$G^*_{2(2)}$ -Structures on pseudo-Riemannian manifolds

I. Kath

Sfb 288 Preprint No. 203

SW9624

Diese Arbeit ist mit Unterstützung des von der Deutschen Forschungsgemeinschaft getragenen Sonderforschungsbereiches 288 entstanden und als Manuskript vervielfältigt worden.

Berlin, April 1996

The list of preprints of the Sonderforschungsbereich 288 is available at:
<http://www-sfb288.math.tu-berlin.de>

$G_{2(2)}^*$ -Structures on pseudo-Riemannian manifolds

I. Kath

April 15, 1996

Abstract

We will give the definition and basic properties of nearly-parallel $G_{2(2)}^*$ -structures on pseudo-Riemannian manifolds of signature (4,3). In particular we explain the equivalence of their existence with that of Killing spinor fields. Furthermore, we will give first examples of pseudo-Riemannian manifolds of signature (4,3) with Killing spinors.

Contents

1	Introduction	1
2	The exceptional non-compact group $G_{2(2)}^*$	2
3	Killing spinors	15
3.1	Geometrical and nearly parallel $G_{2(2)}^*$ -structures	17
3.2	Examples of homogeneous spaces with Killing spinors	19
3.3	Warped products with Killing spinors	24

1 Introduction

This article relates to the paper [4] of Th. Friedrich and others on nearly parallel G_2 -structures. G_2 -structures are topological reductions of the frame bundle of a 7-dimensional manifold to the exceptional group G_2 . They can be described by 3-forms of special algebraic type on the manifold. Since $G_2 \subset SO(7)$ such a structure induces a Riemannian metric and in particular a Levi-Civita connection ∇ on the manifold. It is called nearly parallel if the associated 3-form ω^3 satisfies $\nabla_Z \omega^3 = -2\lambda(Z \lrcorner * \omega^3)$. The existence of

such a 3-form is equivalent to the existence of a spin structure with a Killing spinor field.

Now we are interested in similar structures on pseudo-Riemannian manifolds, more exactly, on manifolds admitting a metric of signature (4,3). There are two real connected non-compact groups of type G_2 . One of them denoted by $G_{2(2)}^*$ is a subgroup of $SO(4, 3)$. $G_{2(2)}^*$ is one of the possible "exceptional" holonomy groups of non-symmetric irreducible pseudo-Riemannian manifolds [2].

The $Spin(4, 3)$ -representation $\Delta_{4,3}$ has some algebraic properties similar to those of the $Spin(7)$ -representation Δ_7 . In particular, both are real. Furthermore, while $Spin(7)$ acts transitively on the sphere S^7 with isotropy group G_2 the action of the connected component $Spin^+(4, 3)$ of $Spin(4, 3)$ on the pseudo-sphere in $\Delta_{4,3}$ is transitive with isotropy group $G_{2(2)}^*$. For a fixed spinor $\psi \neq 0$ in Δ_7 the Clifford multiplication $X \mapsto X \cdot \psi$ is an isomorphism from $\mathbb{R}^7$ to the orthogonal complement of ψ . The same is true in $\Delta_{4,3}$ for any non-isotropic spinor ψ .

These properties will allow us to translate several results from the Riemannian case to signature (4,3). We will give the definition and basic properties of nearly-parallel $G_{2(2)}^*$ -structures. In particular we explain the equivalence of their existence with that of Killing spinor fields. Furthermore, we will give first examples of pseudo-Riemannian manifolds of signature (4,3) with Killing spinors.

2 The exceptional non-compact group $G_{2(2)}^*$

Let $g_{4,3}$ be the symmetric bilinear form on $\mathbb{R}^7$ which is given by $g_{4,3} = \text{diag}(-1, -1, -1, -1, 1, 1, 1)$ with respect to the standard basis $e_1, e_2, \dots, e_7$ of $\mathbb{R}^7$. Define ε_i by $\varepsilon_i = g_{4,3}(e_i, e_i)$. The real Clifford algebra $\mathcal{C}_{4,3} = \text{Cliff}(\mathbb{R}^7, -g_{4,3})$ is the algebra generated by $e_1, e_2, \dots, e_7$ with the relations $e_i^2 = -\varepsilon_i$, $e_i e_j + e_j e_i = 0$ if $i \neq j$. It is isomorphic to the direct sum $\mathbb{R}(8) \oplus \mathbb{R}(8)$ of algebras of matrices. We will use the isomorphism which is defined by

$$\begin{aligned} e_1 &= (\varepsilon \otimes \varepsilon \otimes \sigma, \varepsilon \otimes \varepsilon \otimes \sigma) \\ e_2 &= (-\sigma \otimes \sigma \otimes \tau, -\sigma \otimes \sigma \otimes \tau) \\ e_3 &= (-\sigma \otimes I \otimes \sigma, -\sigma \otimes I \otimes \sigma) \\ e_4 &= (\sigma \otimes \tau \otimes \tau, \sigma \otimes \tau \otimes \tau) \\ e_5 &= (-I \otimes \varepsilon \otimes \tau, -I \otimes \varepsilon \otimes \tau) \end{aligned} \tag{1}$$

$$\begin{aligned}
e_6 &= (-\tau \otimes \varepsilon \otimes \sigma, -\tau \otimes \varepsilon \otimes \sigma) \\
e_7 &= (I \otimes I \otimes \varepsilon, -I \otimes I \otimes \varepsilon),
\end{aligned}$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The projection of this isomorphism onto the first component restricted to $Spin(4, 3) \subset \mathcal{C}_{4,3}$ yields the $Spin(4, 3)$ -representation on $\mathbb{R}^8 =: \Delta_{4,3}$. Furthermore this projection defines the Clifford multiplication of a vector $X \in \mathbb{R}^7 \subset \mathcal{C}_{4,3}$ with a spinor $\psi \in \Delta_{4,3}$ which we will denote by $X \cdot \psi$. Denote by $\psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} =^t (1, 0, \dots, 0)$, $\psi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} =^t (0, 1, \dots, 0)$, $\dots$, $\psi_8 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} =^t (0, \dots, 0, 1)$ the standard basis of $\mathbb{R}^8$. We identify the Lie algebra of $Spin(4, 3)$ with $\mathfrak{spin}(4, 3) = \{\omega = \sum_{i < j} \omega_{ij} e_i e_j \mid \omega_{ij} \in \mathbb{R}\} \subset \mathcal{C}_{4,3}$. We will need the following formulac for the action of $\mathfrak{spin}(4, 3)$ on $\Delta_{4,3}$ which follow from (1). Let D_{ij} be the 8×8 -matrix whose (i, j) -entry is 1 and all of whose other entries are 0. We set $E_{ij} = -D_{ij} + D_{ji}$ and $A_{ij} = D_{ij} + D_{ji}$. Then we have

$$\begin{aligned}
e_1 e_2 &= -E_{12} + E_{34} + E_{56} - E_{78} \\
e_1 e_3 &= E_{13} + E_{24} - E_{57} - E_{68} \\
e_1 e_4 &= -E_{14} + E_{23} + E_{58} - E_{67} \\
e_1 e_5 &= A_{16} - A_{25} + A_{38} - A_{47} \\
e_1 e_6 &= A_{15} + A_{26} + A_{37} + A_{48} \\
e_1 e_7 &= A_{17} - A_{28} - A_{35} + A_{46} \\
e_2 e_3 &= -E_{14} + E_{23} - E_{58} + E_{67} \\
e_2 e_4 &= -E_{13} - E_{24} - E_{57} - E_{68} \\
e_2 e_5 &= A_{15} + A_{26} - A_{37} - A_{48} \\
e_2 e_6 &= -A_{16} + A_{25} + A_{38} - A_{47} \\
e_2 e_7 &= A_{18} + A_{27} + A_{36} + A_{45} \\
e_3 e_4 &= -E_{12} + E_{34} - E_{56} + E_{78} \\
e_3 e_5 &= A_{18} - A_{27} - A_{36} + A_{45} \\
e_3 e_6 &= A_{17} + A_{28} - A_{35} - A_{46}
\end{aligned} \tag{2}$$

$$\begin{aligned}
e_3 e_7 &= -A_{15} + A_{26} - A_{37} + A_{48} \\
e_4 e_5 &= A_{17} + A_{28} + A_{35} + A_{46} \\
e_4 e_6 &= -A_{18} + A_{27} - A_{36} + A_{45} \\
e_4 e_7 &= -A_{16} - A_{25} + A_{38} + A_{47} \\
e_5 e_6 &= E_{12} + E_{34} - E_{56} - E_{78} \\
e_5 e_7 &= E_{14} + E_{23} + E_{58} + E_{67} \\
e_6 e_7 &= -E_{13} + E_{24} + E_{57} - E_{68}
\end{aligned}$$

