Resummation of anisotropic quartic oscillator.
Crossover from anisotropic to isotropic large-order behavior

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Abstract

We present an approximative calculation of the ground-state energy for the anisotropic oscillator with a potential

\[ V(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{g}{4} \left[ x^4 + 2(1 - \delta)x^2y^2 + y^4 \right]. \]

Using an instanton solution of the isotropic action \( \delta = 0 \), we obtain the imaginary part of the ground-state energy for small negative \( g \) as a series expansion in the anisotropy parameter \( \delta \). From this, the large-order behavior of the \( g \)-expansions accompanying each power of \( \delta \) are obtained by means of a dispersion relation in \( g \). These \( g \)-expansions are summed by a Borel transformation, yielding an approximation to the ground-state energy for the region near the isotropic limit. This approximation is found to be excellent in a rather wide region of \( \delta \) around \( \delta = 0 \).

Special attention is devoted to the immediate vicinity of the isotropic point. Using a simple model integral we show that the large-order behavior of an \( \delta \)-dependent series expansion in \( g \) undergoes a crossover from an isotropic to an anisotropic regime as the order \( k \) of the expansion coefficients passes the value \( k_{\text{cross}} \sim 1/|\delta| \).
1 Introduction

Phase transitions in anisotropic systems with cubic symmetry have attracted much interest in the literature [1]-[5]. Especially well studied are corresponding models in quantum mechanics. To gain an analytic insight into the latter, Banks, Bender, and Wu (BBW) [6] studied a Hamiltonian with a potential

\[ V = \frac{g}{4} \left[ x^4 + 2(1 - \delta)x^2y^2 + y^4 \right]. \]  

\( (1) \)

Using multidimensional WKB techniques they derived the large-order behavior of the perturbation series for the ground-state energy

\[ E = \sum_k E_k(\delta) g^k \]  

\( (2) \)

as a function of the anisotropy parameter \( \delta \).

In 1990, Janke [7] derived the same results with more efficiency from a path integral for the imaginary part of the energy \( E \). The imaginary part contains information on the tunneling decay rate of the ground state for \( g < 0 \), and determines directly the large-order behavior of the perturbation coefficients via a dispersion relation in the complex coupling constant plane. Both, BBW and Janke, find a different large-order behavior of the isotropic system \( \delta = 0 \) and the anisotropic system \( \delta \neq 0 \). They do not discuss, however, the interesting question of how the latter goes over into the former as \( \delta \) goes to zero.

It is the purpose of this paper to fill this gap. For an optimal understanding of the expected behavior we shall not attack directly the path integral involving the potential (1), but first only the corresponding simple integral. For this integral, a perturbation expansion of the form (2) yields exactly-determined \( \delta \)-dependent perturbation coefficients. The coefficients \( E_k(\delta) \) are shown to have a large-order behavior which undergoes a crossover between the earlier derived isotropic and anisotropic behaviors when the order \( k \) passes the crossover value \( k_{\text{cross}} \sim 1/|\delta| \).

The expansion terms of a model integral with the potential (1) counts the number of terms in a perturbation expansion of the quantum mechanic and the field theory. Thus the bare model integral is sufficient to derive nontrivial information on the large-order behavior of the eventual object of interest, quantum field theory. It turns out that for resumming the \( g \)-series,
asymptotic large-order estimates for the \( \delta \)-dependent coefficients can be used only in the anisotropic regime \( k|\delta| \gg 1 \). In the isotropic regime \( k|\delta| \ll 1 \), on the other hand, it is impossible to truncate the large-order expansion of the perturbation coefficients after a finite number of terms. Thus the neighbourhood of the isotropic system \( \delta = 0 \) needs an extra investigation. In the context of quantum field theory, this was recently delivered in [8].

The imaginary parts of physical quantities at small negative \( g \) can be calculated with the help of classical solutions called instantons. In systems sufficiently close to the isotropic point it is not necessary to know the exact instanton solutions for all \( \delta \). The knowledge of the solution at the symmetry point \( \delta = 0 \) is perfectly sufficient, around which the imaginary parts can be expanded in powers of \( \delta \).

After having understood the model integral, we shall perform the same analysis for an anisotropic quantum mechanical system, which represents an one-dimensional \( \phi^4 \)-field theory with cubic anisotropy.

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ergy which we derive from an extension of recursion relations first shown by Bender and Wu (BW) [16].

2 Resummation

We begin with developing a practical algorithm for a Borel resummation of a divergent perturbation series

\[ Z(g) = \sum_k Z_k g^k. \]  

Our method will be most efficient under the following conditions:

1. From low-order perturbation theory we know the expansion coefficients \( Z_k \) up to a certain finite order \( N \).
2. From semiclassical methods we are in the possession of the high-order information in the form

\[ Z_k \xrightarrow{k \to \infty} \gamma (-1)^k k! \alpha^k \left( 1 + \frac{\gamma_1}{k} + \frac{\gamma_2}{k} + \cdots \right). \]  

3. By some scaling arguments we are able to assure a power behavior in the strong coupling limit

\[ Z(g) \xrightarrow{g \to \infty} \kappa g^\alpha. \]  

The idea of the algorithm is the following:

It must be possible to construct an infinite, complete set of Borel summable functions \( I_p(g) \) which satisfy the high-order and strong-coupling conditions (4) and (5). These functions can be used as a new basis in which to reexpand \( Z(g) \):

\[ Z(g) = \sum_{p=0}^{\infty} a_p I_p(g), \]  

The series (6) should be such that the knowledge of the first \( (N + 1) \) coefficients in the power series expansion (3) is sufficient to determine directly the first \( (N + 1) \) coefficients \( a_p \), yielding an approximation

\[ Z(g) \approx Z^{(N)}(g) \equiv \sum_{p=0}^{N} a_p I_p(g). \]  

4
This would then be a new representation of the function $Z(g)$ with the same power series up to $g^N$ but which makes use of large-order and strong-coupling informations (4) and (5). In the limit of large $N$, the series (7) is expected to converge towards the exact solution.

The functions $I_p(g)$ being Bore summable have a Borel representation

$$I_p(g) = \int_0^\infty dt e^{-t} t^{b_0} B_p^{b_0}(gt). \quad (8)$$

Parametrized by some $b_0$ and integer $p$, what are the conditions on $B_p(gt)$, such that $I_p(g)$ satisfies (4) and (5) for all $p$? The answer is most easily found with the help of the hypergeometric functions

$$2F_1(a, b; c; -\sigma gt) = \sum_{k=0}^\infty \frac{(a)_k (b)_k (-\sigma gt)_k}{(c)_k k!} \quad (9)$$

with appropriate parameters $a(p)$, $b(p)$ and $c(p)$. The Pochhammer symbol $(a)_k$ stands short for $(a)_k = \Gamma(a+k)/\Gamma(a)$. These functions have the following virtues: First, they are standard special functions of mathematical physics whose properties are well-known. Second, they have a cut running from $t = -1/|\sigma g|$ to minus infinity which is necessary to generate the large-order behavior (4). Third, they have enough free parameters to fit all input-data. The first property permits an immediate calculation of the Borel integral (8), which is simply a Laplace transformation of $t^{b_0} 2F_1(a, b; c; -\sigma gt)$

$$\int_0^\infty dt e^{-t} t^{b_0} 2F_1(a, b; c; -\sigma gt) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} E(a, b, b_0 + 1 : c : 1/\sigma g) \quad (10)$$

