Asymptotic Freedom from Induced Gravity Cosmology

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Abstract

We give conditions to obtain cosmological asymptotic freedom in scalar–tensor theories of gravity. We show that this feature can be achieved in FRW flat space-times since we obtain singularity free solutions where the effective gravitational constant $G_{eff} \to 0$ for $t \to -\infty$ and, for some of them, $G_{eff} \to G_N$ for $t \to \infty$, where $G_N$ is the Newton constant.

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1 Introduction

Nonminimal coupling between gravity and one (or more than one) scalar field is recently become a general ”paradigm” with respect to formulate any effective theory of matter and gravity which tries to take into account cosmology and particle physics [1]. We have to define an effective gravitational constant $G_{\text{eff}}$ and an effective cosmological constant $\Lambda_{\text{eff}}$ which we need for renormalization (at least at one–loop level) and that have to furnish the present observed values $G_{\text{eff}} \to G_N$, $\Lambda_{\text{eff}} \to 0$. Furthermore, nonminimally coupled theories of gravity furnish cosmological models which bring to an end the inflationary stage without any imposed fine–tuning: in other words, the shortcomings of original inflationary models (the so called old and new inflation [2]) are naturally avoided by supposing a variation of Newton constant which regulates the phase transition from the false–vacuum state to the true–vacuum state [3]. In this scheme, gravity is an induced interaction which could result from an average effect of the other fundamental forces. Then, we have to search for similar features between gravity and the other interactions. As it is shown in [4], it is possible to relate the today observed Newton constant with the self coupling constant $\lambda$ of an effective scalar field potential. This constant is related to the gauge coupling $\alpha_{\text{GUT}}$ being $\alpha_{\text{GUT}} = g^2/4\pi$, $\lambda = g^2$ and $G_N \propto \lambda$ [5].

In particular, we know that any force mediated by the exchange of non–Abelian gauge quanta has a property called ”asymptotic freedom”, which means that the effective strength of the interaction tends to zero at short distances, or, in other words, if the energy of the system diverges. The existence of asymptotic freedom of strong interactions was indicated by a series of experiments which are the high energy counterparts of the Rutherford experiments with the alpha particles [6]. This scheme was applied also to high energy electrons which were scattered on proton targets [7]. The goal was to study the internal structure of proton (deep–inelastic scattering). The result was that when the exchanged energy–momentum became larger, the interaction among the quarks in the proton became weaker [8]. In principle, we can seek similar behaviours also in gravitational interaction [9], but the lack of full quantization does not allow us to apply the quantum interpretative scheme which holds in QCD. As a matter of fact, gravity is a ”classical” theory since we have not yet a quantum interpretation of spacetime. However, if we adopt an induced gravity interpretation, we can assume that the average effects of the other material interactions lead, in some sense, the gravitational interaction, as discussed above. These effects have to mimics strong interactions at high energies (i.e. at short distances) but have to make one recover Einstein standard gravity at low energy limits. From a cosmological point of view, ”short distances” mean ”early times” and ”divergence of matter–energy density”, while ”low energy limits” mean that the cosmic time $t \to \infty$. In other words, we shall have a sort of ”gravitational asymptotic freedom” if $\lim_{t \to -\infty} G_{\text{eff}} \to 0$, and recover the standard gravity if $\lim_{t \to \infty} G_{\text{eff}} \to G_N$. However, we are supposing to have singularity free cosmological solutions for which the scale factor of the universe $a(t)$ is defined on the interval $-\infty < t < \infty$.

In this paper, we address the issue to find gravitational asymptotic freedom in a
nonminimally coupled theory. In Sec. 2, we give the conditions of how asymptotic freedom can be found in a cosmological context. In Sec. 3, we discuss cosmological models without singularity, by which it is possible to find the property of asymptotic freedom when cosmic time $t \to -\infty$. Furthermore, some of them allow to recover standard Einstein gravity at present. In Sec. 4, we draw conclusions.

2 Conditions for Cosmological Asymptotic Freedom

We start our analysis to search for gravitational asymptotic freedom from a generic action where a scalar field $\phi$ is nonminimally coupled with gravity:

$$\mathcal{A} = \int d^4x \sqrt{-g} \left[ F(\phi) R + \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - V(\phi) \right].$$  \hspace{1cm} (1)

Here $V(\phi)$ is the potential and $F(\phi)$ the coupling for the field $\phi$. We adopt the units $8\pi G_N = \hbar = c = 1$.

