Thermal versus vacuum magnetization in QED

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The magnetized relativistic Fermi (spin-$\frac{1}{2}$) and Bose (spin-0) gases are studied at finite temperature and density. At high enough magnetic fields the renormalized, paramagnetic vacuum ($T = \mu = 0$) contribution starts to dominate the magnetization in both cases. This happens when the thermal magnetization saturates in the Fermi case, and when the thermal magnetization changes from a diamagnetic to a paramagnetic behavior in the Bose case. For fermions at high temperatures, the nonlinear vacuum part of the effective action is completely canceled by terms in the thermal effective action, so that the effective action becomes quadratic in the field. In the Bose case such a cancellation does not occur. Furthermore, we find for the Bose gas that the effective coupling constant for a weak nonzero external magnetic field is a decreasing function of the temperature.

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I. INTRODUCTION

The thermal Fermi and Bose gases in a high magnetic field have recently been studied in great detail in the literature (see, e.g., Refs. [1–3]). The relevant effective action to be considered has the form

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_{\text{eff}}^{\beta,\mu},$$  (1)

where the tree-level part is $\mathcal{L}_0 = -1/2 B^2$, $\mathcal{L}_1$ corresponds to the one-loop vacuum correction, which was not taken into account in Refs. [1–3], and $\mathcal{L}_{\text{eff}}^{\beta,\mu}$ is the corresponding thermal part. The magnetization of the system under consideration is written in the form

$$M = M^{\beta,\mu} + M_{\text{vac}} = \frac{\partial}{\partial B} \mathcal{L}_{\text{eff}}^{\beta,\mu} + \frac{\partial}{\partial B} \mathcal{L}_1.$$  (2)

In Ref. [1] it was noticed that the thermal contribution $M^{\beta,\mu}$ to the magnetization of a neutral Fermi gas saturates for large magnetic fields. We show in the present paper that the vacuum magnetization $M_{\text{vac}}$ becomes dominant in spinor, as well as scalar, QED, at extremely high field strengths and, therefore, no saturation of the magnetization actually occurs. Such large magnetic fields can, e.g., be associated with collapsed magnetic stellar objects in which case one may encounter fields as high as $B \sim 10^{10}$ T [4] (cf. $m_e^2/e \sim 10^9$ T). Since our numerical algorithms are converging faster at vanishing chemical potential, we shall mostly consider the neutral plasma, but the general behavior is unchanged at finite density.

Effective actions of the form Eq. (1) have been used in the literature (see, e.g., Ref. [5]) to give a physical picture of the scaling properties of massless Abelian as well as non-Abelian gauge theories in the vacuum sector. In terms of a bare coupling constant, asymptotic freedom (antiscreening) is then related to a paramagnetic response to an external (bare) magnetic field. We will, however, work with physical, renormalized parameters. As shown below, the massive plasmas with Abelian interactions then exhibit a paramagnetic behavior at sufficiently large external magnetic fields. Nevertheless, the effective couplings derived are not asymptotically free. This is so since we express the magnetization in terms of a renormalized charge which is measurable at weak fields, while in Ref. [5] screening (or antiscreening) can be expressed in terms of a renormalized coupling measured at sufficiently high fields.

II. THE FERMI GAS

The partition function $Z_f(B,T,\mu)$ for the relativistic ideal $e^+e^-$ plasma in presence of an external magnetic field $B$ in the z direction and in a sufficiently large quantization volume $V$ can be written as [6]

$$\frac{\ln Z_f(B,T,\mu)}{\beta V} \equiv \mathcal{L}_{\text{eff}}^{\beta,\mu} = \frac{e B}{2\pi^2} \sum_{\lambda=1}^{2} \sum_{n=0}^{\infty} \int_0^\infty dk_\perp \frac{k_\perp^2}{E_{\lambda,n}} \left[ f_F^+(E_{\lambda,n}) \right. \left. + f_F^-(E_{\lambda,n}) \right],$$  (3)

where $\mathcal{L}_{\text{eff}}^{\beta,\mu}$ is the corresponding thermal part of the ef-
effective Lagrangian density. The energy spectrum above
is $E_{x,n} = \sqrt{k_x^2 + 2eB(n + \lambda - 1) + m_x^2}$, and $f_{x}^{\pm}(E)$ are
the Fermi-Dirac equilibrium distributions:

$$ f_{x}^{\pm}(E) = \frac{1}{\exp[\beta(E \mp \mu)] + 1}, \tag{4} $$

where $\beta$ is the inverse temperature. Separating the field-

independent part, we write $L_{\text{eff}}^{\beta,\mu} = L_0^{\beta,\mu} + L_1^{\beta,\mu}$, where

$$ L_0^{\beta,\mu} = \frac{\ln Z_f(T, \mu)}{\beta V} = \frac{1}{3\pi^2} \int_{-\infty}^{\infty} d\omega \theta(\omega^2 - m^2)f_{\omega}(\omega)(\omega^2 - m^2)^{3/2}. \tag{5} $$

