On the relation between the connection and the loop representation of quantum gravity

Roberto De Pietri
Dipartimento di Fisica - Sezione di Fisica Teorica, Università di Parma, 43100 Parma, Italy and I.N.F.N., Sezione di Milano, Gruppo Collegato di Parma.
E-mail: depietri@vaxpr.pr.infn.it

Abstract. Using Penrose’s binor calculus for $SU(2)$ ($SL(2,C)$) tensor expressions, a graphical method for the connection representation of Euclidean Quantum Gravity (real connection) is constructed. It is explicitly shown that: (i) the recently proposed scalar product in the loop-representation coincide with the Ashtekar-Lewandowski cylindrical measure in the space of connections; (ii) it is possible to establish a correspondence between the operators in the connection representation and those in the loop representation. The construction is based on embedded spin network, the Penrose’s graphical method of $SU(2)$ calculus, and the existence of a generalized measure on the space of connections modulo gauge transformations.

PACS numbers: 04.60.-m, 02.70.-c, 04.60.Ds, 03.70+k.

Short title: On the relation between the connection and the loop representation

May 30, 1996
1. Introduction

Recently progress has been made in the Ashtekar[1] formulation of Canonical Quantum Gravity[2] and the relation between the two different approach (the connection representation[3] and the loop representation[4]) is becoming more rigorous. The possibility of establishing such a relation relies on the fact that both have a more elegant formulation in terms of spin-network states [3, 5, 6]. However, the formulation in terms of spin-networks arises in the two representations in very different contexts. In the loop-representation the spin-network states originate from the elimination of the so-called Mandelstam-relation[5, 6] while in the connection representation they are the natural cylindrical-functions† in terms of which a generalized measure[7] on the space of connections modulo gauge transformations (\(A/\mathcal{G}\)) is defined. It is worthwhile to recall that the lattice regularization for canonical Quantum Gravity proposed by R. Loll [8] amounts to consider graphs \(\gamma\) that constitute a finite cubic lattice. The idea that it is possible to compare operators defined in the loop and in the connection representation was analyzed by J. Lewandowski in [9].

The main purpose of this article is to make a further step in the direction of proving that the loop and the connection representation are unitarily isomorphic. In particular, we will show that:

- it is possible to construct a graphical method for the description of the connection representation;
- this description is formally identical to the description of the loop representation.
- the recently proposed scalar product in the loop representation [6] coincides with the Ashtekar Lewandowski measure [7, 10].

The essential bridge between the loop representation and the connection representation is given by the so called loop-transformation [4] and its inverse [11] that are rigorously defined, up to now, only in the case of Euclidean general relativity or, which is the same, of a real connection. Consequently, this paper will deal with Euclidean quantum gravity. The possibility of using a real connection also for Lorentzian gravity is object of intensive investigation (see for example [12]). Moreover, Thiemann and Ashtekar [13] have argued that the most promising strategy for implementing the quantum reality conditions is to start from the real Ashtekar connection and circumvent the difficulties due to the complicate form of the Lorentzian Hamiltonian constraint by expressing it in terms of the Riemannian Hamiltonian constraint via a generalized Wick transform.

The structure of the paper is as follows. In section II the basic elements of Penrose’s binor calculus are given in a form suited for quantum gravity. In section

† the term cylindrical function comes from the language of Wiener integration on an infinite-dimensional space.
III the basic elements of the theory of generalized measure in the space of connections modulo gauge transformations $A/G$ are summarized. In section IV the definition of spin-network cylindrical functions is given and their graphical binor representations is constructed, while their normalization is computed in section V. In section VI the graphical representation of the regularized $\tilde{E}_i^a(x)$ operator is given. In section VII these results are used to re-derive the full spectrum of the area operator. Finally, in section VIII the loop transformation is discussed and it is shown that the scalar product proposed in [6] and Ashtekar-Lewandowski’s coincide.

2. Penrose’s graphical method for $SU(2)$ tensor calculus

As shown in reference [6] the loop representation of quantum gravity is strictly related to the graphical methods of $SU(2)$ calculus [14] and more precisely to Penrose’s binor calculus[15]. The existence of this relationship between the loop representation and $SU(2)$ graphical calculus constitutes the basic technical tool that allows us to establish an exact relation between the loop-representation and the connection representation. In [6, 16] it was shown that the most efficient way of dealing with $SU(2)$ ($SL(2,C)$) calculus is the use of the binor-formalism [15] to which we refer for more detail. Here, we only recall the essential elements of the binor graphical calculus to fix the notation. The main idea behind this method is to rewrite any tensor expression in which there are sums of dummy indices in a graphical way [15]. Penrose represents the basic elements of spinor calculus (i.e., tensor expression with indices $A, B, \ldots = 1, 2$) as

\begin{align}
\delta_C^A &= \begin{array}{c}
\delta_C^A
\end{array} \\
\epsilon_{AC} &= \begin{array}{c}
\epsilon_{AC}
\end{array} \\
\eta_A &= \begin{array}{c}
\eta_A
\end{array} \\
\eta^A &= \begin{array}{c}
\eta^A
\end{array} \\
X_{AB}^C &= \begin{array}{c}
X_{AB}^C
\end{array}
\end{align}

(1)

and assigns to any crossing† a minus sign, i.e:

\begin{align}
\delta_C^A \delta_D^B &= - \begin{array}{c}
\delta_C^A \delta_D^B
\end{array}
\end{align}

(2)

Using this rule it is possible to represent any $SU(2)$ ($SL(2,C)$) tensor expression in a graphical way as follows: (1) define the “up” direction in the plane; (2) draw boxes with the name of the $SU(2)$ tensor (except for the $\delta$ and the $\epsilon$ that are simply represented by lines) with as many slots going up as the number of contravariant indices and as many slots going down as the number of covariant indices; (3) for each dummy index connect with a line the corresponding slots of the boxes; (4) assign a “i” factor to each minimum or maximum of the lines; (5) assign a minus sign to each crossings of lines.