The following two bilinear forms on $\Delta_{4,3}$ are related to the $Spin(4,3)$ -representation. On one hand we have the standard inner product of $\mathbb{R}^8$ which we denote by $(\cdot, \cdot)$. It is invariant with respect to the maximal compact subgroup $((Pin(4) \times Pin(3))/\mathbb{Z}_2) \cap Spin(4,3)$ of $Spin(4,3)$ and admits the property $(X \cdot \varphi, \psi) + (\varphi, \theta(X) \cdot \psi) = 0$ for all $X \in \mathbb{R}^7$ and $\varphi, \psi \in \Delta_{4,3}$, where $\theta : \mathbb{R}^7 \longrightarrow \mathbb{R}^7$ denotes the reflection with respect to $span\{e_5, e_6, e_7\}$. On the other hand we consider the product $\langle \cdot, \cdot \rangle_\Delta$ of signature $(4,4)$ defined by $\langle \varphi, \psi \rangle_\Delta := (e_1 e_2 e_3 e_4 \varphi, \psi)$. It is invariant with respect to the connected component $Spin^+(4,3)$ of $1 \in Spin(4,3)$ and the equation $\langle X \cdot \varphi, \psi \rangle_\Delta + \langle \varphi, X \cdot \psi \rangle_\Delta = 0$ holds for all $X \in \mathbb{R}^7$ and $\varphi, \psi \in \Delta_{4,3}$. The matrix of $\langle \cdot, \cdot \rangle_\Delta$ with respect to the standard basis $\psi_1, \dots, \psi_8$ equals $diag(-1, -1, -1, -1, 1, 1, 1, 1)$. In particular, we obtain an imbedding $Spin(4,3) \subset SO(4,4)$.

Because of the $Spin^+(4,3)$ -invariance of $\langle \cdot, \cdot \rangle_\Delta$ the group $Spin^+(4,3)$ acts on $\mathcal{M}_c = \{\psi \in \Delta_{4,3} \mid \langle \psi, \psi \rangle_\Delta = c\}$, $c \in \mathbb{R}$. We will prove that this action is transitive for $c \neq 0$ and has two orbits for $c = 0$.

Proposition 2.1 *The action of $Spin^+(4,3)$ on*

$$S^{1,3} := \{\psi \in \Delta_{4,3} \mid \langle \psi, \psi \rangle_\Delta = 1\}$$

is transitive. The same is valid for

$$H^{3,1} := \{\psi \in \Delta_{4,3} \mid \langle \psi, \psi \rangle_\Delta = -1\}$$

The orbits of the $Spin(4,3)^+$ action on

$$\mathcal{C} := \{\psi \in \Delta_{4,3} \mid \langle \psi, \psi \rangle_\Delta = 0\}$$

are $\{0\}$ and $\mathcal{C} \setminus \{0\}$.

Proof. We consider the subalgebra $\mathfrak{spin}(4, 0) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ of $\mathfrak{spin}(4, 3)$. It is spanned by

$$\begin{aligned} e_1 e_2 + e_3 e_4 &= 2(-E_{12} + E_{34}) & e_1 e_2 - e_3 e_4 &= 2(E_{56} - E_{78}) \\ e_1 e_3 - e_2 e_4 &= 2(E_{13} + E_{24}) & e_1 e_3 + e_2 e_4 &= -2(E_{57} + E_{68}) \\ e_1 e_4 + e_2 e_3 &= 2(-E_{14} + E_{23}) & e_1 e_4 - e_2 e_3 &= 2(E_{58} - E_{67}) \end{aligned} \quad (3)$$

(see (2)). Thus, $\mathfrak{spin}(4, 0) \hookrightarrow \mathfrak{so}(\Delta_{4,3}) = \mathfrak{so}(4, 4)$ equals the standard imbedding of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ into $\mathfrak{so}(4, 4)$. Consequently, given an element $\psi \in \Delta_{4,3}$ we find an element g of $Spin(4, 0) \subset Spin^+(4, 3)$ such that $g\psi = x_1\psi_1 + x_5\psi_5$. Let now $\psi = x_1\psi_1 + x_5\psi_5$ be an element of $S^{4,3}$, i.e. $-x_1^2 + x_5^2 = 1$. Then $(x_1 e_1 - x_5 e_6)e_6 \in Spin^+(4, 3)$ and $(x_1 e_1 - x_5 e_6)e_6\psi_5 = x_1\psi_1 + x_5\psi_5$. Analogously, given a $\psi = x_1\psi_1 + x_5\psi_5 \in H^{3,4}$, i.e. $-x_1^2 + x_5^2 = -1$ we have $e_1(x_1 e_1 + x_5 e_6) \in Spin^+(4, 3)$ and $e_1(x_1 e_1 + x_5 e_6)\psi_1 = x_1\psi_1 + x_5\psi_5$. This proves the transitivity of the $Spin^+(4, 3)$ -action on $S^{4,3}$ and $H^{3,4}$, respectively. Now we consider the case $\psi = x_1\psi_1 + x_5\psi_5 \in \mathcal{C} \setminus \{0\}$. First let ψ be equal to $x\psi_1 + x\psi_5$ ($x \neq 0$). Then

$$e_1\left(\frac{x^2+1}{2x}e_1 + \frac{x^2-1}{2x}e_6\right) \in Spin^+(4, 3)$$

and

$$e_1\left(\frac{x^2+1}{2x}e_1 + \frac{x^2-1}{2x}e_6\right)(\psi_1 + \psi_5) = x\psi_1 + x\psi_5 = \psi$$

For $\psi = x_1\psi_1 - x_5\psi_5$ the assertion follows from $-e_1 e_2 e_3 e_4(\psi_1 + \psi_5) = \psi_1 - \psi_5$.

Corollary 2.2 1. *The isotropy group $H(\psi) = \{h \in Spin^+(4, 3) \mid h\psi = \psi\}$ of a non-isotropic spinor $\psi \in \Delta_{4,3}$ (i.e. $\langle \psi, \psi \rangle_\Delta \neq 0$) with respect to the $Spin^+(4, 3)$ -action is a connected non-compact group of type G_2 with fundamental group $\mathbb{Z}_2$.*

2. *The Lie algebra of the isotropy group of an isotropic spinor is the semidirect sum of a 6-dimensional nilpotent algebra and $\mathfrak{sl}(3, \mathbb{R})$.*

Proof. ad 1. Because of the transitivity of the $Spin^+(4, 3)$ -action it suffices to prove that $H(\psi_1)$ has the required properties. We first consider the Lie algebra $\mathfrak{h}(\psi_1)$ of this group. Because of (2) it equals

$$\mathfrak{h}(\psi_1) = \left\{ \sum_{i < j} \omega_{ij} e_i e_j \mid \begin{aligned} & -\omega_{12} - \omega_{34} + \omega_{56} = 0, \\ & \omega_{13} - \omega_{24} - \omega_{67} = 0, \quad -\omega_{14} - \omega_{23} + \omega_{57} = 0, \\ & \omega_{16} + \omega_{25} - \omega_{37} = 0, \quad \omega_{15} - \omega_{26} - \omega_{47} = 0, \\ & \omega_{17} + \omega_{36} + \omega_{45} = 0, \quad \omega_{27} + \omega_{35} - \omega_{46} = 0 \end{aligned} \right\}. \quad (4)$$

