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Explicit and approximate solutions of the
second order elliptic differential equations
in Hilbert and Banach spaces

by

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Abstract:

An explicit closed-form solution for an abstract elliptic equation in Hilbert or Banach space is presented. Using this explicit representation we propose an approximation with the accuracy automatically depending on the smoothness of the initial data.

Keywords: explicit representation of the solution, Cayley transform, Meixner polynomials

1 Introduction

In this paper we present a framework for construction of explicit representations and for analyzing approximate solutions of the abstract second order differential equation

$$\frac{d^2 u}{dx^2} - Au = -f(x) \quad (1)$$

with boundary conditions (the elliptic case)

$$u(0) = 0, \quad u(1) = u_1. \quad (2)$$

Here $u(x)$ is a vector-valued function with values in a Banach space E , u_1 is a given vector and A is an unbounded operator in E with the domain $D(A)$, the spectral set $\Sigma(A)$ and the resolvent set $\rho(A)$. The operator A is supposed to be densely defined, self-adjoint and positive definite if X is a Hilbert space and densely defined, strongly positive in the case X is a Banach space.

There are many publications concerning the analysis of various properties of the solution of (1), (2) (see for example [11], [13]). On the other hand, relatively little has been done for the construction of explicit and algorithmic representations of the solution of (1) in abstract formulation which allow one to pass to numerical algorithms with desired properties. One of such properties which we are tending to is the automatic dependence of the accuracy of approximation on 'the regularity' of the exact solution (initial data) in the sense of its belonging to the domain of a power of A . Our approach provides an alternative to the spectral and pseudo-spectral methods for stationary equations [4],[5],[6]. But contrary to these methods the ideas and tools of our approach can be used for both stationary and non-stationary problems. The abstract parabolic equation

$$\frac{du}{dt} + Au = 0 \quad u(0) = u_0 \quad (3)$$

in a similar framework was considered in [1] for a bounded operator A , in [2] for an unbounded self-adjoint, positive definite operator A in a Hilbert space and in [9] for a strongly positive operator A in a Banach space. The following representation of the solution was found and justified:

$$\begin{aligned} u(t) &\equiv e^{-At} = e^{-\gamma t} \sum_{p=0}^{\infty} t \left[L_p^{(0)}(2\gamma t) - L_{p-1}^{(0)}(2\gamma t) \right] y_{\gamma,p} \\ &= e^{-\gamma t} \sum_{p=0}^{\infty} (-1)^p L_p^{(0)}(2\gamma t) (y_{\gamma,p} + y_{\gamma,p+1}) \end{aligned} \quad (4)$$

where e^{-At} is the C_0 -semigroup of bounded operators, $y_{\gamma,p} = T_\gamma^p u_0$, $T_\gamma = (\gamma I + A)^{-1}(\gamma I - A)$ is the Cayley transform of the operator A , γ is an arbitrary positive number and $L_p^{(0)}(t)$ are the Laguerre polynomials. A natural

approximation for $u(t)$ is the truncated sum

$$u^N(t) = e^{-\gamma t} \sum_{p=0}^N (-1)^p L_p^{(0)}(2\gamma t) (y_{\gamma,p} + y_{\gamma,p+1}) \quad (5)$$

for which the following estimate was obtained

$$\sup_{t \in [0, \infty)} \|u(t) - u^N(t)\| \leq \begin{cases} q_\gamma^N \|u_0\|, & q_\gamma \in (0, 1) \text{ if } A \text{ is bounded,} \\ N^{-\sigma} \|A^\sigma u_0\|, & \text{if } A \text{ is unbounded, self-adjoint,} \\ & \text{positive definite in a Hilbert} \\ & \text{space and } u_0 \in D(A^\sigma), \\ N^{-\sigma+\varepsilon} \|A^\sigma u_0\|, & \text{if } A \text{ is unbounded, strongly} \\ & \text{positive in a Banach space } E, \\ & \varepsilon \text{ is positive, arbitrary small.} \end{cases} \quad (6)$$

The abstract second order evolution equations (the hyperbolic case) with a self-adjoint definite operator coefficient were studied in [10]. The principal distinction of the present paper from [1],[2],[8],[9],[10] is the method for constructing of the explicit representation of the solution and estimates of the coefficients for this representation.

One can also consider our approach from the following points of view:

1. as a splitting-up method (the problems (1), (2) will be decomposed into a sequence of x -independent problems)
2. as a constructive method to define and to represent the operator sinh-function
3. as a computer-algebraic method for partial differential equations proceeding the solution in a closed analytical form contrary to the convenient pointwise approximations [3],[5],[6],[19].

The paper is organized as follows. In Section 2 we consider the problem (1) in a Hilbert space and present a method for constructing the explicit algorithmic representation of the exact solution based on the Cayley transform and on the Meixner polynomials. The approximate solution and corresponding error estimates are also given. Section 3 is devoted to the case of a Banach space. An example is given in section 4.

Our fundamental tools are the Cayley transform of unbounded operators, the spectral representations of functions of self-adjoint operators and the improper Cauchy-Riesz integral. Throughout the paper c denotes various constants independent of N, n .