Conversely, any curve can be decomposed in a product of $\delta$’s and $\epsilon$’s. It is possible to prove that two curves that are ambient isotopic, i.e., that can be transformed one

† The binor representation is a graphical representation in the plane.
into the other by a sequence of Reidemeister [17] moves, represent the same tensorial expression as a product of epsilons and deltas, and, indeed, two drawings correspond to the same tensor expression if they can be transformed one into the other by a sequence of Reidemeister moves and translations (not rotations) of the boxes representing the true tensors.

A closed loop (with this convention) has value \((-2)\), \((\bigotimes = i\epsilon_{AB} \epsilon^{AB} = -2)\) and the identity \(\epsilon_{AC} \epsilon^{BC} = \delta^C_A\) reads

\[
\left| \right| = - \bigotimes . \tag{3}
\]

A trace of a matrix will be written as

\[
\text{Tr} X^A_B = \delta^B_A X^A_B = - \bigcirc \bigotimes, \tag{4}
\]

and indeed a closed loop denotes the operation of taking \((-1)\) the trace of the corresponding tensor expression. Moreover, the basic binor identity is graphically given by

\[
\bigotimes + \bigotimes = (-1) \delta^C_B \delta^D_A + \delta^A_C \delta^D_B + (-1) \epsilon_{AB} \epsilon^{CD} = 0 \tag{5}
\]

Clearly, in Penrose’s binor calculus there is no meaning in the distinction between over and under crossing. In Penrose’s binor notation it is also possible to write any expression involving representations of the \(SU(2)\) group. In particular, the irreducible representation \(\pi_i(n_i)\) (labeled by an integer \(n\), its color, that is twice the spin: \(n = 2j_n\)) can be constructed as the symmetrization \(\Pi^{(e)}_n\) (in Penrose’s graphical representation an anti-symmetrization because of equation (2)) of the tensor product of \(n\) fundamental representation \(\bigoplus^B_A = U(g)^B_A\) (\(g\) being the group element). Denoting by \(\Pi^{(e)}_n\) the normalized symmetrizer (graphically an anti-symmetrizer), the color \(n\) irreducible representation is given by:

\[
\Pi^{(e)}_n P_\alpha = \frac{1}{n!} \sum_p (-1)^{|p|} P^{(p)}_\alpha = \left| \right| \tag{6}
\]

\[
\pi_i(n_i) = \bigotimes \left| \right| = \bigotimes \left| \right|. \tag{7}
\]

\(|p|\) is the parity of the permutation and a line labeled by a positive integer \(n\) represents \(n\) non-intersecting parallel line). The only additional needed informations about this formalism are: (i) an explicit graphical representation of the Clebsh-Gordon intertwining matrix i.e., of the matrix that represents the coupling of 3 (or more) irreducible representations of the group \(SU(2)\), and (ii) the concept of chromatic evaluation[18] (all the other properties can be deduced from these). The representation of the Clebsh-Gordon intertwining matrix in the binor formalism is given by the special sum of
“tangles” denoted as the 3-vertex. Each line of the vertex is labeled by a positive integer \( a, b \) or \( c \) and is defined as:

\[
\begin{align*}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array{
and the corresponding $b_i$ lines to one another. In this way we obtain a closed scalar expression of $\delta$ and $\epsilon$ tensors in which all the dummy indices are summed, i.e., a number. The evaluation of this number is, by definition, the chromatic evaluation of a network. The chromatic evaluation of a network obtained by joining two vertices will be denoted by $\langle V_n(a_1, \ldots, a_n) | V'_n(b_1, \ldots, b_n) \rangle$ and it is a scalar product in the space of the possible contractors $c_i$ of a vertex. In particular, the chromatic evaluation of a 2-vertex (i.e. of a line) and that of a 3-vertex will be defined as $\Delta$ function and $\theta$ function, respectively:

$$\Delta_n = \langle V_2(n), V'_2(n) \rangle = \left( \begin{array}{c} n \\ n \end{array} \right) = (-1)^n (n+1)$$  \hspace{1cm} (12)

$$\theta(a, b, c) = \langle V_3(a, b, c), V'_3(a, b, c) \rangle = \left( \begin{array}{c} a \\ b \\ c \end{array} \right).$$  \hspace{1cm} (13)

Using this formalism, the anti-hermitian generator of the $SU(2)$ group $\tau_i^A_B = i/2 \sigma_i^A_B$ ($\sigma_i$, Pauli matrix) will be denoted by

$$\tau_i^A_B = \Phi^A_B. \hspace{1cm} (14)$$

From equation (14) we obtain the identity

$$\sum_{i=1}^{3} \tau_i^A_B \tau_i^C_D = \sum_{i=1}^{3} \Phi_i^{AC} = -1/4 \left[ - \bigodot + \bigotimes \right] = -1/2 \bigcirc = -1/2$$

which can be written as

$$\sum_{i=1}^{3} \tau_i^A_B \tau_i^C_D = \frac{1}{2} \bigodot. \hspace{1cm} (15)$$

Analogously, the following relation is also true

$$\sum_{i,j,k=1}^{3} \epsilon_{ijk} \epsilon_i^{AB} = \frac{1}{2} \bigotimes = \frac{1}{2} \bigotimes. \hspace{1cm} (17)$$

The Penrose’s binor formalism allows to graphically write the fundamental property

$$U^{-1}_B^A = \epsilon^{AC} \epsilon_{BD} U_D^C,$$  \hspace{1cm} (18)

valid for any $2 \times 2$ matrix with unit determinant, as

$$\bigwedge_{B}^{B} = \bigwedge_{B}^{B} \bigwedge_{X}^{X} \bigwedge_{A}^{A} \text{ and } \bigwedge_{E_i}^{E_i-1} = \bigwedge_{E_i}^{E_i} \bigwedge_{n_i}^{n_i}. \hspace{1cm} (19)$$
These identities will play an essential role in the following. For any other detail about
binor calculus, its relation to the theory of angular momentum and the explicit value of
the symbols, we refer to [20] and to the appendices of [6].