It is spanned by $X_1 = e_1e_2 - e_3e_4$, $Y_1 = e_3e_4 + e_5e_6$, $X_2 = e_1e_3 + e_2e_4$, $Y_2 = e_2e_4 - e_6e_7$, $X_3 = e_1e_4 - e_2e_3$, $Y_3 = e_2e_3 + e_5e_7$, $X_4 = e_1e_6 - e_2e_5$, $Y_4 = e_1e_6 + e_3e_7$, $X_5 = e_2e_6 + e_1e_5$, $Y_5 = e_2e_6 - e_4e_7$, $X_6 = e_1e_7 - e_3e_6$, $Y_6 = e_1e_7 - e_4e_5$, $X_7 = e_2e_7 + e_4e_6$ and $Y_7 = e_2e_7 - e_3e_5$. Using the isomorphism of $\mathfrak{spin}(4, 3)$ and $\mathfrak{so}(4, 3)$, we see that the Killing form on $\mathfrak{h}(\psi_1)$ is non-degenerate and has index 6. Therefore, $\mathfrak{h}(\psi_1)$ is a non-compact real form of the semisimple Lie algebra $\mathfrak{h}(\psi_1)^\mathbb{C}$. Furthermore one reads from the relations

$$\begin{aligned} [X_1, Y_1] &= 0 \\ [X_1, X_2] &= 4X_3 & [X_1, Y_2] &= 2X_3 \\ [X_1, X_3] &= -4X_2 & [X_1, Y_3] &= 2X_2 \\ [X_1, X_i] &= -2X_{i+1} & [X_1, Y_i] &= -2Y_{i+1} \quad (i = 4, 6) \\ [X_1, X_j] &= 2X_{j-1} & [X_1, Y_j] &= 2Y_{j-1} \quad (j = 5, 7) \\ [Y_1, X_2] &= -2X_3 & [Y_1, Y_2] &= 4Y_3 \\ [Y_1, X_3] &= 2X_2 & [Y_1, Y_3] &= -4Y_2 \end{aligned}$$

that X_1 and Y_1 commute, but no element out of $\text{span}\{X_1, Y_1\}$ commutes with both X_1 and Y_1 , i.e. $\mathfrak{h}(\psi_1)^\mathbb{C}$ has rank 2 and thus it must be simple. Since its dimension is 14 it is of type G_2 . There is only one non-compact real form of the complex Lie algebra of type G_2 (see e.g. [8]). Now we determine $H(\psi_1)$. Recall that there are two non-compact connected groups of type G_2 (see [8]). The simply connected one has centre $\mathbb{Z}_2$. Because of the transitivity of the $Spin^+(4, 3)$ -action $H^{3,4}$ is diffeomorphic to the homogeneous space $Spin^+(4, 3)/H(\psi_1)$. Using the exact homotopy sequence of this fibration we conclude from $\pi_2(H^{3,4}) = \pi_1(H^{3,4}) = \pi_0(H^{3,4}) = 0$ and from $\pi_1(Spin^+(4, 3)) = \mathbb{Z}_2$, $\pi_0(Spin^+(4, 3)) = 0$ that $H(\psi_1)$ is connected and admits fundamental group $\pi_1(H(\psi_1)) = \mathbb{Z}_2$.

ad 2. We calculate the Lie algebra $\mathfrak{h}(\psi_1 + \psi_5)$ of the isotropy group of $\psi_1 + \psi_5$ and obtain using (2)

$$\begin{aligned} \mathfrak{h}(\psi_1 + \psi_5) = & \left\{ \sum_{i < j} \omega_{ij} e_i e_j \mid \begin{aligned} & \omega_{16} + \omega_{25} - \omega_{37} = 0 \ , \\ & \omega_{15} - \omega_{26} - \omega_{12} + \omega_{56} = 0 \ , \ \omega_{34} - \omega_{47} = 0 \ , \\ & \omega_{27} + \omega_{23} + \omega_{35} - \omega_{57} = 0 \ , \ \omega_{14} + \omega_{46} = 0 \ , \\ & \omega_{13} + \omega_{17} + \omega_{36} - \omega_{67} = 0 \ , \ \omega_{24} + \omega_{45} = 0 \end{aligned} \right\}. \end{aligned} \quad (5)$$

Hence, $\mathfrak{h}(\psi_1 + \psi_5)$ is the semidirect sum of the null space $\mathfrak{n}$ of its Killing form, spanned by $e_3e_4 + e_4e_7$, $e_2e_4 - e_4e_5$, $e_1e_4 - e_4e_6$, $e_6e_7 - e_1e_3 + e_1e_7 +$

e_3e_6 , $e_1e_2 - e_5e_6 + e_1e_5 - e_2e_6$, $e_2e_3 - e_5e_7 - e_2e_7 - e_3e_5$ and the 8-dimensional subalgebra $\mathfrak{p}$ spanned by $e_1e_6 + e_3e_7$, $e_1e_6 - e_2e_5$, $e_1e_2 + e_5e_6$, $e_1e_5 + e_2e_6$, $e_1e_3 + e_6e_7$, $e_1e_7 - e_3e_6$, $e_2e_3 + e_5e_7$, $e_2e_7 - e_3e_5$. Obviously, $\mathfrak{n}$ is nilpotent. The Killing form restricted to $\mathfrak{p}$ is non-degenerate and has index 3. Consequently, $\mathfrak{p}$ equals $\mathfrak{sl}(3, \mathbb{R})$.

Definition 2.3 $G_{2(2)}^* := H(\psi_1)$

Remark 2.4 In this notation $H^{3,4}$ is diffeomorphic to $Spin^+(4, 3)/G_{2(2)}^*$.

Corollary 2.5 For a fixed spinor $\psi \in \Delta_{4,3}$ the kernel of the homomorphism

$$\begin{aligned} \mathbb{R}^{\bar{7}} &\longrightarrow \{\psi\}^\perp \subset \Delta_{4,3} \\ X &\longmapsto X \cdot \psi \end{aligned}$$

(i) is trivial iff $\psi \neq 0$ is non-isotropic

(ii) has dimension 3 iff $\psi \neq 0$ is isotropic.

Proof. Using (1) the assertions (i) and (ii) can be easily verified for $\psi = \psi_1$ and $\psi = \psi_1 + \psi_5$, respectively. Hence, they hold for any $\psi \neq 0$.

Now we consider the universal covering $\lambda : Spin(4, 3) \longrightarrow SO(4, 3)$. Because of $-1 \notin G_{2(2)}^*$ there is an isomorphism from $G_{2(2)}^*$ onto a subgroup of $SO(4, 3)$, which we also denote by $G_{2(2)}^*$. We now describe this group using 3-forms on $\mathbb{R}^{\bar{7}}$. The key point is a special relation between non-isotropic spinors in $\Delta_{4,3}$ and generic 3-forms in $\Lambda^3(\mathbb{R}^{\bar{7}})$.

Let $\psi \in \Delta_{4,3}$ be a fixed non isotropic spinor. Then the map

$$\mathbb{R}^{\bar{7}} \ni X \longmapsto X\psi \in \Delta_{4,3}$$

is an isomorphism because of the orthogonality of $X\psi$ and ψ and since $\dim_R \Delta_{4,3} = 8$. We observe now that for $X, Y \in \mathbb{R}^{\bar{7}}$ the spinors ψ and $YX\psi + g_{4,3}(X, Y)\psi$ are orthogonal to each other. Therefore we can define a (2,1)-tensor A_ψ by

$$YX\psi + g_{4,3}(X, Y)\psi = A_\psi(Y, X)\psi. \quad (6)$$

A_ψ has the following properties

1. $A_\psi(X, Y) = -A_\psi(Y, X)$

$$2. \ g_{4,3}(Y, A_\psi(Y, X)) = 0$$

$$3. \ A_\psi(Y, A_\psi(Y, X)) = -\|Y\|_{4,3}^2 X + g_{4,3}(X, Y)Y.$$

It defines a 3-form ω_ψ^3 by $\omega_\psi^3(X, Y, Z) = g_{4,3}(X, A_\psi(Y, Z))$.