2 Abstract elliptic equation in Hilbert space

Let A be a densely defined, self-adjoint, positive definite operator in a Hilbert space H :

$$(Au, u) \geq \lambda_0 \|u\|^2 \quad \forall u \in D(A).$$

Analogously to the case of the scalar homogenous equation (1) when $A \equiv a > 0$, $f(x) \equiv 0$ one can write down formally

$$u(x) \equiv u(x; A) = \sinh^{-1} \sqrt{A} (\sinh \sqrt{A} x) u_1 \quad (7)$$

We will look for a representation of $u(x; A)$ as a series in powers of a rational transform (the Cayley transform) of A . In order to get this series we consider the auxiliary scalar problem

$$\frac{d^2 v(x, z^2)}{dx^2} - \frac{z^2}{1 - z^2} v(x, z^2) = 0, \quad v(0, z^2) = 0, \quad v(1, z^2) = \tilde{u}_1 \quad (8)$$

where $z \in (0, 1)$ is a parameter. It is well known from the theory of ordinary differential equations [12] that there exists an unique function of (8) analytically depending on $z^2 \in (0, 1)$, i.e.

$$v(x, z^2) = \sum_{k=0}^{\infty} v_k(x) z^{2k} \tilde{u}_1. \quad (9)$$

We have from (8) for the coefficients $v_k(x)$

$$\frac{1}{k!} \frac{\partial^k v(x, z^2)}{\partial z^{2k}} \Big|_{z=0} = v_k(x) \quad (10)$$

where $v_k(x)$ are the solutions of the following sequence of boundary value problems

$$\begin{aligned} v_k''(x) &= \sum_{p=0}^{k-1} v_p(x), \quad x \in (0, 1), \quad k \geq 1, \\ v_k(0) &= 0, \quad v_k(1) = 0, \quad v_0(x) = x. \end{aligned} \quad (11)$$

Comparing the abstract problem (1), (2) with the scalar problem (8) one can observe that the solution of (1), (2) can be written in the form

$$u(x, A) \equiv \sinh^{-1} \sqrt{A} (\sinh \sqrt{A} x) u_1 = \sum_{k=0}^{\infty} v_k(x) y_k \quad (12)$$

where

$$y_k = [(I + A)^{-1} A]^{k/2} u_1. \quad (13)$$

In order to find the sequence of vectors $\{y_k\}$ one has to solve the following operator equations (with the same operator but with various right-hand sides)

$$\begin{aligned} y_0 &= u_1, \\ (I + A)y_k &= Ay_{k-1}, \quad k \geq 1. \end{aligned} \quad (14)$$

Let us investigate some properties of the functions $v_k(x)$. First of all we get from (11)

$$(v_k(x) - v_{k-1}(x))'' = v_{k-1}(x), \quad v_k(0) = 0, \quad v_k(1) = 0 \quad (15)$$

what yields

$$v_k(x) = v_{k-1}(x) - \int_0^1 G(x, \xi) v_{k-1}(\xi) d\xi, \quad (16)$$

where

$$G(x, \xi) = \begin{cases} x(1 - \xi), & x \leq \xi \\ \xi(1 - x), & \xi \leq x \end{cases} \quad (17)$$

is the Green function of the problem

$$v''(x) = -f(x), \quad v(0) = v(1) = 0. \quad (18)$$

The equation (16) is equivalent to

$$v_k = (I - G)^{k-1} v_1. \quad (19)$$

where I is the unit operator and the operator G is defined by

$$Gw = \int_0^1 G(x, \xi) w(\xi) d\xi, \quad \forall w \in L^2(0, 1) \quad (20)$$

It is easy to see that $(Gw, w) > 0, \forall w \neq 0$ and

$$\begin{aligned} \|G\|_{L^2 \rightarrow L^2} &= \sup_{w \neq 0} \frac{\int_0^1 \int_0^1 G(x, \xi) w(\xi) w(x) d\xi dx}{\|w\|^2} \\ &\leq \left(\int_0^1 \int_0^1 G^2(x, \xi) dx d\xi \right)^{\frac{1}{2}} \\ &= \left(\int_0^1 \left[\int_0^x \xi^2 (1 - x)^2 d\xi + \int_x^1 (1 - \xi)^2 x^2 d\xi \right] dx \right)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{3}} \left(\int_0^1 x^2 (1 - x)^2 dx \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{90}}. \end{aligned}$$

hence

$$\|I - G\|_{L^2 \rightarrow L^2} = \sup_{w \neq 0} \frac{|(w, w) - (Gw, w)|}{(w, w)} = 1,$$

$$\|v_k\|_{L^2(0,1)} \leq \|v_1\|_{L^2(0,1)} = \left(\int_0^1 \frac{x^2}{36} (1-x^2)^2 dx \right)^{\frac{1}{2}} = \sqrt{\frac{2}{945}}. \quad (21)$$

In fact, a more strong estimate holds true than the estimate (21). It will be improved in the next lemma.

Lemma 1. There exists a constant c independent of x such that

$$\begin{aligned} \int_0^x |v_k(t)| dt &\leq ck^{-\frac{3}{2}+\varepsilon}, \\ \|v_k\|_{C[0,1]} &\leq ck^{-1+\varepsilon}, \\ \|v_k'\|_{C[0,1]} &\leq ck^{-\frac{1}{2}+\varepsilon}, \\ \|v_k''\|_{C[0,1]} &\leq ck^\varepsilon \end{aligned} \quad (22)$$

where ε is an rbitrarily small positive number.