The field equations are obtained by a variation with respect to $g_{\mu\nu}$

$$F(\phi) G_{\mu\nu} = -\frac{1}{2} T_{\mu\nu} - g_{\mu\nu} \Box F(\phi) + F(\phi)_{,\mu\nu},$$  \hspace{1cm} (2)

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu} = \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \phi^{,\alpha} \phi_{,\alpha} + g_{\mu\nu} V(\phi)$, is the "bare" energy-momentum tensor of the scalar field. By a variation with respect to $\phi$, we obtain the generalized Klein-Gordon equation:

$$\Box \phi - R F'(\phi) + V'(\phi) = 0,$$  \hspace{1cm} (3)

where the prime indicates the derivative with respect to $\phi$.

Let us consider now a homogeneous and isotropic cosmology where $a = a(t)$ is the scale factor of the universe and the field $\phi = \phi(t)$ is a function of cosmic time only. The Lagrangian density in (1) becomes

$$\mathcal{L} = 6a \dot{a}^2 F(\phi) + 6a \dot{\phi} a^2 F'(\phi) - 6ka F(\phi) + a^3 \left[ \frac{1}{2} \dot{\phi}^2 - V(\phi) \right],$$  \hspace{1cm} (4)

where the dot is the time derivative and $k$ is the spatial curvature constant. From now on, we assume $k = 0$. The Einstein equations (2) become

$$H^2 = -\frac{1}{F(\phi)} \left[ H \dot{\phi} F'(\phi) + \frac{1}{12} \dot{\phi}^2 + \frac{1}{6} V(\phi) \right],$$  \hspace{1cm} (5)

$$\dot{H} + H^2 = -\frac{1}{2F(\phi)} \left[ H \dot{\phi} F'(\phi) - \frac{1}{3} \dot{\phi}^2 + \frac{1}{3} V(\phi) + \ddot{\phi} F'(\phi) + \dot{\phi}^2 F''(\phi) \right],$$  \hspace{1cm} (6)

and the Klein–Gordon equation is

$$\ddot{\phi} + 6 \left( H + 2H^2 \right) F'(\phi) + 3H \dot{\phi} + V'(\phi) = 0,$$  \hspace{1cm} (7)
where $F'(\phi) \equiv dF(\phi)/d\phi$ and $H = \dot{a}/a$ is the Hubble parameter. We have to note that the effective gravitational constant is

$$G_{\text{eff}} \equiv \frac{1}{2F(\phi)},$$

and, for $F(\phi) = -1/2$, we recover the standard Einstein equations of Friedman cosmology.

We rewrite Eqs.(5) as:

$$H^2 + H \frac{\dot{F}(\phi)}{F(\phi)} + \frac{\rho_\phi}{6F(\phi)} = 0,$$

where $\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi)$;

$$\rho_\phi$$

is the matter–energy density associated to the (minimally coupled) scalar field; it can be considered a sort of "bare" energy density.

As we discussed in [10], from Eq.(2), we can say that, in general, in these nonminimally coupled gravity theories there is no singularity. As consequence, in such cosmologies we have that the physical quantities are all defined in the time interval $(-\infty, +\infty)$.

Anyway, we have to keep in mind the question of whether or not the theory admits an absolutely stable ground state (at classical level) in which $\phi$ assumes a constant value and the Ricci tensor $R_{\mu\nu}$ is proportional to $g_{\mu\nu}$, i.e. spacetime is either Minkowski, de Sitter, or anti–de Sitter. In curved spacetimes (that is if nonminimal coupling terms appears) the energy density is not a good criterion for the stability so that conditions on the parameters in $F(\phi)$ and $V(\phi)$ have to be imposed. In general, the stability request and the recovering of standard gravity restrict the range of possible values of parameters like $\xi$, $\Lambda$, $m^2$ and $\lambda$ in theories where $F(\phi) = 1 - \xi \phi^2$ and $V(\phi) = \Lambda + m^2 \phi^2 + \lambda \phi^4$ as discussed in [11]. In what follows, we have to take into account such issues since asymptotic states have to be stable. In the examples of next section, the ranges of parameters are specified.