The resulting $E(a, b, b_0 + 1 : c : 1/\sigma g)$ is MacRobert’s $E$-function. Using its asymptotic expansion (see Ref. [17], page 203) it is easy to verify that our ansatz reproduces the large-order behavior (4). Indeed, for large $k$ the power series

$$\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} E(a, b, b_0 + 1 : c : 1/\sigma g) \equiv \sum_{k=0}^\infty e_k g^k \quad (11)$$

has coefficients which grow like

$$e_k \xrightarrow{k \to \infty} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} (-1)^k k! k^{a+b-c+b_0-1} \sigma^k. \quad (12)$$
Moreover, this property is unchanged if the original hypergeometric function is multiplied by a power \((\sigma gt)^p\). A possible set of Borel functions are therefore the following functions:

\[
B_p(gt) = (\sigma gt)^p \, _2F_1(a, b; c; -\sigma gt).
\]  

(13)

Looking at (12) we see that the functions (13) are not completely fixed by a given large-order behavior. The parameter \(\beta\) in (4) merely imposes the following relation upon the parameters \(a, b, c\) and \(b_0\)

\[
a + b - c + b_0 - 1 = \beta,
\]  

(14)

and there are many different ways to satisfy this. The specific choice will be suggested by practical considerations. One such consideration is that the \(I_p\)'s should possess a simple integral representation in order to avoid complicated numerical work. In addition, we would like to work with parameters \(a, b\) and \(c\), for which the hypergeometric function \(_2F_1\) reduces to simple algebraic functions. This happens only for special sets of the parameters. A simple possibility is for instance (see Ref. [18], page 556)

\[
_2F_1(a, a + 1; 2a + 1; -z) = 4^a \left(1 + \sqrt{1 + z}\right)^{-2a}
\]  

(15)

which arises by choosing the parameters \(a, b\) and \(c\) which are related by

\[
a + b - c = -\frac{1}{2}, \quad c - 2b = 0.
\]  

(16)

With this, the relation (14) can be satisfied for an arbitrary value of the parameter \(a\) by choosing

\[
b_0 = \beta + \frac{3}{2},
\]  

(17)

and we are left with only one parameter degree of freedom. This freedom may be used to accommodate the strong-coupling behavior of \(Z(g)\) if it is known. The equation (7) yields the condition \(I_p(g) \to \text{const.} \times g^\alpha\) on the functions \(I_p(g)\). From (8) we see that such a power behavior emerges if all Borel functions \(B_p\) satisfy \(B_p(z) \to \text{const.} \times z^\alpha\) and thus \(_2F_1(a, b; c; -z) \to \text{const.} \times z^{-p+\alpha}\) [see (13)]. The explicit representation (15) shows that the parameter \(a\) has to be taken as

\[
a = p - \alpha.
\]  

(18)
Thus we obtain the approximation $Z^{(N)} = \sum_{p=0}^{N} a_p I_p$ with

$$I_p(g) = \int_0^{\infty} dt \frac{e^{-t b_0}}{\Gamma(b_0 + 1)} \frac{(\sigma g t)^p}{4^p} \ _2F_1 \left( p - \alpha, p - \alpha + \frac{1}{2}, 2(p - \alpha) + 1; -\sigma g t \right)$$

$$= \int_0^{\infty} dt \frac{e^{-t b_0}}{\Gamma(b_0 + 1)} \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \sigma g t} \right)^{2\alpha} \frac{(\sigma g t)^p}{\left( 1 + \sqrt{1 + \sigma g t} \right)^{2p}}, \quad (19)$$

where the Borel parameter $b_0$ is fixed by (17). The normalization constant $1/4^p \Gamma(b_0 + 1)$ in front of the expansion functions was introduced for convenience.

Let us now derive equations for the expansion coefficients $a_p$ in terms of the perturbation coefficients $Z_k$. All one has to do is take the asymptotic expansions

$$I_p(g) = \sum_{k=0}^{\infty} I_p^k g^k, \quad (20)$$

insert these into (7), collect terms of equal power $g^k$, and compare these with the perturbation series (3). This gives the $(N + 1)$ algebraic equations

$$Z^{(N)}_k \equiv \sum_{p=0}^{N} I_p^k a_p = Z_k \quad; \quad k = 0, 1, \ldots, N \quad (21)$$

By assumption, the series on the left hand side contains only the coefficients $a_p$ with $p \leq N$. Thus the $a_p$’s can be computed, in principle, by inverting the $(N + 1) \times (N + 1)$ matrix $(I)_{kp} = I_p^k$. Even though this can be done recursively for any given case, it is preferable to find an explicit algebraic solution for $a_p$ in terms of $Z_k$. This is possible using the following trick. We rewrite the asymptotic expansion of $Z^{(N)}$ in Borel form

$$Z^{(N)}(g) \equiv \sum_{p=0}^{N} a_p I_p(g) = \sum_{k=0}^{\infty} Z^{(N)}_k g^k = \int_0^{\infty} dt e^{-t b_0} \sum_{k=0}^{\infty} \frac{Z^{(N)}_k(g t)^k}{\Gamma(k + b_0 + 1)}, \quad (22)$$

insert the expression (19) for $I_p(g)$, and compare directly both integrands

$$\sum_{p=0}^{N} a_p \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + \sigma g t} \right)^{2\alpha} \frac{\sigma g t}{\Gamma(b_0 + 1)} \left[ \frac{\sigma g t}{\left( 1 + \sqrt{1 + \sigma g t} \right)^2} \right]^p = \sum_{k=0}^{\infty} \frac{Z^{(N)}_k(g t)^k}{\Gamma(k + b_0 + 1)}. \quad (23)$$
Introducing the new variable

\[ w \equiv \frac{\sigma gt}{(1 + \sqrt{1 + \sigma gt})^2} = \frac{\sqrt{1 + \sigma gt} - 1}{\sqrt{1 + \sigma gt} + 1}, \] (24)

we obtain from (23) the relation valid for all \( \alpha \):

\[ \sum_{p=0}^{N} a_p w^p = \sum_{k=0}^{\infty} \frac{Z_k^{(N)}}{(b_0 + 1)_k} \left( \frac{4}{\sigma} \right)^k \frac{w^k}{(1 - w)^{2(k-\alpha)}}. \] (25)

In order to compare equal powers in \( w \) we expand on the right hand side

\[ (1 - w)^{-2(k-\alpha)} = \sum_{l=0}^{\infty} \left( -\frac{2(k - \alpha)}{l} \right) (-w)^l \] (26)

which gives after a shift of the summation index from \( l \) to \( p = k + l \) the first \((N + 1)\) coefficients \( a_p \) in terms of the perturbation coefficients \( Z_k \)

\[ a_p = \sum_{k=0}^{p} \frac{Z_k}{(b_0 + 1)_k} \left( \frac{4}{\sigma} \right)^k \left( -\frac{2(k - \alpha)}{p - k} \right) (-1)^{p-k} \] (27)