Proof. Let us extend the function $v_1(x) = -\frac{x}{6}(1-x^2)$ oddly upon the interval $[-1, 1]$ and then periodically upon $\mathbb{R}$. The result function can be represented by the Fourier series

$$v_1(x) = \sum_{k=1}^{\infty} \sqrt{2} a_k \sin k\pi x,$$

where

$$a_k = -\frac{\sqrt{2}}{6} \int_0^1 \xi(1-\xi^2) \sin k\pi\xi d\xi = \frac{\sqrt{2}}{(k\pi)^3} (-1)^k. \quad (23)$$

The operator $I - G$ posess the eigenvalues $1 - 1/\lambda_k \equiv 1 - 1/(k^2\pi^2)$ corresponding to the eigenfunctions $\sqrt{2} \sin k\pi x$, $k = 1, 2, \dots$. Then we get from (19)

$$v_k(x) = \sqrt{2} \sum_{p=1}^{\infty} a_p \left(1 - \frac{1}{\lambda_p}\right)^{k-1} \sin p\pi x \quad (24)$$

Let us consider the function

$$\varphi(\xi) = \left(1 - \frac{1}{\xi}\right)^m \xi^{-\sigma}, \quad \xi \in [\pi^2, \infty).$$

It is easy to verify that

$$\sup_{\xi \in [\pi^2, \infty)} |\varphi(\xi)| \leq c(\sigma) m^{-\sigma}. \quad (25)$$

Now, it follows from (23)-(25) that

$$|v_k(x)| \leq c \left(\sum_{p=1}^{\infty} p^{-1-\delta} \right) \sup_{\xi \in [\pi^2, \infty)} \left(1 - \frac{1}{\xi} \right)^k \xi^{-1+\frac{\delta}{2}} \leq ck^{-1+\varepsilon} \quad (26)$$

where c is a constant independent of k and $\varepsilon = \frac{\delta}{2}$ is an arbitrarily small positive number. Analogously we get the estimates

$$\begin{aligned} \int_0^x |v_k(t)| dt &\leq c \left(\sum_{p=1}^{\infty} p^{-1-\delta} \right) \sup_{\xi \in [\pi^2, \infty)} \left(1 - \frac{1}{\xi} \right)^k \xi^{-\frac{\delta}{2}+\frac{\delta}{2}} \leq ck^{-\frac{3}{2}+\varepsilon} \\ |v'_k(x)| &\leq \sup_{\xi \in [\pi^2, \infty)} \left(1 - \frac{1}{\xi} \right)^k \xi^{-\frac{1}{2}+\frac{\delta}{2}} \leq ck^{-\frac{1}{2}+\varepsilon} \end{aligned}$$

In order to estimate $v''(x)$ we use (11) and (26):

$$|v''(x)| = \left| x + \sum_{p=1}^{k-1} v_p(x) \right| \leq 1 + c \sum_{p=1}^{k-1} p^{-1+\varepsilon} \leq ck^{\varepsilon_1},$$

where ε_1 is an arbitrarily small positive number. $\square$

In order to get an explicit formula for $v_k(x)$ we observe that the solution of the auxiliary problem (8) is given by

$$v(x, z^2) = \frac{\sinh \sqrt{\frac{z^2}{1-z^2}} x}{\sinh \sqrt{\frac{z^2}{1-z^2}}}$$

and we have

$$zv(x, z^2) = f(z) \sinh(u(z)x), \quad (27)$$

where

$$u(z) = \frac{z}{\sqrt{1-z^2}}, \quad f(z) = \frac{z}{\sinh \sqrt{1-z^2}}.$$

Comparing (27) with the generating function for the Meixner polynomials $P_n(x)$ [16], i.e.

$$e^{xu(z)} f(z) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} z^n, \quad f(0) = 1, \quad u(0) = 0, \quad u'(0) = 1$$

$v_k \backslash p$	0	1	2	3	4	5	6	7
$v_1 \cdot 6$	0	-1	0	1				
$v_2 \cdot 360$	0	-53	0	50	0	3		
$v_3 \cdot 15120$	0	-1963	0	1729	0	231	0	3
$v_4 \cdot 1814400$	0	-207339	0	168220	0	38094	0	1020
$v_5 \cdot 119750400$	0	-12019571	0	8821791	0	3069330	0	127182
$v_k \backslash p$	8	9	10	11				
$v_1 \cdot 6$								
$v_2 \cdot 360$								
$v_3 \cdot 15120$								
$v_4 \cdot 1814400$	0	5						
$v_5 \cdot 119750400$	0	1265	0	3				

Table 1: Coefficients of the polynomials $v_k(x) = \sum_p b_p x^p$

and using the properties $P_{2k}(x) = P_{2k}(-x)$, $P_{2k+1}(-x) = -P_{2k+1}(x)$, $k = 0, 1, 2, \dots$ we get

$$v_k(x) = \frac{P_{2k-1}(x) - P_{2k-1}(-x)}{2(2k-1)!} = \frac{P_{2k+1}(x)}{(2k+1)!} \quad (28)$$

Table 1 contains the first five of functions $v_k(x)$ (got with Mathematica [21]). Now we turn to the series (12) and study it's convergence. Using the spectral representation one can write

$$\left[(I + A)^{-1} A \right]^k u_1 = \int_{\lambda_0}^{\infty} \left(\frac{\lambda}{1 + \lambda} \right)^k \frac{1}{\lambda^\sigma} dE_\lambda u_1^\sigma, \quad (29)$$

where E_λ is the spectral family corresponding to A , $u_1^\sigma = A^\sigma u_1$. It holds for the function

$$\varphi(\lambda) = \left(\frac{\lambda}{1 + \lambda} \right)^k \lambda^{-\sigma}, \quad \lambda \in [\lambda_0, \infty)$$

and for k large enough that

$$\varphi_{\max} \equiv \varphi\left(\frac{k - \sigma}{\sigma}\right) = \left(1 - \frac{\sigma}{k}\right)^k \frac{\sigma^\sigma}{(k - \sigma)^\sigma} \leq c(\sigma) k^{-\sigma}. \quad (30)$$