Clearly,

$$\omega_{\alpha\psi}^3 = \omega_\psi^3 \quad \alpha \in \mathbb{R}, \alpha \neq 0. \quad (7)$$

In particular, if $\psi = \psi_1$ then a direct calculation yields $\omega_{\psi_1}^3 = \omega_0^3$, where ω_0^3 is given by

$$\begin{aligned} \omega_0^3 = & -e_1 \wedge e_2 \wedge e_7 - e_1 \wedge e_3 \wedge e_5 + e_1 \wedge e_4 \wedge e_6 \\ & + e_2 \wedge e_3 \wedge e_6 + e_2 \wedge e_4 \wedge e_5 - e_3 \wedge e_4 \wedge e_7 + e_5 \wedge e_6 \wedge e_7. \end{aligned} \quad (8)$$

Definition 2.6 Let ω^3 be a 3-form on $\mathbb{R}^7$. Furthermore let $X_1, \dots, X_7$ be an arbitrary pseudo-orthonormal basis of $(\mathbb{R}^7, g_{4,3})$. We define a 4-form σ^4 by $\sigma^4 = -\sum_{i=1}^4 (X_i \lrcorner \omega^3) \wedge (X_i \lrcorner \omega^3) + \sum_{i=5}^7 (X_i \lrcorner \omega^3) \wedge (X_i \lrcorner \omega^3)$ which does not depend on the choosen basis. We will say that ω^3 defines the orientation of $\mathbb{R}^7$ if $\omega^3 \wedge \sigma^4$ is a positive multiple of the volume form of $\mathbb{R}^7$. Furthermore we will say that ω^3 defines the space and time orientation of $(\mathbb{R}^7, g_{4,3})$ if it defines the orientation of $\mathbb{R}^7$ and if $\omega^3(X_5, X_6, X_7) > 0$ for any positively space and time oriented pseudo-orthonormal basis $X_1, \dots, X_7$.

Now let $\psi \neq 0$ be a fixed non-isotropic spinor and ω_ψ^3 the associated 3-form. With the same notation as above we obtain $\omega_\psi^3 \wedge \sigma_\psi^4 = 42e_1 \wedge \dots \wedge e_7$. Hence, ω_ψ^3 defines the orientation of $\mathbb{R}^7$.

Now fix a spinor ψ with $\langle \psi, \psi \rangle_\Delta = -1$ and let $X_1, \dots, X_7$ be a positively space and time oriented pseudo-orthonormal basis. From the definition of A_ψ we know that $g_{4,3}(A_\psi(X_5, X_6), A_\psi(X_5, X_6)) = 1$ and therefore $A_\psi(X_5, X_6) \notin \{X_5, X_6, X_7\}^\perp$. Since, however, $A_\psi(X_5, X_6) \perp X_5, X_6$ the vectors $A_\psi(X_5, X_6)$ and X_7 can not be orthogonal. Hence, $\omega_\psi^3(X_5, X_6, X_7) \neq 0$. Since on the other hand $\omega_{\psi_1}^3(e_5, e_6, e_7) = 1$ we obtain $\omega_\psi^3(X_5, X_6, X_7) > 0$. Hence ω_ψ^3 defines the space and time orientation of $(\mathbb{R}^7, g_{4,3})$.

Vice versa, let A be a (2,1)-tensor on $\mathbb{R}^7$ which has the properties 1., 2., 3. Then A defines a 3-form $\omega^3 = g_{4,3}(\cdot, A(\cdot, \cdot))$. We can define σ^4 in the same way as above. From the properties 1., 2., 3. we conclude $\omega^3 \wedge \sigma^4 \neq 0$. Suppose that ω^3 defines the orientation of $\mathbb{R}^7$. Furthermore from the properties 1., 2., 3 we deduce as above that $\omega^3(X_5, X_6, X_7) \neq 0$ for any oriented pseudo-orthonormal basis $X_1, \dots, X_7$. Suppose that ω^3 defines the space and time orientation of $(\mathbb{R}^7, g_{4,3})$. Consider now the subspace

$$E = \{\psi \in \Delta_{4,3} \mid XY\psi = -g_{4,3}(X, Y)\psi + A(X, Y)\psi\}.$$

Then E is 1-dimensional and spanned by a spinor ψ_0 with $\langle \psi_0, \psi_0 \rangle_\Delta = -1$. In particular, $\omega^3 = \omega_{\psi_0}$. We obtain

Theorem 2.7 *There is a 1 -1 correspondence between $S^{4,3}/\{1, -1\}$ and those $\omega^3 \in \Lambda^3(\mathbb{R}^7)$ which define the space and time orientation of $(\mathbb{R}^7, g_{4,3})$ and for which the bilinear map A defined by $\omega^3(X, Y, Z) = g_{4,3}(X, A(Y, Z))$ admits the properties 1., 2., 3. .*

Analogously, there is a 1 -1 correspondence between $H^{4,3}/\{1, -1\}$ and those $\omega^3 \in \Lambda^3(\mathbb{R}^7)$ which define the inverse space and time orientation of $(\mathbb{R}^7, g_{4,3})$ and for which the bilinear map A defined by $\omega^3(X, Y, Z) = g_{4,3}(X, A(Y, Z))$ admits the properties 1., 2., 3. .

In particular, since we have for $g \in Spin^+(4, 3)$

$$\omega_{g\psi}^3 = (\lambda(g^{-1}))^* \omega_\psi.$$

we conclude

Corollary 2.8 *The image of $G_{2(2)}^*$ with respect to $\lambda : Spin(4, 3) \mapsto SO(4, 3)$ equals*

$$G_{2(2)}^* = \{A \in SO^+(4, 3) \mid A^* \omega_0 = \omega_0\}.$$

Note that $A \in SO(4, 3)$ and $A^* \omega_0 = \omega_0$ imply $A \in SO^+(4, 3)$ since ω_0 defines a space and time orientation.

On the other hand the equation $A^* \omega_0^3 = \omega_0^3$ for $A \in GL(7)$ implies $A \in SO(4, 3)$. See for a proof in [2]. Consequently, we obtain

$$G_{2(2)}^* = \{A \in GL(7) \mid A^* \omega_0^3 = \omega_0^3\}.$$

Next we investigate in the same way as above the action of $Spin^+(4, 3)$ on some of the Stiefel manifolds

$$\begin{aligned} V(\varepsilon_1, \dots, \varepsilon_l) = \{ (\varphi_1, \dots, \varphi_l) \mid & \varphi_i \in \Delta_{4,3} \quad (i = 1, \dots, l), \\ & \langle \varphi_i, \varphi_i \rangle_\Delta = \varepsilon_i \quad (i = 1, \dots, l), \\ & \langle \varphi_i, \varphi_j \rangle_\Delta = 0 \text{ if } i \neq j \quad (i, j = 1, \dots, l) \}, \end{aligned}$$

where $\varepsilon_i = -1$ for $i = 1, \dots, k$ ($k \leq l$) and $\varepsilon_i = 1$ for $i = k + 1, \dots, l$.

Proposition 2.9 *The action of $Spin^+(4, 3)$ on $V(-1, -1), V(-1, 1)$ and $V(1, 1)$ is transitive.*

Proof. Since $e_1e_5 \in Spin(4, 3)$ maps $S^{4,3}$ one-to-one onto $H^{3,4}$ and

$$(e_1e_5)Spin^+(4, 3)(e_1e_5)^{-1} = (e_1e_5)Spin^+(4, 3)(-e_5e_1) = Spin^+(4, 3) \quad (9)$$

the situation on $V(-1, -1)$ and $V(1, 1)$ is essentially the same.