(recall that \( Z_k^{(N)} = Z_k \) for \( k = 0, 1, \ldots, N \)). Finally, rewriting the binomial coefficients by means of the identity

\[ \left( \begin{array}{c} x \\ p \end{array} \right) = (-1)^p \left( \begin{array}{c} p - x - 1 \\ p \end{array} \right), \] (28)

we obtain the more convenient expression

\[ a_p = \sum_{k=0}^{p} \frac{Z_k}{(b_0 + 1)_k} \left( \frac{4}{\sigma} \right)^k \left( \frac{p + k - 1 - 2\alpha}{p - k} \right). \] (29)

Thus, we have solved the original matrix inversion problem (21) by translating it to a simple problem in function theory, namely that of inverting the function \( w(\sigma gt) \) in (24). For the purpose of calculating the integrals \( I_p(g) \) numerically, we may use the variable \( w \) itself as a variable of integration, and rewrite the integral representation for \( I_p(g) \) in the form:

\[ I_p(g) = \left( \frac{4}{\sigma g} \right)^{b_0+1} \int_{0}^{1} dw \frac{(1 + w)w^{b_0+p}}{\Gamma(b_0 + 1)(1 - w)^{2b_0+2\alpha+3}} \exp \left[ -\frac{4w}{(1 - w)^2 \sigma g} \right]. \] (30)

Together with the explicit formula (29) for the coefficients \( a_p \) we thus have solved the resummation, and it is now straightforward to calculate the approximation (7).
3 Model integral

In order to set up an approximation method for an anisotropic model in the neighborhood of the isotropic point $\delta = 0$, it is instructive to study first a simple toy model whose partition function is defined by a two-dimensional integral:

$$Z = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx dy \exp \left\{ -\frac{1}{2} (x^2 + y^2) - \frac{g}{4} \left[ x^4 + 2(1 - \delta)x^2y^2 + y^4 \right] \right\}. \quad (31)$$

This can be interpreted as a partition function of a $\phi^4$-theory in zero spacetime dimensions with cubic anisotropy. Introducing polar coordinates $x = r \cos \varphi$ and $y = r \sin \varphi$, we obtain the more convenient form of the integral (31):

$$Z = \frac{1}{2\pi} \int_{0}^{\infty} d\rho \int_{0}^{2\pi} d\varphi \exp \left[ -\rho - G(g, \delta, \varphi)\rho^2 \right] \quad (32)$$

with $\rho = r^2/2$ and

$$G(g, \delta, \varphi) = g \left[ 1 - \frac{\delta}{2} \sin^2(2\varphi) \right]. \quad (33)$$

After an integration over the angle $\varphi$, we find the integral

$$Z = \int_{0}^{\infty} d\rho \exp \left[ -\rho - g \left( 1 - \frac{\delta}{4} \right)\rho^2 \right] I_0 \left( \frac{\delta}{4}g\rho^2 \right), \quad (34)$$

where $I_0(x)$ is a modified Bessel function $I_\nu(x)$ for $\nu = 0$. Eq.(34) is useful for a numerical calculation of $Z(g, \delta)$. It will serve as a testing ground for our approximations.

Thanks to the special spacetime dimensionality of the model, the perturbation expansion of $Z(g, \delta)$ can be obtained explicitly and we can calculate the large-order behavior without doing the saddle point approximation, which is unavoidable in quantum mechanics and field theory.

At first glance it seems useful to expand $Z(g, \delta)$ in the form

$$Z(g, \delta) = \sum_{k=0}^{\infty} Z_k(\delta)g^k, \quad (35)$$
where the perturbation coefficients are parametrized by the anisotropy $\delta$. The coefficients $Z_k(\delta)$ may be found by expanding the integrand of (32) in powers of $g$ and performing the integral term by term:

$$Z_k(\delta) = \frac{(-1)^k}{k!} \Gamma(2k + 1) \left(1 - \frac{\delta}{2}\right)^{k/2} P_k \left(\frac{4 - \delta}{2\sqrt{4 - 2\delta}}\right),$$

(36)

where $P_k(x)$ are the Legendre polynomials.

In Figure 1 we have plotted the order dependence of these coefficients for the anisotropy parameter $\delta = 10^{-2}$. What we can see is a crossover of the large-order behavior from an isotropic to an anisotropic regime in the vicinity of an special crossover value $k_{\text{cross}} \sim 1/|\delta| = 10^2$. In the anisotropic regime $k|\delta| \gg 1$, the large-order parameter $\beta$ has the value $\beta = -1$. For $k|\delta| \ll 1$, however, we can read off the large-order behavior of the isotropic case, i.e. $\beta = -1/2$ (see also Figure 2).

Using the duplication formula for Gamma-functions and the expansion

$$\Gamma(k + \varepsilon + 1) = k!k^\varepsilon[1 + \mathcal{O}(1/k)],$$

(37)

the large-$k$ behavior of $\Gamma(2k + 1)$ is given by

$$\Gamma(2k + 1) = \pi^{-1/2}4^k(k!)^2k^{-1/2}[1 + \mathcal{O}(1/k)].$$

(38)

An approximation of the Legendre Polynomials $P_k(x)$ for large $k$ including contributions of the order $\mathcal{O}(1/k)$ can be derived from Hobson (see Ref. [19], page 305):

$$P_k(x) = (2\pi k)^{-1/2} \left(x^2 - 1\right)^{-1/4} \left(x + \sqrt{x^2 - 1}\right)^{k+1/2} \times \left[1 + \frac{1 - \sqrt{x^2 - 1}}{8k} \left(x + \sqrt{x^2 - 1}\right) + \mathcal{O}(1/k^2)\right].$$

(39)

Substituting

$$x = \frac{4 - \delta}{2\sqrt{4 - 2\delta}},$$

(40)

we obtain for $\delta > 0$

$$P_k\left(\frac{4 - \delta}{2\sqrt{4 - 2\delta}}\right) = \sqrt{\frac{2}{\pi}} k^{-1/2} \delta^{-1/2} \left(\frac{1}{1 - \delta/2}\right)^{k/2} \left[1 + \frac{1}{k\delta} \left(\frac{4 - 3\delta}{8}\right) + \mathcal{O}\left(\frac{1}{k^2\delta^2}\right)\right].$$

(41)
Figure 1: Crossover of large-order behavior of the expansion coefficients $Z_k$ in Eq. (35) from the isotropic regime ($\beta = -1/2$) to the anisotropic regime ($\beta = -1$). Plotted is the function $f(k) = \ln \left[ \frac{Z_k}{(-4)^k k!} \right]$ for the anisotropy $\delta = 10^{-2}$. In this case the crossover value is given by $k_{\text{cross}} \sim 1/|\delta| = 10^2$ ($\ln k_{\text{cross}} \approx 4.6$).
Figure 2: Example for the two different large-order regimes of $Z_k$ in Eq. (35), where $f(k)$ is the same function as in Fig. 1.  

a) $\delta = 10^{-4}$, $\beta = -1/2$

b) $\delta = 1.0$, $\beta = -1$

Figure 2: Example for the two different large-order regimes of $Z_k$ in Eq. (35), where $f(k)$ is the same function as in Fig. 1.  