3. Connection representation and integration on the space of connections
modulo gauge transformations

In this section we give a synthetic review of the construction used in [3, 7, 21, 22]. This
construction can be summarized as follows: (i) the space of histories for quantum gauge
field theory $\mathcal{A}/G$ is taken to be the Gel’fand spectrum generated by the Wilson loop
functionals; (ii) in this space a measure $d\mu_0(A)$ is naturally defined as the $\sigma$-additive
extension of the family of measures $d\mu_{0,\gamma}(A) = d\mu_H(g_{e_1}) \ldots d\mu_H(g_{e_n})$ in the space $\mathcal{A}/G_\gamma$ [21] of the cylindrical functions associated to piecewise analytical graphs $\gamma$; (iii) a natural
basis in the space $\mathcal{A}/G$ is given by the spin-network states [10]. The relationship of this
procedure to the standard method of constructive field theory is discussed in [23]. In
more detail, a function $f_\gamma$ ($f_\gamma \in \mathcal{A}/G_\gamma$) is said to be cylindrical with respect to a graph $\gamma$ if it is a gauge invariant function of the finite set of arguments $(g_{e_1}(A), \ldots, g_{e_n}(A))$
where the $g_{e_i} = P \exp(-\int_{e_i} A)$ are the holonomies of $A$ along the edges $e_i$ of the graph
$\gamma$. Given these families (one for each analytic graph $\gamma$ embedded in $\Sigma$) of integrable
functions (the cylindrical ones), the measure on these finite dimensional projections
$\mathcal{A}/G_\gamma$ (which are isomorphic to $G^n$) is given by

$$\int_{\mathcal{A}/G} d\mu_0(A)f_\gamma(A) = \int_{\mathcal{A}/G_\gamma} d\mu_{0,\gamma}(A)f_\gamma(A) =$$

$$= \int_{G^n} d\mu_H(g_{e_1}) \ldots d\mu_H(g_{e_n}) f_\gamma(g_{e_1}, \ldots, g_{e_n}) .$$

Since this family of finite dimensional measures verifies the consistency condition of
[24], it has a $\sigma$-additive extension $d\mu_0(A)$ that defines a generalized measure on the
whole space $\mathcal{A}/G$. The quantum configuration space can be assumed to be the space $L^2[\mathcal{A}/G, d\mu_0(A)]$ of the square integrable functions with respect to this measure.

4. Spin-networks and their graphical representation

Following Baez‡ [10], given a triple $S = (\gamma, \vec{\pi}, \vec{c})$, a spin-network state is the cylindrical map (with respect to the graph $\gamma$) from $\mathcal{A}/G_\gamma$ into $C$ defined by:

$$\mathcal{T}_{\gamma,\vec{c},\vec{\pi}}[A] \overset{\text{def}}{=} \left( \otimes_{i=1}^N \pi_{c_i}(g_{e_i}) \right) \cdot \left( \otimes_{j=1}^M c_j \right) ,$$

† $\gamma$ is assumed to be an analytic, i.e. each of its edges is an analytic embedding of $M^3$.
‡ Precisely, we are using Thiemann’s[11] definition of edge-network. We point out that there is no
privileged or fundamental role in the 3-valent decomposition; it is only a useful way of constructing a
basis in the space of all the inequivalent contractors that can be assigned to a vertex.
where the elements of the triple $S = (\gamma, \vec{\pi}, \vec{c})$ are:

(i) a graph $\gamma$ with $N$ edges $E = \{e_1, \ldots, e_N\}$ and $M$ vertices $V = \{v_1, \ldots, v_M\}$;

(ii) a labeling $\vec{\pi} = (\pi_1, \ldots, \pi_N)$ of the edges $e_1, \ldots, e_N \in E$ of $\gamma$ with irreducible representation $\pi_i$ of $G$;

(iii) a labeling $\vec{c} = (c_1, \ldots, c_M)$ of the vertices $v_1, \ldots, v_M \in V$ of $\gamma$ with contractors $c_j$ (the intertwining matrices $c_j$, in each of the vertices $v_j$, represent the coupling of the $n_j$ representations associated to the $n_j$ edges that start or end in $v_j$).

Since the group involved is $SU(2)$, any irreducible unitary representation is labeled by an integer $n$ (its color $n$ that is twice the spin: $n = 2j_n$) and is given by the completely symmetric tensor product of $n$ irreducible color 1 (spin $1/2$) fundamental representations (see equation 7). Moreover, from the discussion of section II (equations (8) and (9)), it follows that any contractor $c_j$ of the $n_j$ valent vertex $v_j$ is uniquely determined by an ordering of the incoming (outgoing) edges and by the $n_j-3$ integers labeling its trivalent decomposition. To complete the definition, the normalization choice for the irreducible representations $\pi_i(e_i)$ and for the contractors $c_i$ are given by equations (7),(8) and (9), respectively.