$Z_f(T, \mu)$ is the partition function for the field-free ideal
e$^+e^-$ plasma with particle energy $E = \sqrt{k^2 + m^2}$ and
$f_{\omega}(\omega) = \theta(\omega)f_{\omega}(\omega) + \theta(-\omega)f_{\omega}(-\omega)$. Using the identity

$$ \frac{\exp(-|x|)}{|x|} = \int_0^{\infty} \frac{dt}{\sqrt{2\pi t}} \exp \left[ -\frac{1}{2} \left( \frac{x^2}{t} + \frac{1}{t} \right) \right], \tag{6} $$

and expanding the distribution functions under the restriction
$|\mu| < m$, we can write the field-dependent part
$L_1^{\beta,\mu}$ for $|\mu| < m$ as

$$ L_1 + L_1^{\beta,\mu} = -\frac{(eB)^2}{8\pi^2} \int_0^{\infty} \frac{dx}{x^3} \exp \left( -\frac{m^2}{eB} \right) \partial_4 \left[ \frac{i\beta \mu}{2}, \exp \left( -\frac{eB\beta^2}{4x} \right) \right] \left\{ x\coth(x) - 1 - \frac{1}{3} x^2 \right\}, \tag{9} $$

where we have identified a $\partial_4$ function, given by
$\partial_4[z, q] = 1 + 2 \sum_{l=1}^{\infty} (-1)^l \beta^l \cos(2l\pi z)$ and where we,
as above, consider the case $|\mu| < m$.

The effective Lagrangian in Eq. (1) can be used to
define an effective temperature and field-dependent coupling
constant [9]:

$$ \frac{1}{\tau_{\text{eff}}} = -\frac{1}{eB} \partial_4 \left[ \text{e}_{\text{eff}} \right]. \tag{10} $$

For weak fields, $eB \ll m^2$, and for $T \gg m$ the last term
in Eq. (9) dominates and the corresponding effective coupling agrees with a conventional renormalization-group calculation [6], which can be shown using the expansion
$\sum_{l=1}^{\infty} K_l(\beta m)(-1)^{l+1} + \frac{1}{2} \ln x$ as $x \to 0$. In the limit
$T^2 \gg m^2, eB, \mu^2$, we can now perform a Poisson resum-

mation of the $\partial_4$ function in Eq. (9) in order to obtain
the asymptotic expansion

$$ L_1 + L_1^{\beta,\mu} = \frac{(eB)^2}{12\pi^2} \left[ \ln \left( \frac{T}{m} \right) - C \right] + O\left( \frac{(eBm)^2}{T^2} \right), \tag{11} $$

where $C = 0.577 215 \ldots$ denotes Eulers constant. It is
interesting to note that in the sum $L_1 + L_1^{\beta,\mu}$ the vacuum
contribution $L_1$ has been exactly canceled by a contribu-
tion from $L_1^{\beta,\mu}$. From this we conclude that there are no
nonlinear QED one-loop corrections to electromagnetic interac-
tions in the limit $T^2 \gg m^2, eB$; i.e., the corre-
sponding Euler-Heisenberg effective action is described by a
free electromagnetic field, at least in the purely mag-
netic sector. This result is in accordance with Ref. [10]
where it was shown, in terms of a diagrammatic analysis,
that the one-loop four-photon vertex has a finite limit in
the high-temperature limit.