a): Isotropic regime ($k_{\text{cross}} = 10^4$, $\beta = -1/2$).  
b): Anisotropic regime ($k_{\text{cross}} = 1$, $\beta = -1$).
The combination of (38) and (41) yields the large-order behavior of the perturbation coefficients $Z_k(\delta)$:

$$Z_k(\delta) = \frac{2^{1/2}}{\pi} (-1)^k 4^k k! k^{-1} \delta^{-1/2} \left\{ 1 + \frac{1}{k\delta} \left[ \frac{1}{2} + \mathcal{O}(\delta) \right] + \mathcal{O} \left( \frac{1}{k^2 \delta^2} \right) \right\}.$$  (42)

A similar calculation can be done for $\delta < 0$ with the result:

$$Z_k(\delta) = \frac{(2 - \delta)^{1/2}}{\pi} (-1)^k (4 - 2\delta)^k k! k^{-1} (-\delta)^{-1/2} \times \left\{ 1 - \frac{1}{k\delta} \left[ \frac{1}{2} + \mathcal{O}(\delta) \right] + \mathcal{O} \left( \frac{1}{k^2 \delta^2} \right) \right\}.$$  (43)

For resumming the series (35), the perturbation coefficients (42) and (43) can be used only for $k|\delta| \gg 1$. In the regime $k|\delta| \ll 1$, on the other hand, it is impossible to truncate the series in (42) and in (43) after a finite order of $1/k|\delta| \gg 1$. Thus, the isotropic regime cannot be described by resumming the perturbation series (35) using the asymptotic results (42) and (43). Being interested in the region close to the isotropic limit, we therefore use an expansion different from (35), and rewrite $Z(g, \delta)$ as

$$Z(g, \delta) = \sum_{k=0}^{\infty} \sum_{n=0}^{k} Z_{kn} g^k \delta^n.$$  (44)

Then, for the reason given in [8], reasonable results should be obtained by resumming the $g$-series accompanying each power $\delta^n$. The explicit form of the coefficients $Z_{kn}$ is given by

$$Z_{kn} = \pi^{-1/2} (-1)^{k+n} \frac{\Gamma \left( n + \frac{1}{2} \right) \Gamma(2k + 1)}{2^n \Gamma(n + 1) \Gamma(k - n + 1)},$$  (45)

and $Z_{kn} = 0$ for $k < n$. For these coefficients, the expansion of the Gamma-functions yields the behavior for large $k \gg n$:

$$Z_{kn} = (-1)^n \frac{\Gamma \left( n + \frac{1}{2} \right) \Gamma(k - n + 1)}{2^n \Gamma(n + 1) \Gamma(k - n + 1)} \left[ 1 + \mathcal{O}(1/k) \right].$$  (46)

In the following we shall calculate the coefficients (46) by means of the steepest descent method using the saddle points of the limit $\delta \to 0$. This
will serve as a simple preparation for the analogous method in quantum mechanics and field theory.

As a function of a $\delta$ and a complex coupling $g$, the integral (31) is defined in the half-plane $\text{Re } g \geq 0$. For $\text{Re } g < 0$, the integral can be calculated by an analytical continuation from the right into the left half-plane, keeping the integrand in (31) real. This analytical continuation can be achieved by a joint rotation in the complex $g$-plane and of the integration contour in the $\vec{r} = (x, y)$-plane. The convergence of the integral is maintained by the substitution $g \to g \exp(i\theta)$ and $\vec{r} \to \vec{r}\exp(-i\theta/4)$, where $\theta$ is the rotation angle in the complex $g$-plane. Let us assume that the function $Z(g, \delta)$ is analytic in the $g$-plane, with a cut along the negative $g$-axis, and a discontinuity for $g < 0$. Then the rotation in the complex $g$-plane by an angle $\theta = \pm \pi$ yields on the lower lip of the cut:

$$Z_{\pm}(-|g|, \delta) = Z(|g|e^{\pm i\pi}, \delta).$$

(47)

The corresponding rotated integration contours ($\Gamma_{\pm}$) are drawn in Figure 3. The discontinuity across the cut is given by

$$\text{Disc } Z = \int_0^{2\pi} \frac{d\varphi}{2\pi} \int_{\Gamma} r dr \exp \left[ -\frac{r^2}{2} - G(g, \delta, \varphi)\frac{r^4}{4} \right],$$

(48)

where the combined contour $\Gamma = \Gamma_{+} + \Gamma_{-}$ runs for $r > 0$ entirely through the right half-plane.

In a perturbatively expansion in powers of $\delta$, the discontinuity can be computed from an expansion around the saddle point

$$r_0 = \sqrt{\frac{1}{|g|}}$$

(49)

of the isotropic case $\delta = 0$. Since $r > 0$, only the positive square root contributes, the negative one is automatically taken into account by the integration over the angle $\varphi$. Now, the contour of integration $\Gamma$ in the right half-plane can be deformed to run vertically across the saddle point (see Figure 4), i.e., we can integrate along a straight line:

$$r = \sqrt{\frac{1}{|g|}} - i\xi.$$
Figure 3: Analytic continuation $g \rightarrow |g| \exp(\pm \pi)$: a) Rotation by angles $\pm \pi$ in the cut complex $g$-plane. b) Two rotated paths of integration in the $r$-plane ($r > 0$).

Figure 4: Deformation of the contours of integration to make them pass through the saddle point $r_0$. 

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The exponent in (48) plays the role of an action, and the deviations $\xi$ may be considered as fluctuations around the extremal solution. The angle $\varphi$ is analogous to a collective coordinate along the motion of the instanton in the isotropic limit. Expanding the action up to the second order in $\xi$ around the extremum of the isotropic action, we obtain

$$\text{Disc} Z = -\frac{i}{2\pi} \left( \frac{1}{|g|} \right)^{1/2} \int_{-\infty}^{+\infty} \int_{0}^{2\pi} d\xi d\varphi \times \exp \left[ -\frac{1}{4|g|} - \frac{\delta}{8|g|} \sin^2 (2\varphi) - \xi^2 + O \left( \frac{\delta}{\sqrt{|g|}} \right) \right].$$ \hspace{1cm} (51)

Integrating out the fluctuations $\xi$ and the collective coordinate $\varphi$, and using the equation

$$\text{Im} Z = \frac{1}{2i} \text{Disc} Z \hspace{1cm} (52)$$

we obtain the following imaginary part for $Z$:

$$\text{Im} Z = -\sum_{n=0}^{\infty} (-1)^n \delta^n \frac{\Gamma(n + \frac{1}{2})}{2^n \Gamma(n + 1)^2} \left( \frac{1}{4|g|} \right)^{n+1/2} \exp \left( -\frac{1}{4|g|} \right) \left[ 1 + O(g) \right].$$ \hspace{1cm} (53)

Each power $\delta^n$ has its own $n$-dependent imaginary part. Given such an expansion, the large-order estimates for the coefficients $Z_{kn}$ (with $k \ll n$) follows from a dispersion relation in $g$ (see for example Eq. (6) in Ref. [7])

$$Z_{kn} = \frac{1}{\pi} \int_{-\infty}^{0} dg \frac{\text{Im} Z^{(n)}(g + i0)}{g^{k+1}},$$ \hspace{1cm} (54)

where $Z^{(n)}(g)$ is the coefficient of $\delta^n$. In general if a real analytic function $F(g)$ has on top of the cut along $g \in (-\infty, 0)$ an imaginary part

$$\text{Im} F(g + i0) = -\pi \gamma \left( \frac{1}{\sigma |g|} \right)^{\beta+1} \exp \left( -\frac{1}{\sigma |g|} \right) \left[ 1 + O(g) \right],$$ \hspace{1cm} (55)

then a dispersion relation of the form (54) leads to the asymptotic behavior

$$F_k = \gamma (-1)^{k} \sigma^{k} k^{\beta} k! \left[ 1 + O(1/k) \right].$$ \hspace{1cm} (56)

With $\sigma = 4$ and $\beta = n - 1/2$, we obtain again the result (46).
Thus, the steepest descent method using the isotropic saddle point is a perfect tool for calculating the large-order behavior of the expansion coefficients $Z_{kn}$ in the expansion (44). A great advantage of this method with respect to the exact calculation (45) is the fact that it can be generalized to quantum mechanics and field theory where exact calculations would be unpossible.