We calculate the dimension of the isotropy group $H(\varphi_1, \varphi_2)$ of an arbitrary pair (φ_1, φ_2) with $\langle \varphi_1, \varphi_1 \rangle_\Delta = -1$, $\langle \varphi_1, \varphi_2 \rangle_\Delta = 0$ and $\varphi_2 \neq 0$. Clearly (see Proposition 2.1), we may assume $\varphi_1 = \psi_1$. Next we shall explain, why we can assume furthermore $\varphi_2 = x_2\psi_2 + x_5\psi_5$. The isotropy group $G_{2(2)}^*$ of ψ_1 contains $SO(3)$ and $SU(2)$ as subgroups. The Lie algebra $\mathfrak{so}(3) \subset \mathfrak{so}(4, 4)$ is spanned by $e_3e_4 + e_5e_6 = 2(E_{34} - E_{56})$, $e_2e_4 - e_6e_7 = 2(-E_{24} - E_{57})$, $e_2e_3 + e_5e_7 = 2(E_{23} + E_{67})$ and $\mathfrak{su}(2) \subset \mathfrak{so}(4, 4)$ by $e_1e_2 - e_3e_4 = 2(E_{56} - E_{78})$, $e_1e_3 + e_2e_4 = -2(E_{57} + E_{68})$, $e_1e_4 - e_2e_3 = 2(E_{58} - E_{67})$. Therefore we can first achieve that $\varphi_2 = x_2\psi_2 + x_5\psi_5 + x_6\psi_6 + x_7\psi_7 + x_8\psi_8$ using the action of $SO(3) \subset G_{2(2)}^*$ and after that $\varphi_2 = x_2\psi_2 + x_5\psi_5$ using $SU(2)$.

Thus, let φ_2 be $x_2\psi_2 + x_5\psi_5$. The equations (2) imply that the Lie algebra $\mathfrak{h}(\psi_1, x_2\psi_2 + x_5\psi_5)$ of the isotropy group of $(\psi_1, x_2\psi_2 + x_5\psi_5)$ consists of all $\omega = \sum_{i < j} \omega_{ij}e_ie_j \in \mathfrak{spin}(4, 3)$ satisfying

$$\begin{aligned} -\omega_{12} - \omega_{34} + \omega_{56} &= 0 & \omega_{13} - \omega_{24} - \omega_{67} &= 0 & \omega_{14} + \omega_{23} - \omega_{57} &= 0 \\ -\omega_{16} - \omega_{25} + \omega_{37} &= 0 & \omega_{15} - \omega_{26} - \omega_{47} &= 0 & \omega_{17} + \omega_{36} + \omega_{45} &= 0 \\ \omega_{27} + \omega_{35} - \omega_{46} &= 0 \end{aligned}$$

$$\begin{aligned} x_2(\omega_{12} + \omega_{34} - \omega_{56}) &+ x_5(-\omega_{16} - \omega_{25} + \omega_{37}) &= 0 \\ &x_5(\omega_{15} - \omega_{26} + \omega_{47}) &= 0 \\ x_2(\omega_{14} + \omega_{23} + \omega_{57}) &+ x_5(\omega_{17} + \omega_{36} - \omega_{45}) &= 0 \\ x_2(\omega_{13} - \omega_{24} + \omega_{67}) &+ x_5(-\omega_{27} - \omega_{35} - \omega_{46}) &= 0 \\ x_2(-\omega_{15} + \omega_{26} - \omega_{47}) & &= 0 \\ x_5(-\omega_{12} + \omega_{34} + \omega_{56}) &+ x_2(\omega_{16} + \omega_{25} + \omega_{37}) &= 0 \\ x_5(\omega_{13} + \omega_{24} - \omega_{67}) &+ x_2(\omega_{27} - \omega_{35} + \omega_{46}) &= 0 \\ x_5(-\omega_{14} + \omega_{23} - \omega_{57}) &+ x_2(-\omega_{17} + \omega_{36} + \omega_{45}) &= 0. \end{aligned}$$

By simplifying these equations we obtain

$$\mathfrak{h}(\psi_1, x_2\psi_2 + x_5\psi_5) =$$

$$\begin{aligned} = \{ \sum_{i < j} \omega_{ij}e_ie_j \mid & \quad \begin{aligned} & -\omega_{12} - \omega_{34} + \omega_{56} = 0, \\ & \omega_{13} - \omega_{24} - \omega_{67} = 0, \quad \omega_{14} + \omega_{23} - \omega_{57} = 0, \\ & -\omega_{16} - \omega_{25} + \omega_{37} = 0, \quad \omega_{15} - \omega_{26} - \omega_{47} = 0, \end{aligned} \end{aligned} \quad (10)$$

$$\begin{aligned}
&\omega_{17} + \omega_{36} + \omega_{45} = 0 \quad , \quad \omega_{27} + \omega_{35} - \omega_{46} = 0 \quad , \\
&x_5\omega_{47} = 0 \quad , \quad x_2\omega_{47} = 0 \quad , \\
&x_2\omega_{57} - x_5\omega_{45} = 0 \quad , \quad x_2\omega_{67} - x_5\omega_{46} = 0 \quad , \\
&x_5\omega_{34} + x_2\omega_{37} = 0 \quad , \quad x_5\omega_{24} + x_2\omega_{27} = 0 \quad , \\
&x_5\omega_{14} + x_2\omega_{17} = 0 \quad \quad \quad \}.
\end{aligned}$$

Since not $x_2 = x_5 = 0$ the dimension of the Lie algebra $\mathfrak{h}(\varphi_1, \varphi_2)$ of $H(\varphi_1, \varphi_2)$ equals 8 and the one of the orbit of (φ_1, φ_2) equals 13. Hence, all orbits are open sets and the action of $Spin^+(4, 3)$ is transitive.

Corollary 2.10 *The isotropy group of a pair (φ_1, φ_2) of pseudo-orthonormal spinors with respect to the $Spin^+(4, 3)$ -action equals*

1. $SU(1, 2)$ if $(\varphi_1, \varphi_2) \in V(-1, -1)$ or $V(1, 1)$
2. $SL(3, \mathbb{R})$ if $(\varphi_1, \varphi_2) \in V(-1, 1)$.

Proof. The Lie algebra of $H(\psi_1, \psi_2)$ equals

$$\begin{aligned}
\mathfrak{h}(\psi_1, \psi_2) = \{ \sum_{i < j} \omega_{ij} e_i e_j \mid & \quad -\omega_{12} - \omega_{34} + \omega_{56} = 0 \quad , \\
& \omega_{13} - \omega_{24} = 0 \quad , \quad \omega_{14} + \omega_{23} = 0 \quad , \\
& \omega_{16} + \omega_{25} = 0 \quad , \quad \omega_{15} - \omega_{26} = 0 \quad , \\
& \omega_{36} + \omega_{45} = 0 \quad , \quad \omega_{35} - \omega_{46} = 0 \quad , \\
& \omega_{i7} = 0 \quad , \quad i = 1, \dots, 6 \quad \quad \quad \}.
\end{aligned} \tag{11}$$

As a subalgebra of $\mathfrak{so}(4, 4)$ it is spanned by $E_{34} - E_{78}$, $E_{56} - E_{78}$, $E_{57} + E_{68}$, $E_{58} - E_{67}$, $A_{37} + A_{48}$, $A_{38} - A_{47}$, $A_{35} + A_{46}$, $A_{36} - A_{45}$ and equals therefore $\mathfrak{su}(1, 2)$ where $SU(1, 2) \subset SU(2, 2) \subset SO(4, 4)$ is imbedded in the usual way. We conclude that the connected component of $H(\psi_1, \psi_2)$ must be $SU(1, 2)$. On the other hand $V(-1, -1)$ is simply connected. This follows from the exact homotopy sequence of the fibration $SO^+(2, 4) \xrightarrow{i} SO^+(4, 4) \longrightarrow V(-1, -1)$ since

$$\begin{aligned}
\cdots \longrightarrow \pi_1(SO^+(2, 4)) \xrightarrow{i^\#} \pi_1(SO^+(4, 4)) \longrightarrow \pi_1(V(-1, -1)) \longrightarrow \\
\longrightarrow \pi_0(SO^+(2, 4), I) \longrightarrow \cdots
\end{aligned}$$

is the sequence of the following groups

$$\cdots \longrightarrow \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{i^\#} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \pi_1(V(-1, -1)) \longrightarrow 0 \longrightarrow \cdots$$

and $i_{\#}$ sends $(1, 1) \in \mathbb{Z} \oplus \mathbb{Z}_2$ to $(1, 1) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and is therefore surjective. Using now the exact homotopy sequence of $H(\psi_1, \psi_2) \longrightarrow Spin^+(4, 3) \longrightarrow V(-1, -1)$

$$\dots \longrightarrow \pi_1(V(-1, -1)) \longrightarrow \pi_0(H(\psi_1, \psi_2), 1) \longrightarrow \pi_0(Spin^+(4, 3), 1) = 0$$

we deduce from $\pi_1(V(-1, -1)) = 0$ that $H(\psi_1, \psi_2)$ is connected. Thus $H(\psi_1, \psi_2) = SU(1, 2)$.