In general, what is needed for recovering the cosmological asymptotic freedom is that, when $t \to -\infty$,

$$\rho_\phi = \rho_\phi \left(\dot{\phi}(t), V(\phi(t))\right) \longrightarrow \infty, \quad a(t) \longrightarrow 0,$$

which implies

$$F(\phi(t)) \longrightarrow \infty,$$

The condition on $\rho_\phi$ tell us that the "bare" energy density has to diverge in order to follow the analogy with the elementary particle case. The condition on $a(t)$ takes into account that, differently from high energy physics, in cosmology any length varies in connection with the dynamical behaviour of spacetime. That condition then tells us that any lengths approach to zero. Before a rigorous analysis, first we will discuss such problem in a qualitative way. So doing, we will quickly understand some aspects concerning the coupling $F(\phi)$, the potential $V(\phi)$, and the time behaviour of $a(t)$ and $\phi(t)$. Let us assume that we obtain conditions (10)–(11) using exponential functions, that is

$$\rho_\phi(t) \sim \rho_0 \exp(-k_2 t), \quad a(t) \sim a_0 \exp(\Lambda t), \quad F(\phi(t)) \sim F_0 \exp(-k_1 t),$$
where \( k_{1,2}, \Lambda \) are positive constants. Actually, in this qualitative analysis, we are assuming that \( G_{\text{eff}} \to 0 \) for \( t \ll 0 \). From Eq.\((9)\), we get:

\[
\frac{\rho_\phi}{6F} = \frac{\rho_0}{6F_0} e^{(k_1-k_2)t} = \Lambda (k_1 - \Lambda),
\]

and we see that it must be \( k_1 \geq k_2 \). We have to discuss two cases:

i) \( \Lambda = k_1 \) in the case \( k_1 > k_2 \);

ii) \( \Lambda^2 - \Lambda k_1 + (\rho_0/F_0) = 0 \) if \( k_1 = k_2 \). In any case, we find that

\[
\frac{\rho_\phi}{6F} = \text{const} = k_3,
\]

where \( k_3 = 0 \) for \( k_1 > k_2 \) or \( k_3 = \Lambda (k_1 - \Lambda) \) for \( k_1 = k_2 \). From Eq.\((6)\), we find

\[
\Lambda^2 = \frac{1}{2} \Lambda k_1 - \frac{1}{6} \frac{\dot{\phi}^2}{F} - \frac{V}{6F} - \frac{k_1^2}{2}.
\]

By using \((13)\), we get

\[
\frac{V(\phi)}{6F(\phi)} = \sigma_0,
\]

where \( \sigma_0 \) is a constant. We have some different cases:

a) \( k_1 > k_2 \), in this case we get \( \sigma_0 = -\frac{\dot{\phi}^2}{12F} = -\Lambda^2 \).

b) \( k_1 = k_2 \), here we have \( \sigma_0 = \frac{\rho_0}{6F_0} - \frac{k_1^2}{2} (1 - 3\Lambda) \).

c) Choosing from the very beginning \( V(\phi) = \lambda \), we get from \((11)\) that (asymptotically) \( \sigma_0 = 0 \). Of course, we can also choose \( \lambda = 0 \), then (constantly) is \( \sigma_0 = 0 \). In both cases a), b) we get that \( \phi(t) \sim \exp(-k_2t/2) \), that is \( F(\phi) \sim \phi^2 \) and then \( V(\phi) \sim \phi^2 \) (we are not taking into any consideration the role of the constants appearing in \( \sigma_0 \) in those two different cases). The two cases c) are respectively the cosmological constant case in presence of nonminimal coupling, and the Brans–Dicke type model (with no potential).

We will now discuss in a rigorous way the above situations and so doing we will deduce that \( F(\phi(t)) \to \infty \) (i.e. \( G_{\text{eff}} \to 0 \)) for \( t \ll 0 \) in an exponential way.