Before applying the resummation algorithm of the previous section we have to study the strong-coupling behavior, i.e., the limit of large $g$. This can simply be done by rescaling the integral (32)

$$Z = \int_0^\infty dy \int_0^{2\pi} d\varphi G(g, \delta, \varphi)^{-1/2} \exp \left( -\frac{y}{\sqrt{G(g, \delta)}} - y^2 \right)$$  \hspace{1cm} (57)

with $G$ from (33) and $y = \rho \sqrt{G}$. Taking the limit of large $g$ (i.e., large $G$) and integrating out the angle $\varphi$ we find

$$Z(g, \delta) \xrightarrow{g \to \infty} \kappa(\delta) g^{-1/2}$$  \hspace{1cm} (58)

with

$$\kappa(\delta) = \frac{\pi^{1/2}}{2} \sum_{n=0}^{\infty} \frac{(2n)!^2}{(n!)^4 2^{5n}} \delta^n.$$  \hspace{1cm} (59)

Now, a resummation of the $g$-series in (44) yields a generalization of (7):

$$Z^{(N)}(g, \delta) \equiv \sum_{n=0}^{N} \left( \sum_{p=n}^{N} a_{pn} I_{pn}(g) \right) \delta^n$$  \hspace{1cm} (60)

with the complete set of Borel summable functions

$$I_{pn}(g) = \left( \frac{4}{\sigma g} \right)^{b_0(n)+1} \int_0^1 dw \frac{(1 + w) w^{b_0(n)+p}}{\Gamma [b_0(n) + 1] (1 - w)^{2b_0(n)+2\alpha+3}} \exp \left[ -\frac{4w}{(1-w)^2 \sigma g} \right],$$  \hspace{1cm} (61)

and the coefficients

$$a_{pn} = \sum_{k=n}^{p} \frac{Z_{kn}}{(b_0(n) + 1)_k} \frac{4^k}{\sigma^k} \binom{p+k-1-2\alpha}{p-k},$$  \hspace{1cm} (62)
where the perturbation coefficients $Z_{kn}$ are given by (45). The parameters $b_0(n)$, $\sigma$ and $\alpha$ follow from the large-order behavior (46) and the strong-coupling expansion (58), respectively:

$$b_0(n) = n + 1$$
$$\sigma = 4$$
$$\alpha = -\frac{1}{2}.$$  \hspace{1cm} (63)

From (62) it is possible to derive the following closed formula for the coefficients $a_{pn}$:

$$a_{pn} = \frac{2^n \Gamma \left( n + \frac{1}{2} \right)^2 \Gamma(n + 2)}{\pi \Gamma(2n + 2)} (-1)^p (p - n)! \left( \frac{2\alpha + 1}{p! n!} \right) \Gamma(2n + 2) \left( -p \right)^n \left( p - n \right)! \frac{2^n}{n!} \left( 2\alpha + 1 \right).$$  \hspace{1cm} (64)

Inserting the exact strong-coupling parameter $\alpha = -1/2$, we obtain

$$a_{pn} = \begin{cases} 
1 \frac{(n + 1)}{8^n} \frac{2n!}{(2n + 1) (n!)^2} & \text{for } p = n, \\
0 & \text{else}.
\end{cases}$$  \hspace{1cm} (65)

Thus we have a general result that the approximants $\sum_{p=n}^{N} a_{pn} I_{pn}(g)$ in (60) posses no terms with $p > n$, where $n$ is the power of $\delta$. In such a way the $n$-dependent functions of $g$ associated to each $\delta^n$ are recovered exactly.

The approximation $Z^{(N)}(g, \delta)$ may then be compared with the numerically calculated integral (34). In Figures 5 and 6 we have shown the result for various $N$ and coupling constants $g/4$.

### 4 Quantum mechanics

After the integral model, the simplest nontrivial example for our approximation method is a $\phi^4$-theory in one spacetime dimension with a cubic anisotropy, which is equivalent to the quantum mechanics of an anisotropic anharmonic oscillator.
Figure 5: Partition function $Z$ of the simple integral model as a function of the anisotropy parameter $\delta$ with the coupling constant $g/4 = 0.25$. Comparison is made between the precise numerical result and the resummed perturbation series $Z^{(N)}$ [see Eq. (60)] for various order $N$. 
Figure 6: The same functions as in Fig. 5, but with $g/4$ equal to 2.5.
4.1 Recursions relations for the ground-state perturbation coefficients

Consider an anharmonic oscillator with cubic anisotropy and a Hamilton operator

\[ H = -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{\omega^2}{2} (x^2 + y^2) + \frac{g}{4} \left[ x^4 + 2(1 - \delta)x^2y^2 + y^4 \right]. \]  

(66)

Introducing reduced variables by a rescaling

\[ x \rightarrow \sqrt{\frac{1}{\omega}} x, \quad y \rightarrow \sqrt{\frac{1}{\omega}} y, \]
\[ g \rightarrow \omega^3 g, \quad E^{(0)} \rightarrow \omega E^{(0)}, \]

(67)

yields the dimensionless time-independent Schrödinger equation

\[ \left[ -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{2} r^2 + \frac{g}{4} \left( r^4 - 2\delta x^2 y^2 \right) \right] \Psi_n(x, y) = E^{(n)} \Psi_n(x, y). \]

(68)

with the associated boundary condition

\[ |\Psi_n(x, y)| \rightarrow 0, \quad r = \sqrt{x^2 + y^2} \rightarrow \infty. \]

(69)

The boundary condition selects only the discrete energy eigenvalues \( E^{(n)} \).

We now consider the ground-state energy \( E^{(0)} = E \), whose perturbation expansion has the form

\[ E = \sum_{m=0}^{\infty} \sum_{l=m}^{\infty} \left( \frac{g}{4} \right)^l (2\delta)^m E_{lm} \]

(70)

with the unperturbed ground-state energy \( E_{00} = 1 \). In the following, we refer to (70) as the Rayleigh-Schrödinger series and the \( E_{lm} \) as a Rayleigh-Schrödinger coefficient.

In general, the ground-state energy is available from the sum of all connected Feynman diagrams having no external legs. For an efficient computation of the Rayleigh-Schrödinger coefficients at large orders we shall derive recursions for the \( E_{lm} \) following a method introduced by Bender and Wu
In this way we obtain a difference equation generating the Rayleigh-Schrödinger coefficients.