This choice of the contractors has the property of being given in terms of a linear combinations of products of the real tensor $\epsilon$ and $\delta$ tensors. From the reality of the contractor $c_j$, the unitarity of the group $SU(2)$ and equation (19) the reality of the spin-network cylindrical functions follows,

$$\bar{T}_{\gamma, \vec{\pi}, \vec{c}}[A] = T_{\gamma, \vec{\pi}, \vec{c}}[A].$$

(22)

A spin-network is indeed fixed by a graph, a labeling with positive integers of its edges and by trivalent decompositions of the vertices (note that these are exactly the same elements that characterize the spin-network states in the loop representation [5, 6]). The graphical representation of a spin-network state is defined following the construction of the loop representation [6]. As a first step, we consider a projection of the graph $\gamma$ in a plane and its extended planar graph $\Gamma_{ex}$ (see figure 1).

Then, in each of its edges $e_i$, using equation (7), the graphical representation of $\pi_i(g_e)$ is inserted, and in each of the vertices $v_j$, using equations (8) and (9), the graphical representation of the corresponding contractors $c_j$ is inserted. From equation (19) it follows that this planar graphical representation is independent of the orientations of the edges (see figure 2).

At this point the graphical, planar, binor representation of a spin-network state is obtained. It is immediate to note that, were it not for the presence of filled-boxes in the edges of $\gamma$, the resulting graphical representation (see figure 3), would be formally identical to the loop representation of a spin-network state [6].
We finally remark that the notation $T_{\gamma,\vec{\pi},\vec{c}}[A]$ is used instead of the notation $T_{\gamma,\vec{\pi},\vec{c}}[A]$, which is usually adopted in the works on the connection representation, because, in the binor representation, the conventions (1) have the effect of transforming the trace in $(-1)$ times the trace of the corresponding tensor expression (see equation (4)). This means that we have used the $C^*$ algebra generated by $(-1)$ times the trace of the Wilson loop.

5. The normalization of the spin-network states

It is now possible to prove in $L^2[\mathcal{A}/\mathcal{G},d\mu_0(A)]$ the orthogonality of the spin-network states defined in the previous section. Given two spin-network states $s = T_{\gamma,\vec{\pi},\vec{c}}[A]$ and $s' = T_{\gamma',\vec{\pi}',\vec{c}'}[A]$, there is a larger graph $\gamma$ such that $\gamma \subset \gamma'$ and $\gamma' \subset \gamma$ (this is true only for graphs that are analytically embedded, for a discussion of the smooth case see [25]). Since both $s$ and $s'$ are cylindrical functions on $\mathcal{A}/\mathcal{G}_{\gamma}$, the scalar product of these states is given by (see equation 20):

$$\langle s, s' \rangle = \int d\mu_0,\gamma(A) T_{\gamma,\vec{\pi},\vec{c}}[A] T_{\gamma',\vec{\pi}',\vec{c}'}[A] = \int d\mu_0,\gamma(A) T_{\gamma,\vec{\pi},\vec{c}}[A] T_{\gamma',\vec{\pi}',\vec{c}'}[A]$$

where we have used equation (22). From the definition of $d\mu_0,\gamma(A)$ one has to integrate over the group elements associated to each edge. This task could be easily performed using the recoupling theorem (11) and the intermediate result (see Creutz [26]):

$$\int dH(g_e) \Pi_{n}^{(e)} B_{1}^{n} \cdots B_{n}^{n} \left[ U_{A_1}^{B_1} (g_e) \cdots U_{A_n}^{B_n} (g_e) \right] = \delta_{n}^{0} \,,$$

which, in the graphical representation, reads:

$$\int dH(g_e) \begin{array}{l} |n_s> \cdots |n_s> \hline |n_s> \cdots |n_s> \\ \hline |n_s' \end{array} = \delta_{n}^{0} \,.$$  \hspace{1cm} (25)

Two representations of $G$ are associated to each edge of $\gamma$, one of color $n_s$ (eventually 0) from the spin-network $s$ and one of color $n_s'$ from the spin network $s'$. Using equations (11) and (25) one has

$$\int dH(g_e) \begin{array}{l} |n_s> \cdots |n_s> \hline |n_s> \cdots |n_s> \\ \hline |n_s' \end{array} =$$

$$= \int dH(g_e) \sum_{k=|n_s-n_s'|}^{n_s+n_s'} \begin{array}{l} n_s \cdots n_s \cr n_s' \cdots n_s' \end{array} \begin{array}{l} n_s \cdots n_s \cr n_s' \cdots n_s' \end{array} = \delta_{n_s}^{n_s'} \begin{array}{c} \Delta_{n_s} \end{array} \begin{array}{c} n_s \cr n_s' \end{array} \begin{array}{c} n_s \cr n_s' \end{array}.$$  \hspace{1cm} (26)

This formula shows that the scalar product of two spin-network states is different from zero only if they have exactly the same edges and the same coloring (indeed they also
have the same number of vertices and all the vertices have the same valence). The implication of equation (26) is even deeper, since it reduces the computation of the scalar product to the determination of traces of Clebsh-Gordon coefficients (the recoupling of the vertices), i.e., to the chromatic evaluation of the corresponding vertices. Indeed, we have found that the scalar product of two spin-network states is given by:

\[
\langle s, s' \rangle = \int d\mu_0(A) T_{\gamma, \vec{\pi}, \vec{c}}[A] T_{\gamma', \vec{\pi}', \vec{c}'}[A] = \delta_{\gamma, \gamma'} \delta_{\vec{\pi}, \vec{\pi}'} \prod_{e \in E_s} \prod_{i \in V_s} \langle V_i, V'_i \rangle
\]

(27)