Now we turn to the Lie algebra of $H(\psi_1, \psi_5)$. It is equal to

$$\mathfrak{h}(\psi_1, \psi_5) = \left\{ \sum_{i < j} \omega_{ij} e_i e_j \mid \begin{array}{l} -\omega_{16} - \omega_{25} + \omega_{37} = 0 \ , \\ \omega_{12} - \omega_{56} = 0 \ . \ \omega_{13} - \omega_{67} = 0 \ , \\ \omega_{23} - \omega_{57} = 0 \ . \ \omega_{15} - \omega_{26} = 0 \ , \\ \omega_{17} + \omega_{36} = 0 \ . \ \omega_{27} + \omega_{35} = 0 \ , \\ \omega_{i4} = 0 \ . \ i = 1, 2, 3 \\ \omega_{4j} = 0 \ . \ j = 5, 6, 7 \end{array} \right\}. \quad (12)$$

Using the isomorphism of $\mathfrak{spin}(4, 3)$ and $\mathfrak{so}(4, 3)$, we see that the Killing form on $\mathfrak{h}(\psi_1, \psi_5)$ is non-degenerate and has index 3. Therefore, $\mathfrak{h}(\psi_1, \psi_5)$ is a non-compact real form of the semisimple Lie algebra $\mathfrak{h}(\psi_1, \psi_5)^{\mathbb{C}}$. Since, furthermore, $\mathfrak{h}(\psi_1, \psi_5)^{\mathbb{C}}$ has dimension 8 it must be simple and therefore equal to $\mathfrak{sl}(3, \mathbb{C})$. The index of the Killingform distinguishes the various real forms of $\mathfrak{sl}(3, \mathbb{C})$. We conclude that $\mathfrak{h}(\psi_1, \psi_5)$ equals $\mathfrak{sl}(3, \mathbb{R})$. Next we prove that $H(\psi_1, \psi_5)$ is connected and has fundamental group $\mathbb{Z}_2$ what implies immediately $H(\psi_1, \psi_5) = SL(3, \mathbb{R})$ since the center of the universal covering of $SL(3, \mathbb{R})$ equals $\mathbb{Z}_2$. To begin with, we compute the first and second homotopy group of $V(-1, 1)$ using the exact homotopy sequence of the fibration $SO^+(3, 3) \xrightarrow{i} SO^+(4, 4) \longrightarrow V(-1, 1)$. This sequence equals

$$\begin{aligned} \dots \longrightarrow \pi_2(SO^+(4, 4)) \longrightarrow \pi_2(V(-1, 1)) \longrightarrow \pi_1(SO^+(3, 3)) \xrightarrow{i_{\#}} \\ \xrightarrow{i_{\#}} \pi_1(SO^+(4, 4)) \longrightarrow \pi_1(V(-1, 1)) \longrightarrow \pi_0(SO^+(3, 3), I) \longrightarrow \dots \end{aligned}$$

and consists of the groups

$$\begin{aligned} \dots \longrightarrow 0 \longrightarrow \pi_2(V(-1, 1)) \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{i_{\#}} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \\ \longrightarrow \pi_1(V(-1, 1)) \longrightarrow 0 \longrightarrow \dots \end{aligned}$$

Since $i_{\#}$ is an isomorphism we see that $\pi_2(V(-1, 1)) = \pi_1(V(-1, 1)) = 0$. A look at the exact homotopy sequence of the fibration $H(\psi_1, \psi_5) \longrightarrow$

$Spin^+(4, 3) \longrightarrow V(-1, 1)$ now shows that $\pi_1(H(\psi_1, \psi_5)) = \pi_1(Spin^+(4, 3)) = \mathbb{Z}_2$ and $\pi_0(H(\psi_1, \psi_5), 1) = 0$.

Proposition 2.11 *The action of $Spin^+(4, 3)$ on the Stiefel manifolds $V(-1, -1, -1), V(-1, -1, 1), V(-1, 1, 1)$ and $V(1, 1, 1)$ is transitive.*

Proof. As in the proof of Proposition 2.9 it suffices to consider $V(-1, -1, -1)$ and $V(-1, -1, 1)$. Again we calculate the Lie algebras of the corresponding isotropy groups. Let φ_1, φ_2 and φ_3 pseudo-orthonormal spinors with $\langle \varphi_1, \varphi_1 \rangle_\Delta = -1$ and $\langle \varphi_2, \varphi_2 \rangle_\Delta = -1$. Because of Proposition 2.9 we may assume $\varphi_1 = \psi_1$ and $\varphi_2 = \psi_2$. Again the isotropy group of (ψ_1, ψ_2) contains the same subgroup isomorphically to $SU(2)$ as mentioned in the proof of Proposition 2.9 and the group $SO(2) \subset SO(3)$ acting on $span\{\psi_3, \psi_4\}$. Therefore we may set $\varphi_3 = x_3\psi_3 + x_5\psi_5$. Then the isotropy group of $(\varphi_1, \varphi_2, \varphi_3)$ has the Lie algebra

$$\begin{aligned} \mathfrak{h}(\psi_1, \psi_2, x_3\psi_3 + x_5\psi_5) = \\ = \left\{ \sum_{i < j} \omega_{ij} e_i e_j \mid \begin{aligned} & -\omega_{12} - \omega_{34} + \omega_{56} = 0, \\ & \omega_{13} - \omega_{24} = 0, \quad \omega_{14} + \omega_{23} = 0, \\ & \omega_{16} + \omega_{25} = 0, \quad \omega_{15} - \omega_{26} = 0, \\ & \omega_{36} + \omega_{45} = 0, \quad \omega_{35} - \omega_{46} = 0, \\ & x_4\omega_{56} - x_5\omega_{45} = 0, \quad x_5\omega_{34} + x_4\omega_{36} = 0, \\ & x_5\omega_{24} + x_4\omega_{26} = 0, \quad x_5\omega_{14} + x_4\omega_{16} = 0 \\ & x_5\omega_{46} = 0, \quad x_4\omega_{46} = 0 \\ & \omega_{i7} = 0, \quad i = 1, \dots, 6 \end{aligned} \right\}. \end{aligned} \quad (13)$$

Since not $x_3 = x_5 = 0$, the dimension of $\mathfrak{h}(\varphi_1, \varphi_2, \varphi_3)$ equals 3 and the action is transitive.

Corollary 2.12 *The isotropy group of a triple $(\varphi_1, \varphi_2, \varphi_3)$ of pseudo-orthonormal spinors with respect to the $Spin^+(4, 3)$ -action equals*

1. $SU(2)$ if $(\varphi_1, \varphi_2, \varphi_3) \in V(-1, -1, -1)$ or $V(1, 1, 1)$
2. $SL(2, \mathbb{R})$ if $(\varphi_1, \varphi_2, \varphi_3) \in V(-1, -1, 1)$ or $V(-1, 1, 1)$.