Separating out the unperturbed ground-state wave function, \( \Psi_0(x, y) = \exp[-(x^2 + y^2)/2] \), we substitute

\[
\Psi(x, y) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left(-\frac{g}{4}\right)^k (2\delta)^n \exp[-(x^2 + y^2)/2] \Phi_{kn}(x, y),
\]

(71)

where \( \Phi_{kn}(x, y) \) is polynomial in \( x, y \) with \( \Phi_{00} = 1 \). Inserting the perturbation expansions (70), (71) into the differential equation (68), and collecting powers of \( g \) and \( \delta \), we find

\[
\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \Phi_{kn} + \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \Phi_{kn} - r^4 \Phi_{k-1,n} + x^2 y^2 \Phi_{k-1,n-1} = \\
\sum_{l=1}^{k} (-1)^l E_{l0} \Phi_{k-l,n} + \sum_{m=1}^{n} \sum_{l=1}^{k} (-1)^l E_{lm} \Phi_{k-l,n-m}
\]

(72)

Finally, the ansatz

\[
\Phi_{kn}(x, y) = \sum_{i,j=0}^{2k-n} A_{ij}^{kn} x^{2i} y^{2j}
\]

(73)

with

\[
A_{ij}^{kn} = 0 \quad \text{for} \quad i, j > 2k - n; \quad i, j, k, n < 0; \quad k < n
\]

(74)

gives the desired difference equation

\[
2(i + j)A_{ij}^{kn} = (2i + 1)(i + 1)A_{i+1,j}^{kn} + (2j + 1)(j + 1)A_{i,j+1}^{kn} \\
+ A_{i-2,j}^{kn} + A_{i,j-2}^{kn} + 2A_{i-1,j-1}^{kn} - A_{i-1,j+1}^{kn} \\
- \sum_{l=1}^{k} \left(A_{10}^{l0} + A_{01}^{l0}\right) A_{ij}^{k-l,n} - \sum_{m=1}^{n} \sum_{l=1}^{k} \left(A_{10}^{lm} + A_{01}^{lm}\right) A_{ij}^{k-l,n-m}.
\]

(75)

The \( A_{ij}^{kn} \) yield the desired Rayleigh-Schrödinger coefficients \( E_{kn} \) via the simple formula

\[
E_{kn} = -(-1)^k \left(A_{10}^{kn} + A_{01}^{kn}\right).
\]

(76)

These can be determined recursively via (75). The recursion must be initialized with

\[
A_{00}^{kn} = \delta_{k0} \delta_{n0},
\]

(77)
and solved at increasing $k = 0, 1, 2, \ldots$; $n = 0, 1, 2, \ldots, k$ and, for each set $k$ and $n$, with decreasing $i = 2k - n, \ldots, 0$; $j = 2k - n, \ldots, 0$ (omitting $i=j=0$). The procedure is most easily performed with the help of an algebraic computer program. The list of the first Rayleigh-Schrödinger coefficients up to $k = 12$ ($n = 0, \ldots, k$) is given Table 1.

4.2 Large-order coefficients and resummation

Working with Langer’s formulation [20] (which is related to Lipatov’s [21] by a dispersion relation) and making use of known results for the isotropic anharmonic oscillator, we shall derive the large-order behavior of perturbation expansion for the ground-state energy.

The method is based on the path-integral representation of the quantum partition function

$$Z = \int \mathcal{D}x \mathcal{D}y \exp[-A(x, y)] \xrightarrow{\beta \to \infty} \exp(-\beta E)$$ (78)

where

$$A = \int_{-\beta/2}^{+\beta/2} d\tau \left\{ \frac{1}{2} \left( \dot{x}^2 + \dot{y}^2 \right) + \frac{1}{2} \left( x^2 + y^2 \right) + \frac{g}{4} \left[ x^4 + 2(1 - \delta)x^2y^2 + y^4 \right] \right\}$$ (79)

is the Euclidean action corresponding to the Hamiltonian (66). For $g > 0$, the system is stable and $Z$ is real. On the other hand, if the coupling constant $g$ is negative the system becomes unstable and $Z$ develops an exponentially small imaginary part related to the decay-rate $\Gamma$ of the ground-state resonance. The imaginary part of the ground-state energy may be obtained by taking the large $\beta$ limit in (78),

$$\text{Im} E = -\frac{1}{2} \frac{\Gamma}{\beta} \text{Re} Z, \quad \beta \to \infty.$$ (80)

In the above equation the fact was used that $\text{Im} Z \propto \exp(-\beta) \exp[-1/(\sigma |g|)]$ is much smaller than $\text{Re} Z = \exp\{-\beta[1 + \mathcal{O}(g)]\}$. For small $g < 0$, this imaginary part can be computed perturbatively in the anisotropic parameter $\delta$ by the expansion around the isotropic instanton solution $r_c(\tau)$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix} (r_c + \xi) + \begin{pmatrix} -\sin \varphi \\ \cos \varphi \end{pmatrix} \eta, \quad r_c = \sqrt{\frac{2}{|g|} \frac{1}{\cosh(\tau - \tau_0)}}.$$ (81)
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Table 1: Coefficients $E_{kn}$ in the perturbation series (70) for the ground-state energy up to $k = 12$ ($n = 0, \ldots, k$).
For simplicity, we shall set $\tau_0 = 0$ in the sequel. In (81) we have separated out the rotation angle $\varphi$ of the isotropic instanton in the $(x,y)$-plane and $\xi, \eta$ are the degrees of freedom, orthogonal to this rotation. Inserting the expansion (81) into the action (79) we obtain the expression

$$A = \frac{4}{3|g|} + \frac{\delta}{|g|} \frac{2\sin^2(2\varphi)}{3} + \frac{1}{2} \int d\tau \left[ \xi \left( -\frac{d^2}{d\tau^2} + 1 - \frac{6}{\cosh^2 \tau} \right) \xi + \eta \left( -\frac{d^2}{d\tau^2} + 1 - \frac{2}{\cosh^2 \tau} \right) \eta \right] + \mathcal{O}(\delta) \left( \frac{\delta}{\sqrt{|g|}} \right),$$

(82)

where we have splitted of the action into the terms responsible for the leading contributions in an expansion of the form (53) and a remainder $\mathcal{O}(\delta/\sqrt{|g|})$. Then the $\delta$-dependence of the transversal quadratic fluctuations belongs to the omitted terms. Expanding (78) in $\delta$ and integrating out the quadratic fluctuations we obtain

$$Z = f_\xi f_\eta \int_0^{2\pi} d\varphi \sum_{n=0}^{\infty} \left( -\frac{\delta}{4} \right)^n \left( 2\sin^2(2\varphi) \right)^n \left( \frac{4}{3|g|} \right)^n \exp \left( -\frac{4}{3|g|} \right) \left[ 1 + \mathcal{O}(g) \right]$$

(83)

where the angle integral can be done with

$$\int_0^{2\pi} d\varphi \left( 2\sin^2(2\varphi) \right)^n = 8^n \frac{\Gamma^2 \left( n + \frac{1}{2} \right)}{\Gamma \left( 2n + 1 \right)} = 8^n 2B \left( n + \frac{1}{2}, n + \frac{1}{2} \right).$$

(84)