\((\langle V_i, V'_i \rangle \) is the chromatic evaluation obtained gluing the vertex \(V_i\) and \(V'_i\)). From this expression, it follow that the norm of a spin network state \(s\) is given by:

\[
N[\gamma, \vec{\pi}, \vec{c}] = \sqrt{\langle s, s \rangle} = \sqrt{\prod_{i \in V} \prod_{e \in E} \theta(a_i, b_i, c_i) \Delta_p} \]

(28)

where the product is extended to all edges (including the virtual ones, i.e., those that come from the 3-valent decomposition) and to all the three-vertices \((a_i, b_i, c_i)\) denote the integers that label the three representations in the three-vertex \(i\). This result, differing from 1, is the consequence of the particular normalization chosen for the fundamental representations and for the Clebsh-Gordon coefficients; this representation is characterized by the fact that there are not square root in the expressions for the 3\(n\)-\(J\) Wigner coefficients. It is immediate to note that this is the same expression found in the loop-representation, equation (8.7) of [6]. We will return on this point in section VIII.

6. Quantization and the operators regularization

The quantization procedure amounts to choosing an Hilbert space and a realization of the Poisson algebra of the observables in terms of Hermitian operators. In the connection representation the Hilbert space \(L^2[\mathcal{A}/\mathcal{G}, d\mu_0(A)]\) is chosen and the operators realizing the Poisson algebra are

\[
\hat{A}^i_a(x) \cdot f_\gamma(A) = A^i_a(x) f_\gamma(A)
\]

(29)

\[
\hat{E}^i_a(x) \cdot f_\gamma(A) = -i l_0^2 \frac{\delta}{\delta A^i_a(x)} f_\gamma(A) .
\]

(30)

The first task that one has to consider in this construction is the definition of a regularization procedure for the operator valued distribution \(\hat{E}^i_a(x)\). From the early work on loop quantum gravity [4] it was realized that in order to regularize expressions involving this operator it is necessary to smear \(\hat{E}^i_a(x)\) over a surface \(\Sigma\), and a different regularization surface \(\Sigma\) for any distinct component of the \(\hat{E}^i_a(x)\) operator has to be chosen. Supposing one wants to regularize the components \(c_a(x)\hat{E}^i_a(x)\) of the \(\hat{E}^i_a(x)\) operator at the point \(x\), one has to consider a particular two dimensional embedding \(\Sigma\)
in $M^3$, with $x \in \Sigma$, $x^a = z^a(\sigma_x)$, such that $c_a(x) = n_a(z(\sigma_x))$. The symbol $\sigma^u = (\sigma^1, \sigma^2)$, $(u, v = 1, 2)$ denotes a coordinate system over $\Sigma (S : \Sigma \rightarrow M^3, \sigma^u \rightarrow z^a(\sigma))$ and

$$n_a(z(\sigma)) = \frac{1}{2} \epsilon^{huv} \epsilon_{abc} \frac{\partial z^b}{\partial \sigma^u} \frac{\partial z^c}{\partial \sigma^v},$$

(31)
denotes the normal one-form of the embedding $\Sigma$. Following [27], the regularization over a surface $\Sigma$ will be based on a family of functions, dependent on a parameter $\epsilon$, such that

$$\lim_{\epsilon \rightarrow 0} \int_{\Sigma} d^2 \sigma^f \phi(\sigma, \sigma') g(\sigma') = g(\sigma)$$

(32)
The smeared version of $n_a\tilde{E}^a_i(x)$ is indeed defined as:

$$[c_a\tilde{E}^a_i]_f(x) := \int_{\Sigma} d^2 \sigma f_e(\sigma_x, \sigma) n_a(\sigma)\tilde{E}^a_i(z(\sigma)),$$

(33)
so that, when $\epsilon$ goes to zero, $[c_a\tilde{E}^a_i]_f$ goes to the $c_a\tilde{E}^a_i(x)$ operator. This point-splitting strategy provides a regularized expression for the $c_a(x)\tilde{E}^a_i(x)$. Using the terminology and the graphical representation of the previous sections, the regularized version of the $c_a(x)\tilde{E}^a_i(x)$ operator is given by:

$$[c_a\tilde{E}^a_i]_f(x) \cdot \Psi_\gamma(A) = -i \frac{\epsilon}{2} \sum_{i=1}^{N} \int_{\Sigma} d^2 \sigma f_e(\sigma_x, \sigma) \frac{\delta g_{el} A_B}{\delta A_i^R(y)} \bigg|_{y = y(\sigma)} n_a(\sigma) \frac{\partial \Psi_\gamma}{\partial g_{el} A_B}$$

(34)
where

$$\frac{\delta g_{el}}{\delta A_i^R(y)} = -\int_0^1 d\lambda \dot{e}^a_i(\lambda) \frac{\delta e^a_i(\lambda)}{\delta \gamma} g_{el}[1, \lambda] \tau_i g_{el}[\lambda, 0].$$

Now, without any loss of generality, it is possible to assume that the graph $\gamma$, defining the state $\Psi_\gamma$, has intersections with $\Sigma$ only at its vertices†. Under this assumption it is possible to perform the integration in (34) and one obtains

$$[c_a\tilde{E}^a_i]_f(x) \cdot \Psi_\gamma(A) = \frac{i \epsilon}{2} \sum_{i=1}^{N} f_e(\sigma_x, v_k) \tilde{X}^i_{v_k e_i} \cdot \Psi_\gamma(A)$$