Proof. The Lie algebra of the isotropy group $H(\psi_1, \psi_2, \psi_3)$ of (ψ_1, ψ_2, ψ_3) equals

$$\mathfrak{h}(\psi_1, \psi_2, \psi_3) = \left\{ \sum_{i < j} \omega_{ij} e_i e_j \mid \begin{aligned} & \omega_{12} + \omega_{34} = 0, \\ & \omega_{13} - \omega_{24} = 0, \quad \omega_{14} + \omega_{23} = 0, \\ & \omega_{i5} = \omega_{i6} = \omega_{i7} = 0 \end{aligned} \right\}. \quad (14)$$

As a subalgebra of $\mathfrak{so}(4,4)$ it is spanned by $E_{56} - E_{78}$, $E_{57} + E_{68}$ and $E_{58} - E_{67}$ and equals therefore $\mathfrak{su}(2)$ where $SU(2) \subset SU(2,2) \subset SO(4,4)$ is imbedded in the usual way. In particular, the connected component of the unity of $H(\psi_1, \psi_2, \psi_3)$ is isomorphic to $SU(2)$. It remains to prove that $H(\psi_1, \psi_2, \psi_3)$ is connected. A look at the exact homotopy sequence of the fibration $H(\psi_1, \psi_2, \psi_3) \longrightarrow Spin^+(4,3) \longrightarrow V(-1, -1, -1)$ shows that

$$\begin{aligned} \dots \longrightarrow \pi_1(H(\psi_1, \psi_2, \psi_3), 1) &\longrightarrow \mathbb{Z}_2 \longrightarrow \pi_1(V(-1, -1, -1)) \longrightarrow \\ &\longrightarrow \pi_0(H(\psi_1, \psi_2, \psi_3)) \longrightarrow 0 \longrightarrow \dots \end{aligned}$$

is exact. Since $\pi_1((H(\psi_1, \psi_2, \psi_3), 1) = \pi_1(SU(2)) = 0$ it suffices to prove that $\pi_1(V(-1, -1, -1))$ equals $\mathbb{Z}_2$. But this is clear from the homotopy sequence of $SO^+(1,4) \xrightarrow{i} SO^+(4,4) \longrightarrow V(-1, -1, -1)$. Indeed,

$$\begin{aligned} \dots \longrightarrow \pi_1(SO^+(1,4)) &\xrightarrow{i_\#} \pi_1(SO^+(4,4)) \longrightarrow \pi_1(V(-1, -1, -1)) \longrightarrow \\ &\longrightarrow \pi_0(SO^+(1,4), I) \longrightarrow \dots \end{aligned}$$

is the sequence of the following groups

$$\dots \longrightarrow \mathbb{Z}_2 \xrightarrow{i_\#} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \pi_1(V(-1, -1, -1)) \longrightarrow 0 \longrightarrow \dots$$

where $i_\#$ sends $1 \in \mathbb{Z}_2$ to $(0, 1) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

We now prove the second assertion in the same way. The Lie algebra of the isotropy group $H(\psi_1, \psi_5, \psi_6)$ is

$$\mathfrak{h}(\psi_1, \psi_5, \psi_6) = \left\{ \sum_{i < j} \omega_{ij} c_i c_j \mid \begin{aligned} &\omega_{12} - \omega_{56} = 0 \ , \\ &\omega_{16} + \omega_{25} = 0 \ , \ \omega_{15} - \omega_{26} = 0 \ , \\ &\omega_{i3} = \omega_{i4} = \omega_{i7} = 0 \end{aligned} \right\}. \quad (15)$$

As a subalgebra of $\mathfrak{so}(4,4)$ it is spanned by $E_{34} - E_{78}$, $A_{37} + A_{48}$ and $A_{38} - A_{47}$ and equals therefore $\mathfrak{su}(1,1)$ where $SU(1,1) \subset SU(2,2) \subset SO(4,4)$ is imbedded in the usual way. In particular, the connected component of $H(\psi_1, \psi_2, \psi_5)$ is isomorphic to $SU(2)$ which is on the other hand isomorphic to $SL(2; \mathbb{R})$. To show that $H(\psi_1, \psi_5, \psi_6)$ is connected it suffices to verify that the Stiefel manifold is simply connected. But this follows again from the exact homotopy sequence of the corresponding fibration $SO^+(3,2) \xrightarrow{i} SO^+(4,4) \longrightarrow V(-1, 1, 1)$. Indeed,

$$\begin{aligned} \dots \longrightarrow \pi_1(SO^+(3,2)) &\xrightarrow{i_\#} \pi_1(SO^+(4,4)) \longrightarrow \pi_1(V(-1, 1, 1)) \longrightarrow \\ &\longrightarrow \pi_0(SO^+(3,2), I) \longrightarrow \dots \end{aligned}$$

is the sequence of the following groups

$$\dots \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z} \xrightarrow{i_{\#}} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \pi_1(V(-1, 1, 1)) \longrightarrow 0 \longrightarrow \dots$$

where $i_{\#}$ sends $(1, 1) \in \mathbb{Z}_2 \oplus \mathbb{Z}$ to $(1, 1) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and is therefore surjective.

The rest of this section is devoted to real representations of $G_{2(2)}^*$. Recall that all complex representations of $\mathfrak{g}_{2(2)}$ are of real type [8]. Therefore, the real irreducible representations of the universal covering $\widetilde{G_{2(2)}}$ of $G_{2(2)}^*$ correspond to the real forms of the complex irreducible representations of $\mathfrak{g}_{2(2)}$. On the other hand the fundamental representations of $\widetilde{G_{2(2)}}$, i.e. the standard representation on $\mathbb{R}^7$ and the adjoint representation are in fact representations of $G_{2(2)}^*$. Thus all representations of $\widetilde{G_{2(2)}}$ are representations of $G_{2(2)}^*$. We conclude that the real irreducible representations of $G_{2(2)}^*$ correspond exactly to the complex irreducible representations of $\mathfrak{g}_{2(2)}$. In particular, the dimensions of the irreducible real representations are 1, 7, 14, 27, ... Furthermore the decomposition of $\Lambda^p(R^7)$ into irreducible components of the $G_{2(2)}^*$ -action is similar to that with respect to the action of the compact group G_2 . Denote by $*$ the Hodgeoperator of the pseudo-Euclidean space $(\mathbb{R}^7, g_{4,3})$ and let ω_0^3 be the 3-form defined by (8). Then we have

Proposition 2.13 1. $R^7 = \Lambda^1(R^7) =: \Lambda_7^1$ is irreducible.

2. $\Lambda^2(R^7) = \Lambda_7^2 \oplus \Lambda_{14}^2$, where

$$\begin{aligned} \Lambda_7^2 &= \{\alpha^2 \in \Lambda^2 \mid *(\omega_0^3 \wedge \alpha^2) = 2\alpha^2\} = \{X \lrcorner \omega_0^3 \mid X \in R^7\} \\ \Lambda_{14}^2 &= \{\alpha^2 \in \Lambda^2 \mid *(\omega_0^3 \wedge \alpha^2) = -\alpha^2\} = \mathfrak{g}_{2(2)} \end{aligned}$$

3. $\Lambda^3(R^7) = \Lambda_1^3 \oplus \Lambda_7^3 \oplus \Lambda_{27}^3$, where

$$\begin{aligned} \Lambda_1^3 &= \{t\omega_0^3 \mid t \in \mathbb{R}\} \\ \Lambda_7^3 &= \{*(\omega_0^3 \wedge \alpha^1) \mid \alpha^1 \in \Lambda_7^1\} \\ \Lambda_{27}^3 &= \{\alpha^3 \in \Lambda^3 \mid \alpha^3 \wedge \omega_0^3 = 0, \alpha^3 \wedge *\omega_0^3 = 0\} \end{aligned}$$

3 Killing spinors

Now let $(M^{4,3}, g_{4,3})$ be a 7-dimensional pseudo-Riemannian spin manifold of signature (4,3) which is space and time oriented. Assume that $M^{4,3}$ admits a spin structure $Q(M^{4,3})$. This is a $Spin^+(4,3)$ -reduction of the bundle

$R(M^{4,3})$ of all space and time oriented pseudo-orthonormal frames. Then the spinor bundle S of $M^{4,3}$ is the associated bundle $Q(M^{4,3}) \times_{Spin^+(4,3)} \Delta_{4,3}$. Furthermore ∇ denotes the Levi-Civita-connection on the tangent bundle $TM^{4,3}$ as well as the induced covariant derivative on S . The pseudo-Euclidean product $\langle \cdot, \cdot \rangle_\Delta$ on $\Delta_{4,3}$ induces a product of signature (4,4) on S .