3 Cosmological models with asymptotic freedom

Now we discuss models where the above hypotheses hold. We have to note that the gravitational asymptotic freedom can depend or not on the initial conditions, this conditioned on the value of \( \sigma_0 \). In fact, as we shall see below, we have models where asymptotic freedom holds for general solutions (in the sense that initial conditions have not to be specified) and models where it holds for a restricted range of initial data. This feature depends on the ratio of \( V(\phi) \) and \( F(\phi) \), that is on \( \sigma_0 \).
3.1 The case with \( V(\phi) = \lambda \phi^2 \)

This is the case of the so called "free" effective potential coming from the one–loop approximation of a scalar field. The method to seek the general solutions (we have called it the Noether Symmetry Approach) is discussed in [13]. In that case, the coupling and the potential have the forms

\[
F(\phi) = k_0 \phi^2, \quad V(\phi) = \lambda \phi^2, \tag{17}
\]

where \( k_0 \) and \( \lambda \) are free parameters, and, furthermore, it has to be \( k_0 \neq 1/12 \) in order to avoid the degeneration (i.e the \( \phi \)–part of Hessian determinant is zero) of the Lagrangian (4). Solving exactly the system (5)–(7) we find that the scale factor and the scalar field evolve as

\[
a(t) = \left[ c_1 e^{\Lambda_0 t} + c_2 e^{-\Lambda_0 t} \right] \times \\
\exp \left\{ -\frac{2}{3} \left[ c_3 \arctan \sqrt{\frac{c_1}{c_2}} e^{\Lambda_0 t} + c_4 \ln(c_1 e^{\Lambda_0 t} + c_2 e^{-\Lambda_0 t}) \right] \right\}, \tag{18}
\]

and

\[
\phi(t) = \frac{\exp \left[ c_3 \arctan \sqrt{\frac{c_1}{c_2}} e^{\Lambda_0 t} + c_4 \ln(c_1 e^{\Lambda_0 t} + c_2 e^{-\Lambda_0 t}) \right]}{c_1 e^{\Lambda_0 t} + c_2 e^{-\Lambda_0 t}}. \tag{19}
\]

We see, from (18), that the asymptotic behaviour of \( a(t) \) (for \( t \to \pm \infty \)) is de Sitter like. Then the Hubble parameter is

\[
H = \Lambda_0 \left( 1 - \frac{2}{3} \frac{\xi_1}{\xi_2} \right) \left( \frac{c_1 e^{\Lambda_0 t} - c_2 e^{-\Lambda_0 t}}{c_1 e^{\Lambda_0 t} + c_2 e^{-\Lambda_0 t}} \right) - \left( \frac{c_3}{c_1 e^{\Lambda_0 t} + c_2 e^{-\Lambda_0 t}} \right), \tag{20}
\]

where

\[
\Lambda_0 = \sqrt{\frac{2\lambda \xi_2}{\xi_1 (\xi_1 - \xi_2)}}, \quad c_3 = \frac{\mathcal{F}_0 \sqrt{c_1 c_2}}{\xi_2 \Lambda_0}, \quad c_4 = \frac{\xi_1}{\xi_2}, \quad \xi_1 = 1 - 12k_0, \quad \xi_2 = 1 - \frac{32}{3} k_0. \tag{21}
\]

The constants \( c_1, c_2, c_3 \) are the initial data and \( \mathcal{F}_0 \) is a constant of motion [13]. The asymptotic behaviour of these solutions are

\[
\lim_{t \to -\infty} a(t) = a_0 \exp \left[ -\sqrt{\frac{\lambda (1 - 8k_0)^2}{2k_0 (12k_0 - 1) (3 - 32k_0)}} t \right], \tag{22}
\]

\[
\lim_{t \to -\infty} \phi(t) = \phi_0 \exp \left[ -\sqrt{\frac{8 \lambda k_0}{(12k_0 - 1) (3 - 32k_0)}} t \right]. \tag{23}
\]

Coherently with (18), we see that \( a(t) \) diverges for \( t \to -\infty \), that is it has to be a de Sitter behaviour. From (22), we immediately recover asymptotic freedom \((G_{eff} \to 0)\) for \( t \ll 0 \) (actually, we know the complete integral of the model and we recover the same result for \( t \gg 0 \)). Furthermore, we have to say that (16) is not a condition to obtain asymptotic freedom in this case, since \( V(\phi)/(6F(\phi)) = \sigma_0 = \lambda/6k_0 \) always holds.
3.2 The string–dilaton cosmology case