(35)
where

$$\tilde{X}^i_{v_k e_i} \cdot \Psi_\gamma(A) = \begin{cases} 
\varepsilon_I (g_{el} \tau_i)^A B \frac{\partial \Psi_\gamma}{\partial g_{el} A_B} & \text{if } v_k = e_I(0) \\
\varepsilon_I (\tau_i g_{el})^A B \frac{\partial \Psi_\gamma}{\partial g_{el} A_B} & \text{if } v_k = e_I(1) \\
0 & \text{otherwise}
\end{cases}$$

and

$$\varepsilon_I = \begin{cases} 
0 & \text{if } \dot{e}_I \text{ is tangent to } \Sigma \\
+1 & \text{if } \dot{e}_I \text{ is directed as } n_a \\
-1 & \text{if } \dot{e}_I \text{ is directed opposite to } n_a.
\end{cases}$$

† If this is not the case, we can always consider $\Psi_\gamma$ as a cylindrical function $\Psi_\gamma'$ of a larger graph $\gamma'$ ($\gamma \subset \gamma'$) that has intersections with $\Sigma$ only at its vertices.
We finally remark some facts about the $\tilde{X}_{i_v k e_I}^i$ vertices operators. In order to compute $\varepsilon_I$, in the case when $\dot{e}_I$ is tangent to $\Sigma$, additional regularization is needed, as shown in reference [27, 28]. The relation between these vertices operator and the related ones used by Ashtekar and Lewandowski [27] is $\tilde{X}_{i_v k e_I}^i = \chi_I X_I^i$.

Considering the graphical representation of a cylindrical function it is possible to write the action of the $\tilde{X}_{i_v k e_I}^i$ vertices operators in a graphical way. Let us consider a planar representation in which the regularizing surface $\Sigma$ is represented by a horizontal line in the page (the surface in which we construct the representation) and the direction of $n_a$ is directed from the bottom to the top of the page. With this conventional choice, the four possible cases of the action of the operators (35) are:

\begin{align}
\tilde{X}_{i_v k e_I}^i \cdot \begin{array}{c} \gamma \\ v_k \end{array} & = \begin{array}{c} \gamma \\ v_k \end{array} \quad (36) \\
\tilde{X}_{i_v k e_I}^i \cdot \begin{array}{c} \gamma \\ v_k \end{array} & = \begin{array}{c} \gamma \\ v_k \end{array}, \quad (37)
\end{align}

when the orientation of the edge $e_I$ is from the bottom to the top of the page, and

\begin{align}
\tilde{X}_{i_v k e_I}^i \cdot \begin{array}{c} \gamma \\ v_k \end{array} & = - \begin{array}{c} \gamma \\ v_k \end{array} = \begin{array}{c} \gamma \\ v_k \end{array} \quad (38) \\
\tilde{X}_{i_v k e_I}^i \cdot \begin{array}{c} \gamma \\ v_k \end{array} & = - \begin{array}{c} \gamma \\ v_k \end{array} = \begin{array}{c} \gamma \\ v_k \end{array} \quad (39)
\end{align}

in the other cases.

From the previous formulas it is immediately deduced that the graphical expression of the operators $\tilde{X}_{i_v k e_I}^i$ is independent of the orientation of the edges of the graph (it was already noted that the graphical representation is independent of the orientation of the edges of $\gamma$). The action of $\tilde{X}_{i_v k e_I}^i$ depends only on the structure of the contractor associated to the vertex $v_k$:

\begin{align}
\tilde{X}_{i_v k e_I}^i \cdot \begin{array}{c} n_i \\ c_k \\ \ldots \end{array} & = n_I \begin{array}{c} n_i \\ c_k \\ \ldots \end{array} \quad (40)
\end{align}
Using equations (16) and (17) it is easy to compute the action of the two gauge invariant vertices operators

\[ \sum_i \tilde{X}^i_{e_j} \tilde{X}^i_{e_l} = n_I n_J \sum_i \frac{n_j n_I}{2} \]

and,

\[ \sum_{i,j,k} \epsilon_{ijk} \tilde{X}^i_{e_j} \tilde{X}^j_{e_l} \tilde{X}^k_{e_K} = \frac{n_J n_K}{2} \]

7. The binor representation of the area operator

Two interesting operators for the construction of a quantum theory of gravity are the operators associated to the classical expression of the area and volume. The particular interest for these two operators comes from the suggestion that they will be true observables in a complete quantum theory of gravity coupled to matter [29]. We now discuss, in the context of the binor graphical representation, the operator associated to the classical area of a surface Σ in the connection representation of canonical quantum gravity. We essentially repeat the derivation of the spectrum of [27]. As a consequence, it will be shown that the mathematical steps required in the computation are exactly the same as those of the loop representation [6, 16, 28].

A surface Σ in M is an embedding of a 2-dimensional manifold Σ, with coordinates \( \sigma^u = (\sigma^1, \sigma^2) \), into M. We write \( S : \Sigma \rightarrow M^3, \sigma^u \rightarrow x^a(\sigma) \). Denoting the normal one-form with \( n_a \) and the induced metric on Σ with \( g_{uv}^\Sigma \), the classical expression for the area of Σ is

\[ A[\Sigma] = \int_{\Sigma} d^2\sigma \sqrt{\det g^\Sigma} = \int_{\Sigma} d^2\sigma \sqrt{n_a n_b \tilde{E}^{ai} \tilde{E}^{bj}}. \]