The contribution $f_\xi$ and $f_\eta$ from the quadratic longitudinal and transversal fluctuations coincide with those appearing in the isotropic oscillator problem and are therefore known. With the isotropic classical action $A_{0c} = 4/(3|g|)$ the well-known results are

$$f_\xi = -\frac{i}{2} \sqrt{\frac{A_{0c}}{2\pi}} \beta \left| \frac{\det'(-d^2/d\tau^2 + 1 - 6/\cosh^2 \tau)}{\det(-d^2/d\tau^2 + 1)} \right|^{-1/2} Z_{osc}$$

$$= -\frac{i}{2} \sqrt{\frac{A_{0c}}{2\pi}} \beta \sqrt{12} \exp(-\beta/2)$$

26
and
\[
\eta = \sqrt{3} A_0 c^2 \pi \left[ \det \left( -d^2 / d\tau^2 + 1 - 2/cosh^2 \tau \right) \right]^{-1/2} Z_{osc}
\]
\[
= 2 \sqrt{3} A_0 c \exp(-\beta/2)
\]
(85)

where we have used the partition function of the harmonic oscillator

\[
Z_{osc} \equiv \det \left( -d^2 / d\tau^2 + 1 \right)^{-1/2} = \frac{1}{2 \sinh(\beta/2)} \exp(-\beta/2)
\]
(86)

to normalize the determinants. In the upper determinants, the zero eigenvalues are excluded. This fact is recorded by the prime.

The longitudinal fluctuations \( \xi \) contain a negative eigenmode, this being responsible for the factor \(-i/2\) and the absolute value sign, and a zero eigenmode associated with the translation invariance which is spontaneously broken by the special choice \( \tau_0 = 0 \). The separation of this zero eigenmode in the framework of collective coordinates yields the factor \( \beta \sqrt{A_0c/(2\pi)} \) (see Chapter 17 in Ref. [11]). Collecting the contributions of the negative and all positive eigenmodes one obtains the remaining factor \( \sqrt{12} \).

In contrast to the longitudinal case the transversal fluctuations \( \eta \) do not contain any negative mode. The transversal fluctuation operator has one zero eigenvalue due to the rotational invariance in the limit \( \delta \to 0 \). The associated eigenmode is extracted from the integration measure via the change of variables (81). The Jacobian of this coordinate transformation can be deduced from the isotropic system. It contributes the factor \( \sqrt{3A_0c/(2\pi)} \). The remaining factor 2 results from all other modes with positive eigenvalues.

Collecting all contributions to the imaginary part of the ground-state energy (80), an cancellation of all \( \beta \) dependent factors leads to:

\[
\text{Im} E \xrightarrow{g \to 0} \frac{2 |f_\xi f_\eta|}{\beta \exp(-\beta)} \sum_{n=0}^{\infty} \frac{(-2\delta)^n}{n!} B \left( n + \frac{1}{2}, n + \frac{1}{2} \right) \left( \frac{4}{3|g|} \right)^n \exp \left( -\frac{4}{3|g|} \right)
\]
\[
= \frac{6}{\pi} \sum_{n=0}^{\infty} \frac{(-2\delta)^n}{n!} B \left( n + \frac{1}{2}, n + \frac{1}{2} \right) \left( \frac{4}{3|g|} \right)^{n+1} \exp \left( -\frac{4}{3|g|} \right).
\]
(87)

Finally, by means of the dispersion relation (54) we find the corresponding
large-order behavior of the coefficients in the expansion (70):

\[ E_{kn} \xrightarrow{k \to \infty} -\frac{6}{\pi^2} \frac{(-2)^n}{n!} B \left( n + \frac{1}{2}, n + \frac{1}{2} \right) (-1)^k \left( \frac{3}{4} \right)^k k^k k^n. \]  

After having derived the large-order behavior of \( E_{kn} \) and the low-order perturbation coefficients via the Bender and Wu-like recursions (75) and (76), we are in the position to resum the \( g \)-series accompanying each power \( \delta^n \) in the expansion (70).

The remaining strong-coupling expansion follows from Symanzik scaling [22]:

\[ E(g, \delta) = \sum_{m=0}^{\infty} \kappa_m(\delta) g^{(1-2m)/3} \]  
i.e., the power behavior in the strong-coupling limit is given by

\[ E(g, \delta) \xrightarrow{g \to \infty} \kappa_0(\delta) g^{1/3}, \]

where the 1/3 coincides with that appearing in the one-dimensional oscillator problem. The \( \delta \)-dependence enters by the prefactor \( \kappa_0 \).

Combining (17) and the formulas (60), (61) and (62), the resummation procedure must be worked through with the parameters

\[ b_0(n) = n + \frac{3}{2} \]
\[ \sigma = \frac{3}{4} \]
\[ \alpha = \frac{1}{3}. \]

In Figures 7 and 8 we have plotted the \( \delta \)-dependence of the resummed ground-state energy \( E \) for two different values of the coupling constant \( g/4 \) and for various orders \( N \). For a reference plot we have used the very accurate dotted curve which we have obtained numerically from the variational perturbation theory described in the next subsection.

### 4.3 Variational perturbation theory

It is useful to compare the above results with those of another recently-developed resummation procedure known as variational perturbation theory.
Figure 7: Ground-state energy $E$ of the anisotropic anharmonic oscillator with $g/4 = 0.1$ as a function of the anisotropy $\delta$. Shown are the resummed perturbation series for various order of approximation $N$ and the approximation $W_5(\Omega_5)$ (see the text after Eq. (99)) from the variational perturbation theory (VPT). Differences between $W_5(\Omega_5)$ and the exact ground-state energy are existent only on a finer energy scale.
Figure 8: The same functions as in Fig. 7, but with $g/4 = 1.0$. 
[for an introduction see Ref. [11], Chapter 5]. Consider first the Rayleigh-
Schrödinger expansion of the ground-state energy:

\[ E(g, \delta) = \omega \sum_{l=0}^{\infty} \sum_{m=0}^{l} E_{lm}(2\delta)^m \left( \frac{g/4}{\omega^3} \right)^l, \]  

(92)

where the Rayleigh-Schrödinger coefficients \( E_{lm} \) are obtained from the recursion relation (75) via (76).

The variation is done as follows: First, the potential is separated into an arbitrary harmonic term and a remainder:

\[ \frac{\omega^2}{2} (x^2 + y^2) = \frac{\Omega^2}{2} (x^2 + y^2) + \frac{\omega^2 - \Omega^2}{2} (x^2 + y^2). \]  

(93)

In contrast to ordinary perturbation theory, an interacting potential \( V_{\text{int}} \) is defined by

\[ V(x, y) = \frac{\Omega^2}{2} (x^2 + y^2) + V_{\text{int}}(x, y) \]  

(94)

and setting

\[ V_{\text{int}}(x, y) = \frac{g}{4} \left( \rho r^2 + r^4 - 2\delta x^2 y^2 \right) \quad ; \quad \rho = \frac{2}{g} \left( \omega^2 - \Omega^2 \right). \]  

(95)

A perturbation expansion is now found in powers of \( g \) at fixed \( \rho \) and \( \delta \):

\[ E_k(g, \delta, \rho) = \Omega \sum_{l=0}^{k} \epsilon_l(\rho, \delta) \left( \frac{g/4}{\Omega^3} \right)^l. \]  

(96)