A string–dilaton four–dimensional effective action, neglecting the torsion terms and other scalar fields except the dilaton $\varphi$, is [14]

$$ A = \int d^4x \sqrt{-g} e^{-2\varphi} \left\{ \frac{1}{2} \left[ R + 4g^{\mu\nu} \varphi_{,\mu} \varphi_{,\nu} - 2\Lambda \right] \right\} . $$

(24)

This action is nothing else but a particular case of the most general action (1), when we take the positions

$$ \phi = 2e^{-\varphi}, \quad F(\varphi) = \frac{1}{8} \phi^2 = \frac{1}{2} e^{-2\varphi}, \quad V(\varphi) = e^{-2\varphi} \Lambda, $$

(25)

(we see that $F(\varphi) \sim \phi^2$, $V(\varphi) \sim \phi^2$ and $V(\varphi)/(6F(\varphi)) = \sigma_0$).

In a FRW flat metric, the action (24) gives rise to a Lagrangian density:

$$ L = e^{-2\varphi} \left[ 3\dot{a}^2 - 6\dot{a}a \dot{\varphi} + 2a^3 \dot{\varphi}^2 - a^3 \Lambda \right]. $$

(26)

whose equations of motion, by the transformations (25), can be recast in the form of system (5)–(7). Also here, condition (16) holds at any time (with $\sigma_0 \neq 0$). The general solution of the dynamics is (using also here the Noether Symmetry Approach)

$$ a(t) = a_0 \exp \left\{ \pm \frac{1}{\sqrt{6}} \arctan \left[ \frac{1 - 2e^{4\lambda \tau}}{2e^{2\lambda \tau} \sqrt{1 - e^{4\lambda \tau}}} \right] \right\}, $$

(27)

$$ \varphi(t) = \frac{1}{4} \ln \left[ \frac{2\lambda^2 e^{4\lambda \tau}}{(1 - e^{4\lambda \tau})} \right] \pm \frac{1}{\sqrt{6}} \arctan \left[ \frac{1 - 2e^{4\lambda \tau}}{2e^{2\lambda \tau} \sqrt{1 - e^{4\lambda \tau}}} \right] + \varphi_0, $$

(28)

where $\tau = \pm t$, $\lambda^2 = \Lambda/2$. In this solution the "scale factor duality" (i.e. the property that if $a(t)$ is a solution $a(t)^{-1}$ is a solution too) is evident [14]. Using (27) and (28) at $t \ll 0$, the asymptotic freedom is easily recovered.

3.3 The case with $V(\phi) = \Lambda$

This is a very interesting case since the solutions allow to recover the standard Einstein gravity at $t \to \infty$, and the asymptotic freedom at $t \to -\infty$.

Also for $V(\phi) = \Lambda$, the dynamical equations (5)–(7) can be exactly solved using the Noether Symmetry Approach [12]. The existence of the Noether symmetry selects a coupling of the form

$$ F(\phi) = \frac{1}{12} \phi^2 + F_0' \phi + F_0, $$

(29)

where $F_0'$ and $F_0$ are integration constants. The general solution of the system (5)–(7) is [12]

$$ a(t) = \left[ c_1 e^{\lambda t} + c_2 e^{-\lambda t} \right]^{1/2}, \quad \phi(t) = \frac{J}{\sqrt{c_1 e^{\lambda t} + c_2 e^{-\lambda t}}}K + \frac{c_3}{\sqrt{c_1 e^{\lambda t} + c_2 e^{-\lambda t}}} - 6F_0', $$

(30)
where $c_1$, $c_2$ and $c_3$ are integration constants and $\lambda = \sqrt{-2\Lambda/3\mathcal{H}}$ with $\mathcal{H} = F_0 - 3F_0^2$ (the $\phi$–part of the Hessian determinant of $\mathcal{L}$). $\mathcal{J}$ is a constant of motion [12] and

$$\mathcal{K} = \int \frac{dt}{\sqrt{c_1 e^{\lambda t} + c_2 e^{-\lambda t}}},$$

is an elliptical integral of first kind. It is easy to see that the asymptotic freedom (that is $F_{-\infty} \to \infty$) is recovered by choosing the initial condition $c_2 = 0$, otherwise we have $F_{\pm\infty} \to \text{const}$. In other words, we recover always standard gravity for $t \to \infty$, being $G_{\text{eff}} \to \text{const}=G_N$, but asymptotic freedom is recovered only for a certain set of initial conditions. Furthermore, we have to note that, for $c_2 = 0$, this model allows to recover both standard gravity and asymptotic freedom with the "same" cosmological de Sitter behaviour. Finally, referring to the discussion in Sec.2, this case corresponds to get asymptotically $\sigma_0 = 0$.