The quantum area operator will be defined using the regularization of the \( \tilde{E}^{ai} \) operators defined in the previous section, and the regularizing surface will be chosen to be Σ itself. From equation (35) the following expression for the regularized area operator is obtained:

\[ \hat{A}^2[\Sigma] \cdot \Psi_\gamma(A) = \int_{\Sigma} d^2x \sqrt{\sum_i [n_a \tilde{E}^{ai}_f(x)] [n_a \tilde{E}^{ai}_f(x)]} \cdot \Psi_\gamma(A) \]

\[ = \int_{\Sigma} l_0^2 d^2x \sum_{k=1}^M f_\epsilon(\sigma_x, v_k) \sqrt{\hat{A}^2[\Sigma, v_k]} \cdot \Psi_\gamma(A). \]
Here, $\hat{A}^2[\Sigma, v_k]$ is the recoupling vertex operator corresponding to the area contribution due to the vertex $v_k$ of the spin-network state $\gamma$

$$\hat{A}^2[\Sigma, v_k] \cdot \Psi_\gamma(A) = \sum_{I,J=1}^{N} \sum_{i=1}^{3} \frac{-1}{4} \tilde{X}^i_{v_k e_I} \tilde{X}^i_{v_k e_J} \cdot \Psi_\gamma(A). \quad (45)$$

In the previous section it was shown how a generic cylindrical function can be written in a graphical planar representation inside $\Gamma_{ex}$ and how the action of the vertices operator $\tilde{X}^i_{v_k e_I}$ can be written in a graphical way. Indeed, given a graph $\gamma$, consider one of its bidimensional graphical representations. In this plane, the set of the edges at the vertex $v_k$ (that lies over $\Sigma$) are naturally decomposed in 3 distinct subclasses: (i) those that lie above $\Sigma$; (ii) those that lie below $\Sigma$; (iii) those that lie tangential to $\Sigma$. From the recoupling theorem it is always possible to parameterize the contractor $c_k$ associated to the vertex $v_k$ as

$$\sum_{k=1}^{u} n_k \cdot = $$

From the identity†

$$\sum_{k=1}^{u} n_k \cdot = $$

it is possible to reduce the computation of the area vertex operator to the three-valent vertex with only three edges (possibly of color 0) in which one lies above $\Sigma$, one lies below $\Sigma$ and one is tangent to $\Sigma$. Indeed, one has to compute its action only on the three valent vertex $V_3(d, u, t)$ where the line of color $u$ correspond to the edges above $\Sigma$, the line of color $d$ to the edges below $\Sigma$ and the line of color $t$ to the edges tangent to $\Sigma$. Now, from the result of section VII one has that the operators $X^i_{v_k e_I}$ give contributions only when they are applied to an edge which is not tangent to $\Sigma$. Using this fact and the identities (41) one has the following three contribution to the action of the $\tilde{A}^2[\Sigma, v_k]$ vertex operator

$$\sum_{i=1}^{3} \tilde{X}^i_{v_k e_u} \tilde{X}^i_{v_k e_u} \cdot = A^2_u \quad (48)$$

† This identity can be easily obtained using the definition of the contractors (9) and the expansion of the three-vertices (8). For a deduction based only on the recoupling theorem see [28].
\[
\sum_{i=1}^{3} \tilde{X}^{i}_{e d} \tilde{X}^{i}_{e d} \begin{pmatrix} u \\ \bar{d} \\ t \end{pmatrix} = \frac{d^2}{2} \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{t} \end{pmatrix} = A^{2}_{d} \begin{pmatrix} u \\ \bar{d} \\ t \end{pmatrix}
\]

(49)

\[
\sum_{i=1}^{3} \tilde{X}^{i}_{e u} \tilde{X}^{i}_{e d} \begin{pmatrix} u \\ \bar{d} \\ t \end{pmatrix} = \frac{ud}{2} \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{t} \end{pmatrix} = A^{2}_{t} \begin{pmatrix} u \\ \bar{d} \\ t \end{pmatrix}
\]

(50)

where the number \(A^{2}_{u}, A^{2}_{d}\) and \(A^{2}_{t}\) can be calculated using the scalar product in the space of the contractors of a vertex defined by the chromatic evaluation (see section II). The explicit expression for \(A^{2}_{u}, A^{2}_{d}\) and \(A^{2}_{t}\) are

\[
A^{2}_{u}(u,d,t) = \frac{u^2}{2} \left( \frac{\bar{u}}{\bar{u}+1} \right) = -\frac{u(u+2)}{4}
\]

(51)

\[
A^{2}_{d}(u,d,t) = \frac{ud}{2} \left( \frac{\bar{u}}{\bar{u}+1} \right) = -\frac{u(u+2)+d(d+2)-t(t+2)}{8}
\]

(52)

and \(A^{2}_{d}(u,d,t) = A^{2}_{u}(d,u,t)\). In this way, one has obtained that the vertices written in the r.h.s. of equation (46) are eigenvectors of the \(\hat{A}^{2}[\Sigma,v_{k}]\) operator corresponding to the eigenvalues:

\[
A^{2}[\Sigma,v_{k}[u,d,t]] = -\frac{A^{2}_{u} + A^{2}_{d} + 2A^{2}_{t}}{4}
\]

(53)

\[
= -\frac{1}{4} \left[ -\frac{u(u+2)}{4} - \frac{d(d+2)}{4} - 2\frac{u(u+2)+d(d+2)-t(t+2)}{8} \right]
\]

\[
= \frac{u(u+2)}{8} + \frac{d(d+2)}{8} - \frac{t(t+2)}{16} = j_{u}(j_{u}+1) + j_{d}(j_{d}+1) - j_{t}(j_{t}+1)
\]

The full spectrum of the area operator† is indeed

\[
\hat{A}^{2}[\Sigma] = l^{2}_{0} \sum_{v_{k} \in \gamma \cap \Sigma} \sqrt{A^{2}[\Sigma,v_{k}[u,d,t]]} \Psi_{\gamma}(A)
\]

(54)

where \(\Psi_{\gamma}(A)\) is a spin network state \(\mathcal{T}_{\gamma,\pi,\vec{c}}[A]\) in which all the intersection \(\gamma \cap \Sigma\) are vertices of the spin-network state and all the vertices \(v_{k} \in \gamma \cap \Sigma\) are decomposed according to the r.h.s. of equation (46).