An external field is included by adding a term $L_{\text{ext}} = j_{\text{ext}} \cdot A$ to $L_{\text{eff}}$. Here $j_{\text{ext}}$ is the external current which is independent of the dynamics of the system considered,
such that $\nabla \times \mathbf{H} = j_{\text{ext}}$, and $A$ is the vector potential
such that $\mathbf{B} = \nabla \times \mathbf{A}$. Neglecting a boundary term at infinity we rewrite $L_{\text{ext}} = \mathbf{B} \cdot \mathbf{A}$. The mean-field equation

$$ \mathbf{B} = \mathbf{H} + M(\mathbf{B}) \tag{12} $$

is then obtained by minimizing the effective action with
respect to $\mathbf{B}$, where the magnetization $M(\mathbf{B})$ is given by
Eq. (2). We find that the paramagnetic thermal part $M^{\beta,\mu}$ of the magnetization saturates for high fields ($\alpha = e^2/4\pi$):

$$ eM^{\beta,\mu} \approx \frac{\alpha \pi}{3} T^2, \quad \sqrt{eB} > T \gg m. \tag{13} $$
This was discussed in Ref. [1], but without considering the paramagnetic vacuum contribution

\[ eM_{\text{vac}} \approx \frac{\alpha}{3\pi} eB \ln(eB/m^2), \quad eB \gg m^2, \]  

(14)

that starts to dominate over \( eM^{\beta,\mu} \) when \( eB \ln(eB/m^2) \approx e^2 T^2 \). We have numerically evaluated \( M_{\text{vac}} \) and \( M^{\beta,\mu} \) and the result is presented in Fig. 1. In order to improve the convergence in the numerical calculation we have found it convenient to write Eq. (9) in the form \( \int_0^\infty dx f(x) = \int_0^\infty dx f(x) + \int_0^\infty dx f(x) \) and to perform a modular transformation \( x \to 1/x \) in the last integral. One can then choose \( \epsilon = 1/\pi \) in order to obtain a numerically rapidly converging integral. As a curiosity we observe that Eq. (12) has a solution for vanishing external field \( H = 0 \), but a nonzero microscopic field \( B \) at \( eB/m^2 \approx \exp(3\pi/\alpha) \). This would mean that a spontaneous magnetic field would be generated at the Landau-ghost pole and therefore lead to a breakdown of Lorentz invariance. Perturbation theory cannot, however, be naively extrapolated to such very large magnetic fields so any spontaneous vacuum magnetization cannot be concluded from the present calculation.

The magnetic susceptibility \( \chi = \partial M/\partial B \) is a measure of the fluctuations of the magnetization. If only the thermal part of the free energy is retained one would erroneously conclude that \( \chi \to 0 \) [1] and that there would be no fluctuations in \( M \) in the large \( B \) limit.

III. THE BOSE GAS

The formalism used in the previous section applies also to scalar QED. The energy spectrum is now given by \( E_n = \sqrt{k^2 + (2n + 1)eB + m^2} \) and the one-particle distributions are

\[ f^\pm(E) = \frac{1}{\exp[\beta(E \pm \mu)] - 1}. \]  

(15)

It is rather straightforward to obtain the renormalized vacuum correction to the effective action [8]

\[ \mathcal{L}_1 = \frac{1}{16\pi^2} \int_0^\infty \frac{ds}{s^3} \exp(-m^2s) \times \left\{ \frac{eBs}{\sinh(eBs)} - 1 + \frac{(eBs)^2}{6} \right\}. \]  

(16)

For weak fields \( \mathcal{L}_1 \) has the expansion

\[ \mathcal{L}_1 = \frac{7}{120} \frac{(eB)^2}{48\pi^2} \left[ \frac{(eB)^2}{m^2} \right] + O \left( \left( \frac{(eB)^4}{m^2} \right) \right). \]  

(17)

For large fields we obtain

\[ \mathcal{L}_1 \simeq \frac{(eB)^2}{96\pi^2} \ln \left( \frac{eB}{m^2} \right). \]  

(18)

Proceeding as in the previous section we find the field-independent thermal part

\[ \mathcal{L}^{\beta,\mu}_0 = \frac{\ln Z_b(T, \mu)}{\beta V} \]

\[ = \frac{1}{6\pi^2} \int_{-\infty}^\infty d\omega \theta(\omega^2 - m^2) f_B(\omega)(\omega^2 - m^2)^{3/2}, \]  

