quad \text{if} \quad t \rightarrow +\infty$$

$$\lim_{t \rightarrow +\infty} U^+(t) = 1 \quad \text{if} \quad t \rightarrow +\infty.$$  \hspace{1cm} (A.10)

The final result will be

$$\delta(A(t)) \mid_{\delta J(x_1, \tau), \tau x_1>0} = \delta(t - \tau)\langle [A(t), B(\tau, x_1)] \rangle.$$  \hspace{1cm} (A.11)
References

N.N. Bogoliubov, Selected Works in Statistical Physics (Moscow State University Press, 1979);