8. The loop transformation

As noted in [5] the loop representation \(R_{l}\) has to be identified with the representation \(\overline{R}_{c}\) dual to the connection representation \(R_{c}\). For convenience we will denote by \(\langle d\mu \rangle\) the bra states of \(R_{c}\) that, by definition, are the ket states \(|d\mu\rangle\) of \(\overline{R}_{c}\). Clearly, in absence of

† In this way we have re-derived the spectrum computed in [27, 28].
an inner product, there is no canonical map between the connection representation $\mathcal{R}_c$ and the dual-representation $\mathcal{R}_c^*$ and, indeed, between the connection $\mathcal{R}_c$ and the loop $\mathcal{R}_l$ representation. In detail we have that any functional of the connection representation $\psi(A) = \langle A|\psi \rangle$ defines, by double duality, a linear map on the state space $\mathcal{R}_c$

$$\langle \psi|d\mu\rangle = \overline{\langle d\mu|\psi \rangle} = \int d\mu(A)\psi(A) .$$  \hfill (55)

Given a loop state $|\psi_\alpha\rangle$, which is defined in the connection representation $\mathcal{R}_c$ by

$$\langle A|\psi_\alpha\rangle = \mathcal{T}[A,\alpha] = -\text{Tr}[\mathcal{P}e^{-\int_\alpha dx^\alpha A_\alpha}] ,$$  \hfill (56)

it determines a dual state $\langle \psi_\alpha|$ in $\mathcal{R}_c^*$, via

$$\langle \psi_\alpha|d\mu\rangle = \overline{\langle d\mu|\psi_\alpha \rangle} = \int d\mu(A)\mathcal{T}[A,\alpha] ,$$  \hfill (57)

and this bra state must be identified precisely with the loop state

$$L\langle \alpha| = \langle \psi_\alpha \rangle .$$  \hfill (58)

In the case that a scalar product is defined in the connection representation, it is possible to associate to any functional $\psi(A) \in \mathcal{R}_c$ a unique dual state $\langle d\mu_\psi|$ such that, for any $\psi'(A)$,

$$\langle d\mu_\psi|\psi'\rangle = \int d\mu_\psi(A)\psi'(A) = \int d\mu_0(A)\psi(A)\psi'(A).$$  \hfill (59)

From equation (56) and the duality defined by equation (59), the definition of the loop transform is given by

$$\psi(\alpha) = L\langle \alpha| = \int d\mu_0(A)\mathcal{T}[A,\alpha]|\psi(A) \rangle .$$  \hfill (60)

In particular, if a scalar product is given in the connection representation, a scalar product in the loop representation $\mathcal{R}_l$ is defined by

$$L\langle \alpha|\beta\rangle_L = \int d\mu_0(A)_L\langle \alpha|A\rangle\langle A|\beta\rangle_L = \int d\mu_0(A)\overline{\mathcal{T}[A,\alpha]}\mathcal{T}[A,\beta] .$$  \hfill (61)

A direct examination of the definition of the spin-network states in the loop representation (equations (2.12),(2.15) and section V of [6]) and a comparison with the definition (21), shows that

$$\langle A|\gamma,\vec{\pi},\vec{c}\rangle = \mathcal{T}_{\gamma,\vec{\pi},\vec{c}}[A] .$$  \hfill (62)

Using equations (27), (61) and (62) one has that the scalar product induced in the loop representation is

$$\langle \gamma,\vec{\pi},\vec{c}|\gamma',\vec{\pi}',\vec{c}'\rangle = \delta_{\gamma,\gamma'}\delta_{\vec{\pi},\vec{\pi}'}N^2[\gamma,\vec{\pi},\vec{c}]$$  \hfill (63)

which is exactly the result for the proposed scalar product in the loop representation of [6], section VIII.
9. Conclusions

In this article a graphical representation for the cylindrical functions in $L^2[\mathcal{A}/\mathcal{G}, d\mu_0(A)]$ and for the spin-network basis has been explicitly constructed. Using a particular choice of irreducible representations of $G$ and of contractors, a graphical representation for the regularized operator $[c_a \hat{E}_a^q]f(x)$ has been obtained. Moreover, using these constructions, the equivalence of the area operator in the loop and in the connection representation of Euclidean quantum gravity has been proven. Finally, it has been shown that the scalar product proposed in the loop representation of Ref. [6] is exactly the one induced by the loop-transformation of the Ashtekar-Lewandowski measure $d\mu_0(A)$.

Acknowledgments

I am particularly grateful to Carlo Rovelli for valuable criticisms and insight. Moreover, I would like to thank: Simonetta Frittelli and Luis Lehner for having shared with me the results contained in Ref. [28] prior to publication; Massimo Pauri and Luca Lusanna for their continuous support and encouragement during these years. This work has been partially supported by the INFN grant “Iniziativa specifica FI-41” (Italy), and by the Human Capital and Mobility Program “Constrained Dynamical Systems” (European Union).

References

Figure captions

**Figure 1.** The graph $\gamma$ and a possible extended-planar projection $\Gamma_{ex}$.

**Figure 2.** The binor representation of a cylindrical function at the vertex $V_1$ of graph $\gamma$ of Fig. 1 on the extended-planar projection $\Gamma_{ex}$, and its independence of the orientations of the edges.

**Figure 3.** The binor representation of a cylindrical function of the graph $\gamma$ of Fig. 1.