(19)

where \( Z_b(T, \mu) \) is now the partition function for the field-independent ideal charged boson gas with particle energy \( E = \sqrt{k^2 + m^2} \) and \( f_B(\omega) = \theta(\omega)f_B^+(\omega) + \theta(-\omega)f_B^-(\omega) \). In the bosonic case we always have \( |\mu| \leq \sqrt{m^2 + eB} \). In analogy to the previous section, the field-dependent part of the effective action can then generally be written in the form

\[ \mathcal{L}_1 + \mathcal{L}^{\beta,\mu}_1 = \frac{(eB)^2}{16\pi^2} \int_0^\infty \frac{dx}{x^3} \exp \left( -\frac{x^2}{eB} \right) \times \theta_3 \left[ \frac{i\beta \mu}{2}, \exp \left( -\frac{eB \beta^2}{4x} \right) \right] \times \left\{ \frac{x}{\sinh(x)} - 1 + \frac{1}{6x^2} \right\} \]

\[ - \frac{(eB)^2}{24\pi^2} \sum_{l=1}^\infty K_0(l\beta m) \cosh(l\beta \mu), \]  

(20)

where \( \theta_3[z, q] = 1 + 2 \sum_{l=1}^\infty q^l \cos(2lz) \). For weak fields, \( eB \ll m^2 \), and for \( T \gg m, \mu \) the last term in Eq. (20) dominates and can be used to calculate the effective temperature-dependent coupling as in the previous section by Eq. (10). One then uses the expansion \( \sum_{l=1}^\infty K_0(l\beta m) \to \pi/(2x) + 1/\ln x \) as \( x \to 0 \). Even though the coefficient in front of the logarithmic term in this case agrees with an asymptotic renormalization-group analysis, the term linear in \( T \) leads to an effective coupling which is a decreasing function of the temperature. We obtain the asymptotic expansion

FIG. 1. The vacuum \( M_{\text{vac}} \) (dotted line) and thermal \( M^{\beta,\mu} \) (solid line) parts of the magnetization for a neutral Fermi gas. Notice that the thermal contribution saturates for large values of the magnetic field.
\begin{align}
\mathcal{L}_1 + \mathcal{L}_1^{\beta,\mu} &= \frac{(eB)^2}{48\pi^2} \left[ \ln \left( \frac{4T\pi}{m} \right) - C + O \left( \left( \frac{m}{T} \right)^2 \right) \right] \\
&\quad + \frac{T}{m} \frac{(eB)^2}{48\pi} \left[ 1 - \frac{7}{40\pi} \left( \frac{eB}{m^2} \right)^2 \right] \\
&\quad + O \left( \frac{eB}{T^2} \right)^4 \\
&\quad + O \left( \frac{eB}{T^2} \right)^2 \frac{m}{T} \right] .
\end{align}

(21)

As in the case of the Fermi gas we notice that in the sum \( \mathcal{L}_1 + \mathcal{L}_1^{\beta,\mu} \) the vacuum contribution \( \mathcal{L}_1 \) has been exactly canceled by a contribution from \( \mathcal{L}_1^{\beta,\mu} \). In the case of the scalar Bose gas we, however, notice that in contrast with the Fermi gas, all even powers of \( eB/m^2 \) appear with a linear temperature dependence.

The difference between bosonic and fermionic QED, in this respect, can be understood from a one-loop calculation of the photon polarization tensor. Since we consider a static magnetic field we do not obtain any dominant thermal mass of order \( e^2 T^2 \). For fermions the leading high-\( T \) term is a logarithm which is related to the UV divergence and the coefficient is the same as for the standard \( \beta \) function. In the bosonic case there is an IR divergence at high \( T \) from the Bose-Einstein distribution which generates a term linear in \( T \). One can easily verify that the coefficient from the polarization tensor agrees with the one obtained from Eq. (20).

Furthermore, one can verify that the vacuum contribution \( \mathcal{L}_1 \) dominates over the thermal parts of the effective action if the magnetic field is sufficiently large. In the limit \( eB \gg m^2 \) using Eq. (18) we find that

\[
eM_{\text{vac}} \approx \frac{\alpha}{12\pi} eB \ln(eB/m^2) ,
\]

which is a paramagnetic contribution ( unlike the diamagnetic magnetization of scalar particles at small magnetic fields). A paramagnetic magnetization in the high-field limit is expected since the theory is Abelian and therefore the effective coupling should increase as a function of the scale \( eB \). We have numerically evaluated \( M_{\text{vac}} \) and \( M^{\beta,\mu} \) and the result is presented in Fig. 2. We see that the vacuum contribution starts to dominate over the thermal magnetization when the gas changes from diamagnetic to paramagnetic behavior.

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