### 3.4 The Brans–Dicke case

A pure Brans–Dicke action, without ordinary matter contributions, can be written as

$$\mathcal{A} = \int d^4x \sqrt{-g} \left[ \phi R + \frac{\omega(\phi)}{\phi^2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \right].$$

(32)

It can be recast in a usual nonminimally coupled form, like (1), with $V(\phi) = 0$ by the transformations [12]

$$\phi = F(\phi), \quad \omega(\phi) = \frac{F(\phi)}{2F'(\phi)},$$

(33)

(where we are not specifying the function $F(\phi)$). In a homogeneous and isotropic metric, we recover the Lagrangian (4) and the equations (5)–(7) (in which we take $V(\phi) = 0$ or $\sigma_0 = 0$ constantly, as in case c) with $k_1 = k_2$). By imposing into (5)–(7)

$$a(t) = a_0 e^{\Lambda t},$$

(34)

we get

$$\phi(t) \sim e^{-\frac{3\Lambda}{2} t} + \phi_0 \sim e^{-\frac{3\Lambda}{2} t}, \quad F(\phi(t)) \sim c_2 e^{-3\Lambda t},$$

(35)

we see that also here, asymptotically, $F(\phi) \sim \phi^2$. Actually, the presence of the constants $(c_1, c_2, \phi_0)$ tells us that $F(\phi)$ has to be a more complicated function of $\phi$. Even if we do not have the complete control of the time evolution of the model, we have shown that it has the feature of asymptotic freedom. We have to note that (34) is only a particular solution of the system; for a discussion of the definition of $\Lambda$ in this case without potential see [13].
4 Discussion and Conclusions

In our first qualitative considerations, we have used exponential asymptotical behaviour for \( a(t) \) and \( \rho_\phi(t) \), getting \( H = \dot{a}/a \) constant. Let us approach the issue to get asymptotic freedom from a more general point of view.

Eq.(9) can be rewritten in an integral form as:

\[
F(\phi(t)) = \frac{\tilde{F}_0}{a} \exp \left[ - \left( \frac{\rho_\phi}{6HF(\phi)} \right) dt. \right. \] (36)

Let us now assume that asymptotically (i.e. for \( t \to -\infty \)),

\[
\frac{\rho_\phi}{6HF(\phi)} = \Sigma_0, \] (37)

where \( \Sigma_0 \) is a positive constant. Then we have

\[
F(\phi(t)) = \frac{\tilde{F}_0}{a} \exp (-\Sigma_0 t). \] (38)

Hypothesis (37) is more general with respect to what we have realized untill now using exponential asymptotic funtions for \( a(t) \) and \( \rho_\phi(t) \). In fact, we are now assuming that \( H \) is not, a priori, a constant. Of course, if we assume, or if we show that \( H \) is constant, we get that \( \rho_\phi/(6F(\phi)) \) is a constant too, that is we restore (14). Hypothesis (37), being a relation among \( (a, \dot{a}, \phi, \dot{\phi}) \), has to be compatible with the Klein–Gordon equation (7), then we get

\[
6\dot{H}F(\phi)+6HF_0 + 3H\dot{\phi}^2 + 6\dot{H}F + 12H^2\dot{F} = 0. \] (39)

Eq.(6), by Eq.(5), can be recast in the form \( \ddot{\phi}^2 = 4F\dot{H} - 2H\dot{F} + 2\ddot{F} \). With a little algebra, we obtain

\[
\dot{H} + 2H^2 - 3\Sigma_0 H + \Sigma_0^2 = -\frac{V(\phi)}{F(\phi)}. \] (40)