Hence

$$\|y_k\| = \left\| \left[(I + A)^{-1} A \right]^k u_1 \right\| \leq c k^{-\sigma} \|u_1^\sigma\|. \quad (31)$$

Taking this estimate and the estimates of Lemma 1 into account we get that the series (12) is majorized by the number series

$$c \sum_{p=1}^{\infty} p^{-(\sigma+1-\varepsilon)}$$

and consequently converges uniformly in x provided that $\sigma > 0$. Together with (12) we consider also the series

$$\tilde{u}(x, A) = \sum_{k=0}^{\infty} v_k'(x) y_k \quad (32)$$

and

$$\tilde{\tilde{u}}(x, A) = \sum_{k=0}^{\infty} v_k''(x) y_k = \sum_{k=0}^{\infty} \sum_{p=0}^{k-1} v_p(x) y_k, \quad (33)$$

which one gets by formal differentiation of (12). It follows from Lemma 1 and (31) that these series are majorized by the number series

$$c \sum_{p=1}^{\infty} p^{-(\sigma+\frac{1}{2}-\varepsilon)} \text{ and } c \sum_{p=1}^{\infty} p^{-(\sigma-\varepsilon_1)}$$

and are uniformly convergent on $[0, 1]$ provided that $\sigma > \frac{1}{2}$ and $\sigma > 1$ respectively. Thus

$$\tilde{u}(x, A) = \frac{du}{dx}, \quad \tilde{\tilde{u}}(x, A) = \frac{d^2u}{dx^2}.$$

Similarly one can show that $u(x, A) \in D(A)$ provided that $\sigma > 1$. Now we can verify that the series (12) satisfies the homogenous equation (1). By substituting (12) into (1) with $f(x) \equiv 0$ we get

$$\begin{aligned} \frac{d^2u}{dx^2} - Au &= \sum_{k=0}^{\infty} \sum_{p=0}^{k-1} v_p(x) y_k - \sum_{k=0}^{\infty} v_k(x) A y_k \\ &= \sum_{k=0}^{\infty} \sum_{p=0}^{k-1} v_p(x) (w_k - w_{k-1}) - \sum_{k=0}^{\infty} v_k(x) A y_k, \end{aligned}$$

where $w_k = -\sum_{p=k+1}^{\infty} y_p$. Summing by parts we further get

$$\begin{aligned}
\frac{d^2 u}{dx^2} - Au &= \sum_{k=0}^{\infty} \sum_{p=0}^{k-1} v_p(x) w_k - \sum_{k=0}^{\infty} \sum_{p=0}^{k-1} v_p(x) w_{k-1} - \sum_{k=0}^{\infty} v_k(x) A y_k \\
&= \sum_{k=0}^{\infty} \sum_{p=0}^{k-1} v_p(x) w_k - \sum_{k=0}^{\infty} \sum_{p=0}^k v_p(x) w_k - \sum_{k=0}^{\infty} v_k(x) A y_k \\
&= - \sum_{k=0}^{\infty} v_k(x) w_k - \sum_{k=0}^{\infty} v_k(x) A y_k \\
&= \sum_{k=0}^{\infty} v_k(x) \int_{\lambda_0}^{\infty} \lambda^{-\sigma} \left[\sum_{p=k+1}^{\infty} \left(\frac{\lambda}{1+\lambda} \right)^p - \lambda \left(\frac{\lambda}{1+\lambda} \right)^k \right] dE_{\lambda} u_1^{\sigma} \\
&= \sum_{k=0}^{\infty} v_k(x) \int_{\lambda_0}^{\infty} \lambda^{-\sigma} \left[\left(\frac{\lambda}{1+\lambda} \right)^{k+1} \left(1 - \frac{\lambda}{1+\lambda} \right)^{-1} \right. \\
&\quad \left. - \lambda \left(\frac{\lambda}{1+\lambda} \right)^k \right] dE_{\lambda} u_1^{\sigma} = 0.
\end{aligned}$$

The solution of the inhomogenous problem (1) with homogenous boundary conditions can be written in the form

$$u(x) = \int_0^1 G(x, \xi, A) f(\xi) d\xi, \quad (34)$$

where

$$G(x, \xi, A) = \frac{1}{2} \begin{cases} \int_{x+\xi-1}^{x-\xi+1} \sinh^{-1} B \cdot \sinh Bt dt, & x \leq \xi \\ \int_{x+\xi-1}^{\xi-x+1} \sinh^{-1} B \cdot \sinh Bt dt, & \xi \leq x. \end{cases} \quad (35)$$

This can be checked by substitution of (12) into (35) and of (34) into the inhomogenous equation (1) and taking into account that corresponding operations with series make sense provided that $f(x) \in D(A^{\sigma-\frac{1}{2}})$, $\sigma > 1$.

Thus we have proved the following result.

Theorem 1. Let A be a densely defined, self-adjoint, positive definite linear operator in a Hilbert space H and $u_1 \in D(A^{\sigma})$, $f \in D(A^{\sigma-\frac{1}{2}})$, $\sigma > 1$. Then the series (12) is the solution of the homogenous problem (1) and (34) is the solution of the corresponding inhomogenous problem.

Remark 1. One can consider the series (12) for $\sigma \in (0, 1]$ as a generalized (weak) solution.