The calculation of the new coefficients \( \epsilon_l(\rho, \delta) \) up to a specific order \( k \) does not require much additional work, since they are easily obtained from the ordinary perturbation series (92). We simply replace \( \omega \) by the identical expression

\[ \omega = \sqrt{\Omega^2 + \omega^2 - \Omega^2} = \sqrt{\Omega^2 + g\rho/2}, \]  

(97)

reexpand \( E(g, \delta) \) in powers of \( g \), and truncate the series after an order \( l > k \). This yields the reexpansion coefficients

\[ \epsilon_l(\rho, \delta) = \sum_{j=0}^{l} \sum_{n=0}^{j} E_{jn}(2\delta)^n \left( \frac{(1-3j)/2}{l-j} \right) \left( \frac{2\rho\Omega}{l-j} \right)^{l-j}. \]  

(98)
The truncated power series

\[ W_k(g, \delta, \Omega) := E_k \left[ g, \delta, 2 \left( \omega^2 - \Omega^2 \right) / g \right] \]  (99)

is certainly independent of \( \Omega \) for \( k \) going to infinity. However, at any finite order it depends on \( \Omega \). The optimal value of \( \Omega \) is found by calculating all extrema and the turning points. The smallest among these order-dependent points is used as an optimal trying value and is denoted by \( \Omega_k(g, \delta) \). The associated energy \( W_k[g, \delta, \Omega_k(g, \delta)] \) constitutes the desired approximation to the ground-state energy. In Figures 9 and 10 we have plotted the \( \Omega \)-dependence of \( W_{5,6} \) for various anisotropy parameters \( \delta \) at the coupling constant \( g/4 = 0.1 \).

The shape of the curves depends little on \( \delta \). Only in the case of odd \( k \) does a minimum exist. For even \( k \), there is no extremum and the optimal \( \Omega \)-value lies at a turning point.

For an isotropic \( gx^4 \)-model, the precision of the variational perturbation method has been illustrated by a comparison with accurate numerical energies [15]. With increasing \( k \) the approach of \( W_k \) to the exact energy is quite rapid and its mechanism is well understood.

In Table 2 we display the ground-state energies for odd \( k \), which we have obtained for the anisotropic model at various \( \delta \) and \( g/4 \). The convergence to fixed energy values is comparable to the case of the simple \( gx^4 \)-interaction. So we assume that these numbers coincide with the exact ground-state energy values at least up to the first four digits.

## 5 Summary

With the help of a simple model integral containing a quadratic and two quartic terms of different symmetry, we have investigated in detail the large-order behavior of the \( \delta \)-dependent \( g \)-series in a function \( f(g, \delta) = \sum_k f_k(\delta)g^k \) for the region near the isotropic limit \( \delta \rightarrow 0 \). We have shown that the large-order behavior of \( f_k(\delta) \) undergoes a crossover from the anisotropic to an isotropic regime near the order of perturbation theory \( k_{\text{cross}} \approx 1/|\delta| \).

In quantum mechanics, the extreme large-order behavior of perturbation theory for the anisotropic regime \( k|\delta| \gg 1 \) is identical with earlier results of BBW [6] and Janke [7]. In displaying the crossover-behavior we have gone beyond these earlier works.
Figure 9: The function $W_5$ at constant coupling strength $g/4 = 0.1$ for various anisotropy parameters $\delta$. 
Figure 10: The function $W_6$ for $g/4 = 0.1$ and various $\delta$. 
Table 2: Convergence of the ground-state energy in the variational perturbation expansion for various anisotropy parameters $\delta$. 

<table>
<thead>
<tr>
<th>$g/4$</th>
<th>$\delta$</th>
<th>-2.5</th>
<th>-1.5</th>
<th>-0.5</th>
<th>0.5</th>
<th>1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g/4=0.1$</td>
<td>1</td>
<td>1.222923</td>
<td>1.19626</td>
<td>1.167751</td>
<td>1.137</td>
<td>1.103438</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>1.217193</td>
<td>1.192062</td>
<td>1.164807</td>
<td>1.134739</td>
<td>1.100658</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>1.217109</td>
<td>1.192032</td>
<td>1.164801</td>
<td>1.134734</td>
<td>1.100607</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>1.217107</td>
<td>1.192033</td>
<td>1.164803</td>
<td>1.134735</td>
<td>1.100604</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>1.217107</td>
<td>1.192034</td>
<td>1.164810</td>
<td>1.134736</td>
<td>1.100604</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>1.217107</td>
<td>1.192035</td>
<td>1.164811</td>
<td>1.134739</td>
<td>1.100604</td>
</tr>
</tbody>
</table>

| $g/4=1.0$ | 1 | 1.969986 | 1.88556 | 1.791636 | 1.684863 | 1.559412 |
|           | 3 | 1.941934 | 1.863112 | 1.773978 | 1.669261 | 1.536823 |
|           | 5 | 1.941196 | 1.862803 | 1.773867 | 1.669156 | 1.535609 |
|           | 7 | 1.941172 | 1.862806 | 1.773888 | 1.669172 | 1.535454 |
|           | 9 | 1.941172 | 1.862815 | 1.773909 | 1.669188 | 1.535425 |
|           | 11| 1.94118  | 1.862823 | 1.773924 | 1.669199 | 1.535418 |
In particular, our resummation algorithm is shown to work very well in the vicinity of $\delta = 0$ and for $\delta > 0$, the latter being relevant to the question of a stable cubic fixed point in field theory. With increasing coupling constant $g/4$, the error of the result for the ground-state energy becomes larger. However, for $N = 6$ (this is the largest available order for the $\beta$-functions in quantum field theory, see Ref. [5]) and in the wide region $\delta \in (-0.5, 2)$ and $g/4 \in (0, 1)$ the error remains smaller than 0.8%. The increasing error for large negative values of $\delta$ can intuitively be understood by comparing the first two terms in the action (82): For $\delta < 0$, the “tunneling-paths” of extremal action are obviously straight lines along the two diagonales in the $(x,y)$-plane ($\varphi = \pi/4$). Along these diagonal rays, the basic factor $\exp[-1/(c|g|)]$ related to the decay-rate disappears for $\delta \to -2$, and the ensuing expansion of (78) in powers $\delta^n$ becomes meaningless. An improved fit for $\delta < 0$ can be obtained by choosing larger values of the large-order parameter $\sigma$. In Figures 11 and 12 we display the result for $\sigma = 3$ and $N = 6$, where for $g/4 = 0.1$ the accurate and the resummed curve coincide.

To obtain the correct description of the neighbourhood of the isotropic system $\delta = 0$ we have used the method developed in the context of an anisotropic quantum field theory in [8]: By replacing the series $\sum_k f_k(\delta)g^k$ by $\sum_n \sum_k f_{kn} g^k \delta^n$ and resumming the $g$-series accompanying each power $\delta^n$, we obtain very good results for the model integral and the ground-state energy of the anisotropic anharmonic oscillator. In this way our results justify the earlier field theoretic analysis and should be useful for understanding similar problems in other systems.
Figure 11: The same functions as in Fig. 7, but with the large-order parameter $\sigma = 3$ (explained in the text).
Figure 12: The same functions as in Fig. 8, but with the large-order parameter $\sigma = 3$. 
References


[18] See, for example, Handbook of Mathematical Functions, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965).