Let us now suppose that in the above limit \( (t \to -\infty) \) the condition (16) holds. Eq.(40) becomes

\[
\dot{H} + 2H^2 - 3\Sigma_0 H + \Sigma_0^2 + 6\sigma_0 = 0, \] (41)

which is exactly solvable. The solution of this (asymptotic) equation is

\[
H = \frac{\lambda_1 Ce^{\lambda_1 t} + \lambda_2 e^{\lambda_2 t}}{2 \left[ Ce^{\lambda_1 t} + e^{\lambda_2 t} \right]} \] (42)

where \( C \) is the integration constant and

\[
\lambda_{1,2} = \frac{3}{2} \Sigma_0 \pm \frac{1}{2} \sqrt{\Sigma_0^2 - 48\sigma_0}. \] (43)
It is worthwhile to note that, asymptotically for $t \to -\infty$, $H$ converge to a constant then de Sitter behaviour is recovered (it is important to stress that in this way we get (14) under the hypothesis $V(\phi)/(6F(\phi)) = \text{constant}$ as is clear from (40)).

Being $H = \dot{a}/a$, we get from (42) the scale factor of the universe

$$a(t) = a_0 \sqrt{Ce^{\lambda_1 t} + e^{\lambda_2 t}}.$$  \hspace{1cm} (44)

whose asymptotic behaviour strictly depends on the signs and the values of $\Sigma_0$ and $\sigma_0$. Inserting (44) into (38), we get

$$F(\phi(t)) = \left( \frac{\tilde{F}_0}{a_0} \right) \frac{e^{-\frac{7}{4} \Sigma_0 t}}{\sqrt{Ce^{\frac{\mu}{2} t} + e^{-\frac{\mu}{2} t}}},$$ \hspace{1cm} (45)

where $\mu = \sqrt{\Sigma_0^2 - 48\sigma_0}$, is a positive definite constant (it has to be $\Sigma_0 \geq 48\sigma_0$ since $H$ is a real number). Eq.(45) has to diverge for $t \ll 0$ to get asymptotic freedom. This situation, which is always true, is, in any case, compatible with the reality condition then we always get

$$F(\phi(t)) \sim e^{-\gamma t}, \quad \text{for} \quad t \ll 0,$$ \hspace{1cm} (46)

where $\gamma = \gamma(\Sigma_0, \sigma_0)$ is a constant determined by a $\Sigma_0$ and $\sigma_0$. On the other side, the scale factor of the universe converges exponentially to zero for any combination of $\Sigma_0$ and $\sigma_0$, but diverges as $a(t) \sim e^{-\mu t/4}$ when $\Sigma_0 > 0$, $\sigma_0 < 0$ and $\Sigma_0 < 6|\sigma_0|$ (the behaviour (46) is not altered by this last condition). It is interesting to note that in both cases (that is when $a(t) \to 0$ and $a(t) \to \infty$ for $t \ll 0$) we loose the gravitational interaction; in other words, if a given length converges or diverges the result is the same: the first situation can be seen as an analog of QCD, the second one as the lack of interaction due to the fact that test particles are brought to infinite distance. We see that also using the more general hypothesis (37), the dynamics leads again to exponential functions for $a(t)$ and $\phi(t)$ as well as to a nonminimal coupling which, in general, is still $F(\phi) \sim \phi^2$ as we can easily obtain putting the above results into (7). Of course such behaviours are controlled by the two parameters $\Sigma_0$, $\sigma_0$.

As it emerges from these last considerations, it is clear that Eqs.(39)–(41) are the asymptotic form of the system of equation (5)–(7): however, solving the system (39)–(41) does not mean that these ”asymptotic” solutions are the asymptotic behaviour of the solutions of the system (5)–(7). Anyway, we are able to solve exactly some important cosmological cases, then we can perfectly control these two different asymptotic behaviours. In particular, we can understand how the asymptotic freedom depends upon initial data of the problem (fine tuning).

In conclusion, we can say that, at least at a classical level, asymptotic freedom seems to be a fundamental feature also for gravity, if the gravitational ”constant” is supposed to be a function of a scalar field (and then of time). A further step in our analysis is to see how the presence of ordinary matter affects all the above considerations. Finally, another important goal related to what we have done is to find the most general conditions for
a cosmological model to obtain asymptotic freedom. That is, if we impose only that
$G_{\text{eff}} \to 0$ for $t \ll 0$, which are the cosmological models satisfying such a conditions?
These last topics are the subjects which we will try to understand in a forthcoming paper.

REFERENCES


    S. Capozziello and R. de Ritis, *Class. Quantum Grav.* 11 (1994) 107;