We get a natural approximation for the exact solution of (1), (2) if we use the truncated sum

$$u^N(x, A) = \sum_{k=0}^N v_k(x) y_k \quad (36)$$

instead of (8). Using Lemma 1 and (31) we get

$$\begin{aligned} \sup_{x \in [0,1]} \|u(x, A) - u^N(x, A)\| &\leq c \sum_{p=N+1}^{\infty} p^{-(\sigma+1-\varepsilon)} \|u_1^\sigma\| \\ &\leq c N^{-(\sigma-\varepsilon)} \|u_1^\sigma\| \end{aligned} \quad (37)$$

for an arbitrarily small $\varepsilon > 0$ provided that $u_1 \in D(A^\sigma)$, $\sigma > 0$. Proceeding analogously in the case $f \neq 0$, $u_1 = 0$ and representing the solution of (1), (2) as the corresponding sum we arrive at the following result.

Theorem 2. Let the assumptions of Theorem 1 hold and the truncated sum (36) is used to get the approximate solution $u^N(x, A)$ of (1), (2). Then

$$\begin{aligned} \sup_{x \in [0,1]} \|u(x, A) - u^N(x, A)\| \\ \leq c N^{-(\sigma-\varepsilon)} (\|A^\sigma u_1\| + \sup_{x \in [0,1]} \|A^{\sigma-\frac{1}{2}} f(x)\|) \end{aligned} \quad (38)$$

for any arbitrarily small $\varepsilon > 0$ provided that $u_1 \in D(A^\sigma)$, $f \in D(A^{\sigma-\frac{1}{2}})$, $\sigma > 0$.

Remark 2. Let us consider together with (1), (2) the problem

$$\frac{d^2 \tilde{u}}{dx^2} - A \tilde{u} = -\tilde{f}(x), \quad u(0) = 0, \quad u(1) = \tilde{u}_1. \quad (39)$$

Proceeding as done for the proof of Theorems 1 and 2 one can get the following stability result:

$$\begin{aligned} \sup_{x \in [0,1]} \|u(x) - \tilde{u}(x)\| \\ \leq c (\|A^\sigma (u_1 - \tilde{u}_1)\| + \sup_{x \in [0,1]} \|A^{\sigma-\frac{1}{2}} (f(x) - \tilde{f}(x))\|) \end{aligned} \quad (40)$$

provided that $u_1, \tilde{u}_1 \in D(A^\sigma)$, $f, \tilde{f} \in D(A^{\sigma-\frac{1}{2}})$, $\sigma > 0$.

3 Abstract elliptic equation in Banach space

The assumption of the self-adjointness and of the positive definiteness of A as well as the spectral decomposition as the associated mathematical tool can

not be used if the problem (1), (2) is considered in a Banach space E . An appropriate assumption in this case is the strong positiveness of A [3] and a suitable tool is the improper Cauchy-Riesz integral [3],[13]. We remind that an operator $A : E \rightarrow E$ is called positive if there exist positive constants φ, γ, M such that

$$\begin{aligned} \mathcal{R}(\varphi, \gamma) &\equiv \{z | 0 < \varphi \leq |\arg z| \leq \pi\} \cup \{z | |z| \leq \gamma\} \subset \rho(A), \\ \|(z - A)^{-1}\| &\leq \frac{M}{1 + |z|} \quad \forall z \in \mathcal{R}(\varphi, \gamma) \end{aligned} \quad (41)$$

The lower bound $\varphi(A) \equiv \varphi(A, E)$ of all such φ is called the spectral angle of the operator A . The operator A is called strongly positive if $\varphi < \frac{\pi}{2}$. Throughout this section we assume that the operator A in (1), (2) is densely defined and strongly positive. As an example of such operators one can mention the strongly elliptic operators of order $2m$ in $L^p(\Omega)$, $1 \leq p \leq \infty$ with appropriate domains [17].

If $F(z)$ is an analytical function in $\rho(A)$ then one can define the function $F(A)$ of the operator A by

$$F(A) = \frac{1}{2\pi i} \int_C F(z)(z - A)^{-1} dz$$

where the path C runs in positive direction in $\rho(A)$ (the integral is improper if A is unbounded). In particular we can define fractional powers of A by $A^\sigma = (A^{-\sigma})^{-1}$, $\sigma > 0$, where

$$A^{-\sigma} = \frac{1}{2\pi i} \int_C z^{-\sigma}(z - A)^{-1} dz$$

and the path C runs in the resolvent set of the unbounded operator A from $\infty e^{i\varphi}$ to $\infty e^{-i\varphi}$, avoiding the negative real axis and the origin and $z^{-\sigma}$ is taken to be positive for real positive values of z . We use for convenience $C \equiv \Gamma$, where Γ is the boundary of $\mathcal{R}(\varphi, \gamma)$.

In order to show that the series (12) is the solution if (1), (2) is considered in a Banach space E we proceed in general as done for the Hilbert case. As

opposed to the previous analysis we estimate $\|y_k\|$ in the following way

$$\begin{aligned}
\|y_k\| &= \left\| [(I + A)^{-1}A]^k u_1 \right\| \\
&= \left\| \frac{1}{2\pi i} \int_{\Gamma} \left(\frac{z}{1+z} \right)^k z^{-\sigma} (z - A)^{-1} dz u_1^\sigma \right\| \\
&= \frac{1}{2\pi} \left\| - \int_{\gamma}^{\infty} \left(\frac{\rho e^{i\varphi}}{1 + \rho e^{i\varphi}} \right)^k \rho^{-\sigma} e^{-i\sigma\varphi} (\rho e^{i\varphi} - A)^{-1} d\rho e^{i\varphi} \right. \\
&\quad \left. + \int_{\gamma}^{\infty} \left(\frac{\rho e^{-i\varphi}}{1 + \rho e^{-i\varphi}} \right)^k \rho^{-\sigma} e^{i\sigma\varphi} (\rho e^{-i\varphi} - A)^{-1} d\rho e^{-i\varphi} \right. \\
&\quad \left. + \int_{-\varphi}^{\varphi} \left(\frac{\gamma e^{i\theta}}{1 + \gamma e^{i\theta}} \right)^k \gamma^{-\sigma} e^{-i\sigma\theta} (\gamma e^{i\theta} - A)^{-1} d\theta \gamma i \right\| \\
&\leq c \left(\int_{\gamma}^{\infty} \frac{\rho^k}{|1 + \rho e^{i\varphi}|^k} \rho^{-\sigma} (1 + \rho)^{-1} d\rho + \int_0^{\varphi} (1 + \gamma e^{i\theta})^{-k} d\theta \right).
\end{aligned}$$

Further we use the inequalities for $\varphi \in [0, \frac{\pi}{2})$, $\rho \in [\gamma, \infty)$

$$\left| \frac{\rho}{1 + \rho e^{i\varphi}} \right|^2 = \frac{\rho^2}{\rho^2 + 2\rho \cos \varphi + 1} \leq \frac{\rho^2}{1 + \rho^2}$$

and

$$\left(\frac{\rho^2}{1 + \rho^2} \right)^{\frac{k}{2}} (\rho^2)^{-\frac{\sigma - \varepsilon_1}{2}} \leq c k^{-(\sigma - \varepsilon_1)},$$

where $c = c(\gamma, \sigma - \varepsilon_1)$ and ε_1 is an arbitrarily small positive number. Then we get

$$\|y_k\| \leq c k^{-(\sigma - \varepsilon_1)} \|u_1^\sigma\|. \quad (42)$$

Using Lemma 1 and (42) we come to the following statement.

Theorem 3. Let A be a densely defined, strongly positive linear operator in a Banach space E and $u_1 \in D(A^\sigma)$, $f \in D(A^{\sigma - \frac{1}{2}})$, $\sigma > 1$. Then the series (12) is the solution of the homogenous problem (1) and (34) is the solution of the corresponding inhomogenous problem. Besides, for the approximate solution $u^N(x, A)$ which one gets using the truncated sum (36) the estimate (38) holds.

The proof is very similar to the one of Theorems 1 and 2.

Remark 3. For the solution of the abstract boundary value problem

$$\frac{d^2 u}{dx^2} - Au = 0, \quad u(0) = u_0, \quad u(1) = 0$$

we get the formula

$$u(x) = (I + \sinh^{-1} \sqrt{A} \sinh \sqrt{A}) e^{-\sqrt{A}x} u_0$$

where $v \equiv e^{-\sqrt{A}x} u_0$ can be represented analogously to (4) and $(\sinh^{-1} \sqrt{A} \sinh \sqrt{A} x)v$ by (12). Further, using the truncated sums of the kinds (5) and (36) one gets by the estimates (6), (38) and (40) approximations with accuracy automatically depending on the regularity of the initial data.

Remark 4. In the case A is a bounded operator one can show analogously to [1], [2] that $\|y_k\| \leq cq^k$ for some $q \in (0, 1)$ and get that

$$\sup_{x \in [0,1]} \|u(x, A) - u^N(x, A)\| \leq cq^N (\|u_1\| + \sup_{x \in [0,1]} \|f(x)\|).$$

4 An example

Let us consider the boundary value problem

$$\begin{aligned} \Delta u(x) &\equiv \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0, \quad x = (x_1, x_2) \in \Omega \\ u(x) &= 0, \quad x \in \Gamma_{-1} \cup \Gamma_{-2} \cup \Gamma_{+2}, \\ u(x) &= u_1(x), \quad x \in \Gamma_{+1}, \end{aligned} \tag{43}$$

where $\Omega = \{x = (x_1, x_2) \mid -1 < x_\alpha < +1, \alpha = 1, 2\}$, $\Gamma_{\pm\alpha} = \{x = (x_1, x_2) \mid x_\alpha = \pm 1, x_{3-\alpha} \in [-1, 1]\}$, $\alpha = 1, 2$ and $u_1(x) \equiv u_1(x_1)$ is a given function. If we define the operator A by

$$\begin{aligned} D(A) &= \{v \mid v \in H^2(-1, 1), v(-1) = v(1) = 0\}, \\ Av &= -\frac{d^2 v}{dx_2^2} \quad \forall v \in D(A) \end{aligned} \tag{44}$$

then the problem (43) is of the kind (1),(2) with the densely defined, positive definite operator A in $L^2(-1, 1)$. By (12) we have

$$u(x) = \sum_{k=0}^{\infty} r_k(x_1) y_k(x_2),$$

where $y_0(x_2) = u_1(x_2)$ and $u_k(x_2), k \geq 1$ satisfy the one-dimensional equations

$$-\frac{d^2 y_k}{dx_2^2} + y_k(x_2^2) = \frac{d}{dx_2^2} y_{k-1}(x_2), \quad y_k(-1) = y_k(1) = 0, \quad k \geq 1. \tag{45}$$

In order to solve (45) numerically we use the Legendre collocation method. Let

$$\omega = \{x_2^{(i)} \mid x_2^{(0)} = -1, x_2^{(n)} = 1, L_n'(x_2^{(i)}) = C_{n-1}^{(\frac{1}{2})}(x_2^{(i)}) \equiv 0, 1 \leq i \leq n\}$$

be the grid of the Legendre-Gauß-Lobatto nodes, where $L_n(x)$ are the Legendre and $C_n^{(\lambda)}(x)$ the Gegenbauer polynomials. Given the grid ω , the collocation solution $\hat{y}_k = (1 - x_2^2) \sum_{j=0}^{n-2} a_j^{(n)} x_2^j \in \mathbb{P}_n$ is defined by the equations

$$\begin{aligned} \hat{y}_0(x_2^{(i)}) &= u_1(x_2^{(i)}) \\ -\frac{d^2 \hat{y}_k(x_2^{(i)})}{dx_2^2} + \hat{y}_k(x_2^{(i)}) &= -\frac{d^2 \hat{y}_{k-1}(x_2^{(i)})}{dx_2^2}, \quad 1 \leq i \leq n-1, k \geq 1. \end{aligned} \quad (46)$$

Here $\mathbb{P}_n$ is the set of all polynomials of degree $\leq n$. We denote by $V_n \subset \mathbb{P}_n$ the subspace of all polynomials vanishing at $x_0 = -1$ and $x_n = 1$. Let I_n be the interpolation operator related to ω , i.e.

$$I_n \varphi \in \mathbb{P}_n, \quad (I_n \varphi)(x_2^{(i)}) = \varphi(x_2^{(i)}), \quad 0 \leq i \leq n, \quad \forall \varphi \in C[-1, 1].$$

The operator $P \equiv I_n$ is a projection operator $C[-1, 1] \rightarrow \mathbb{P}_n$, in particular $\mathbb{P}_n$ is the projector from $D(A)$ into V_n . One can write (46) in the operator form

$$\begin{aligned} \hat{y}_0 &= P_n u_1, \\ (I + P_n A) \hat{y}_k(x_2) &= P_n A \hat{y}_{k-1}(x_2), \quad k \geq 1, \quad \hat{y}_k \in V_n \end{aligned} \quad (47)$$

It holds in our case that $P_n A v \equiv A v \quad \forall v \in V_n$, so that (47) takes the form

$$(I + A) \hat{y}_k(x_2) = A \hat{y}_{k-1}(x_2), \quad k \geq 1, \quad \hat{y}_k \in V_n. \quad (48)$$

We call the appropriate solution

$$u_n^N(x_1, x_2) = \sum_{k=0}^N r_k(x_1) \hat{y}_k(x_2) \quad (49)$$

the fully discrete approximate solution for the problem (43) with the discretisation parameters N with respect to x_1 and n with respect to x_2 . Let us consider the error $z(x_1, x_2) = u_n^N(x_1, x_2) - u(x_1, x_2)$. We have

$$\|z(x_1, x_2)\| \leq \|u_n^N - u^N\| + \|u^N - u\| \quad (50)$$

and due to Theorem 2

$$\begin{aligned} \sup_{x_1 \in [-1, 1]} \|u^N(x_1, \cdot) - u(x_1, \cdot)\|_{L^2(-1, 1)} &\leq c N^{-(\sigma-\varepsilon)} \|A^\sigma u_1\| \\ &\leq c N^{-(\sigma-\varepsilon)} \|u_1\|_{H^{2\sigma}(-1, 1)} \end{aligned} \quad (51)$$

provided that $u_1 \in D(A^\sigma)$, where ε is an arbitrary small positive number. Using the spectral representation of functions of A and Lemma 1 we get

$$\begin{aligned} & \|u_n^N(x_1, \cdot) - u^N(x_1, \cdot)\|_{L^2(-1,1)} \\ &= \left\| \int_{\pi^2}^{\infty} \sum_{k=0}^N v_k(x_1) \left(\frac{\lambda}{\lambda+1} \right)^k dE_\lambda (P_k u_1 - u_1) \right\|_{L^2(-1,1)} \\ &\leq c \sum_{k=1}^N k^{-(1-\varepsilon)} \|P_k u_1 - u_1\|_{L^2(-1,1)} \\ &\leq c N^{\varepsilon_1} \|P_n u_1 - u_1\|_{L^2(-1,1)}, \end{aligned} \quad (52)$$

where $\lambda_0 = \pi^2$ is the minimal eigenvalue of A , E_λ is the spectral family associated with A and ε_1 is an arbitrary small positive number. The inequality (see [5], pp. 77-78)

$$\|u_1 - I_n u_1\|_{L^2(-1,1)} \leq c N^{-s} \|u_1\|_{H^s(-1,1)}, \quad s \geq 1$$

and (50)-(52) imply

$$\begin{aligned} & \sup_{x_1 \in [-1,1]} \|u_n^N(x_1, \cdot) - u(x_1, \cdot)\|_{L^2(-1,1)} \\ & \leq c(N^{-\sigma+\varepsilon} + N^{\varepsilon_1} n^{-2\sigma}) \|u_1\|_{H^{2\sigma}(-1,1)} \end{aligned} \quad (53)$$

provided that $u_1 \in D(A^\sigma)$, $\sigma \geq \frac{1}{2}$, where $\varepsilon, \varepsilon_1$ are arbitrarily small positive numbers. In particular we have for $u_1 \in D(A)$

$$\sup_{u_1 \in [-1,1]} \|u_n^N(x, \cdot) - u(x, \cdot)\|_{L^2(-1,1)} \leq c(N^{-1+\varepsilon} + N^{\varepsilon_1} n^{-2}) \|u_1\|_{H^{2\sigma}(-1,1)}.$$

Remark 5. The condition $u_1 \in D(A^\sigma)$ for $\sigma > 1$ is rather restrictive. In order to overcome this difficulty in the case of the homogenous Dirichlet boundary conditions one can treat in the following way.

Let N_0 be a fixed integer and ξ_j^α , $1 \leq j \leq N_0$ be the zeros of the polynomials

$$J_{N_0}^\alpha(\xi) \equiv \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} \frac{\Gamma(N_0+2\alpha+1)}{\Gamma(N_0+\alpha+1)} P_{N_0}^{(\alpha,\alpha)}(\xi) = C_{N_0}^{\alpha+\frac{1}{2}}(\xi),$$

where $P_n^{(\alpha,\beta)}(\xi)$ and $C_n^\lambda(\xi)$ denote the Jacobi and the Gegenbauer polynomials respectively.

We denote by $i_{N_0}^\alpha$ the interpolation operator related to $\{\xi_j^\alpha\}$ which is defined as follows

$$i_{N_0}^\alpha \varphi \in \mathcal{P}_{N_0-1}, \quad (i_{N_0}^\alpha \varphi)(\xi_j^\alpha) = \varphi(\xi_j^\alpha), \quad 1 \leq j \leq N_0 \quad \forall \varphi \in C[-1,1].$$

For each parameter $\alpha > -1$ we introduce a family of Sobolev spaces associated with the measure $\rho_\alpha(\xi)d\xi = (1 - \xi^2)^\alpha d\xi$ by

$$H_\alpha^0(-1, 1) \equiv L_\alpha^2(-1, 1) = \{v \mid \|v\|_{0,\alpha}^2 = \int_{-1}^1 v^2(\xi) \rho_\alpha(\xi) d\xi < +\infty\},$$

$$H_\alpha^k(-1, 1) = \{v \mid \|v\|_{k,\alpha}^2 = \sum_{l=0}^k \|v^{(l)}\|_{0,\alpha}^2 < +\infty\}, \quad k \geq 1$$

and for any real number $s \geq 0$ which is not an integer the space $H_\alpha^s(-1, 1)$ is defined by interpolation between $H_\alpha^{\lfloor s \rfloor}(-1, 1)$ and $H_\alpha^{\lfloor s \rfloor + 1}(-1, 1)$, where $\lfloor s \rfloor$ is the integer part of s . Let us define k as being equal to $m/2$ or $(m+1)/2$ for any integer $m \geq 1$. Let α satisfy $-1 < \alpha < 1$. Then for any real number $\sigma > \max\{\frac{1}{2}, \frac{1+\alpha}{2}\}$ and for any φ such that the function $(1 - \xi^2)^k \varphi$ belongs to $H_\alpha^\sigma(-1, 1)$ the following inequality holds [4], p.98

$$\|(1 - \xi^2)^k (\varphi - i_{N_0-1}^{m+\alpha} \varphi)\|_{0,\alpha} \leq c N_0^{\frac{1}{2}-\sigma} \|(1 - \xi^2)^k \varphi\|_{\sigma,\alpha}. \quad (54)$$

Now we suppose that for the operator A from (44) $u_1 \in D(A)$, $u_1 \in H^{2\sigma}(-1, 1)$ but $u_1 \notin D(A^\sigma)$ for $\sigma > 1$. In order to simplify the exposition we assume that $\sigma = 2s$ for an integer s (the case $\sigma \in \mathbb{R}$ can be regarded analogously using results from [14]). The domain of the operator A^σ consists of all $u \in H^{2\sigma}(-1, 1)$ such that $u^{(2j)}(-1) = u^{(2j)}(1) = 0$, $0 \leq j \leq \sigma - 1$. We choose $\alpha = 0$ and $k = 2\sigma - 1$, i.e. $m = 4\sigma - 2$. Then we have that

$$u_1(x_2) = (1 - x_2^2)^k \varphi(x_2),$$

where

$$\varphi(x_2) = (1 - x_2^2)^{-k} u_1(x_2)$$

and the function $(1 - x_2^2)^k \varphi$ belongs to $H^{2\sigma}(-1, 1) \equiv H_0^{2\sigma}(-1, 1)$. Obviously it holds

$$\tilde{u}_1(x_2) = (1 - x_2^2)^k i_{N_0-1}^m \varphi \in D(A^\sigma)$$

and due to (54) we have

$$\begin{aligned} \|(1 - x_2^2)(\varphi - i_{N_0-1}^m \varphi)\|_{L^2(-1,1)} &= \|u_1(x_2) - \tilde{u}_1(x_2)\|_{L^2(-1,1)} \\ &\leq c N_0^{\frac{1}{2}-2\sigma} \|u_1\|_{H_{(-1,1)}^{2\sigma}}. \end{aligned}$$

Hence, we can replace the problem (43) by

$$\frac{d^2 \tilde{u}}{dx^2} - A \tilde{u} = 0, \quad \tilde{u}(-1) = 0, \quad \tilde{u}(1) = \tilde{u}_1 \quad (55)$$

and due to (40) with $\sigma = 1$ we have

$$\sup_{x_1 \in [-1, 1]} \|u(x_1, \cdot) - \tilde{u}(x_1, \cdot)\|_{L^2(-1, 1)} \leq c N_0^{\frac{1}{2} - 2\sigma} \|u_1\|_H^{2\sigma}(-1, 1).$$

The problem (55) can be treated by the method described above for which $\tilde{u}_1 \in D(A^\sigma)$ and the estimate (53) holds.

Remark 6. The idea to construct an approximate solution of a multidimensional stationary problem by solving one-dimensional problems was used for various equations in many numerical methods [7],[15],[18],[19],[20]. As opposed to these methods which are of iterative nature for stationary processes our splitting-up procedure is realized on the stage of discretization.

Remark 7. The same technique is available to solve problems of the kind

$$L_x u(x) - Au(x) = f(x), \quad u(0) = u_0, \quad u(1) = u_1,$$

where $L_x = \sum_{i=0}^m a_i(x) \frac{d^i}{dx^i}$.

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