Definition 3.1 *A section $\psi \in \Gamma(S)$ is called Killing spinor if there is a real number $\lambda \neq 0$ such that the differential equation*

$$\nabla_X \psi = \lambda X \cdot \psi$$

is satisfied for all vector fields $X \in \mathfrak{X}(M^{4,3})$. We call λ the Killing number of ψ .

The following properties of Killing spinors are well known [1]. Let $\psi \in \Gamma(S)$ be a Killing spinor on $M^{4,3}$ with Killing number λ . Then $\langle \psi, \psi \rangle_\Delta$ is constant on $M^{4,3}$. For the Ricci map $Ric : TM^{4,3} \rightarrow TM^{4,3}$ of the tangent bundle the equation $Ric(X)\psi = 24\lambda^2 X \cdot \psi$ holds. If ψ is non-isotropic this means that $M^{4,3}$ is an Einstein manifold of scalar curvature $\tau = 168\lambda^2$. Now let W be the Weyl tensor of $M^{4,3}$. Then $W(X, Y) \cdot \psi = 0$ for all $X, Y \in \mathfrak{X}(M^{4,3})$, where this product is defined in the following way. Let $s_1, s_2, \dots, s_7$ be a local pseudo-orthonormal frame, $\varepsilon_i = g(s_i, s_i)$ and $W_{ijkl} = W(s_i, s_j, s_k, s_l)$. Then set

$$W(s_i, s_j) \cdot \psi = \sum_{k < l} \varepsilon_k \varepsilon_l W_{ijkl} s_k \cdot s_l \cdot \psi \quad .$$

Of course, parallel spinors have the same properties. We now turn to the question how many Killing spinors can exist on $(M^{4,3}, g_{4,3})$.

Theorem 3.2 *If there exist four orthogonal non-isotropic Killing spinors with the same Killing number on $(M^{4,3}, g_{4,3})$ such that at least three of them have the same length then $M^{4,3}$ is conformally flat.*

Proof. Let $\varphi_1, \dots, \varphi_4$ be four such Killing spinors. Let $\langle \varphi_\alpha, \varphi_\alpha \rangle_\Delta = -1$ for $\alpha = 1, 2, 3$. Because of the transitivity of the $Spin^+(4, 3)$ -action on $V(-1, -1, -1)$ we may assume that for some local time and space oriented pseudo-orthonormal frame $s_1, \dots, s_7$ the spinor φ_α equals ψ_α for $\alpha = 1, 2, 3$. Moreover, since the isotropy group of (ψ_1, ψ_2, ψ_3) equals $SU(2)$ acting on $span\{\psi_5, \psi_6, \psi_7, \psi_8\}$ we can assume $\varphi_4 = x_4 \psi_4 + x_5 \psi_5$ where x_4 and x_5 are real functions. The condition $W(s_i, s_j) \cdot \varphi_\alpha = 0$ ($\alpha = 1, 2, 3$) implies

$$W_{ij12} + W_{ij34} = 0, \quad W_{ij13} - W_{ij24} = 0, \quad W_{ij14} + W_{ij23} = 0$$

and $W_{ijkl} = 0$ for any other k, l . Furthermore, we have

$$\begin{aligned} 0 &= W(s_i, s_j) \cdot (x_4\psi_4 + x_5\psi_5) = \sum_{k < l} \varepsilon_k \varepsilon_l W_{ijkl} s_k \cdot s_l \cdot (x_4\psi_4 + x_5\psi_5) \\ &= x_5 \{ (-W_{ij12} + W_{ij34})\psi_6 + (W_{ij13} + W_{ij24})\psi_7 + (W_{ij14} + W_{ij23})\psi_8 \}. \end{aligned}$$

Consequently, in case $x_5 \neq 0$ the Weyl tensor must vanish and we are done. Consider now the case $x_5(m) = 0$ for $m \in M^{4,3}$. If there is any sequence $m_n \in M^{4,3}$ which converges to m and such that $x_5(m_n) \neq 0$ then by continuity of the Weyl tensor we have again $W(m) = 0$. Assume now that $x_5(m) = 0$ on an open set containing m , i.e. $\varphi_4 = \psi_4$. Since $\varphi_1, \dots, \varphi_4$ are Killing spinors we have $\nabla_{s_1} \psi_\alpha = \lambda s_1 \cdot \psi_\alpha$ ($\alpha = 1, \dots, 4$). We can calculate the covariant derivative using the local connection forms $\omega_{ij} = \varepsilon_i \varepsilon_j g_{4,3}(\nabla s_i, s_j)$ and obtain

$$\nabla_{s_1} \psi_\alpha = \frac{1}{2} \sum_{i < j} \varepsilon_i \varepsilon_j \omega_{ij}(s_1) s_i \cdot s_j \cdot \psi_\alpha = \lambda s_1 \cdot \psi_\alpha \quad (\alpha = 1, \dots, 4)$$

In particular,

$$\begin{aligned} -\omega_{27}(s_1) - \omega_{35}(s_1) + \omega_{46}(s_1) &= 2\lambda \\ -\omega_{27}(s_1) + \omega_{35}(s_1) - \omega_{46}(s_1) &= 2\lambda \\ -\omega_{27}(s_1) + \omega_{35}(s_1) + \omega_{46}(s_1) &= -2\lambda \\ -\omega_{27}(s_1) - \omega_{35}(s_1) - \omega_{46}(s_1) &= -2\lambda \end{aligned}$$

which is impossible if $\lambda \neq 0$. The assertion can be proved similarly if $\langle \varphi_\alpha, \varphi_\alpha \rangle_\Delta = 1$ for $\alpha = 1, 2, 3$.

3.1 Geometrical and nearly parallel $G_{2(2)}^*$ -structures

Let M^7 be a 7-dimensional manifold and $R(M^7)$ the frame bundle of M^7 . We define the bundle $\Lambda_*^3(M^7)$ by

$$\Lambda_*^3(M^7) := R(M^7) \times_{GL(7)} \Lambda_*^3(R^7) \subset R(M^7) \times_{GL(7)} \Lambda^3(R^7) = \Lambda^3(M^7).$$

where $\Lambda_*^3(R^7)$ is the open subset $\{A^* \omega_0^3 \mid A \in GL(7)\}$ of $\Lambda^3(R^7)$.

Definition 3.3 *A topological G_2 -structure ($Spin^+(4, 3)$ -structure) on M^7 is a G_2 -reduction ($Spin^+(4, 3)$ -reduction) of the frame bundle $R(M^7)$.*

The fact that $G_{2(2)}^*$ is a subset of $SO^+(4,3)$ and of $Spin^+(4,3)$ implies that a $G_{2(2)}^*$ -structure $P \subset R(M^7)$ on M^7 induces an orientation of M^7 (i.e. $\omega_1 = 0$) , a pseudo-Riemannian metric $g_{4,3}$ of index 4 on M^7 together with a space and time orientation such that the corresponding $SO^+(4,3)$ -bundle equals $P \times_{G_{2(2)}^*} SO^+(4,3)$ and a spin structure $P \times_{G_2} Spin^+(4,3)$. Furthermore it defines the following spinor $\psi \in \Gamma(S)$ of length -1 in the real spinor bundle $S = P \times_{G_{2(2)}^*} \Delta_{4,3}$ of M^7 . Since $G_{2(2)}^* \subset Spin^+(4,3)$ is the isotropy group of $\psi_1 \in \Delta_{4,3}$ the map $\psi : P \longrightarrow \Delta_{4,3}; \psi(p) = \psi_1$ has the property $\psi(pg) = g^{-1}\psi$ for all $g \in G_{2(2)}^*$ and is therefore a section in S . Because of the $G_{2(2)}^*$ -invariance of ω_0 the $G_{2(2)}^*$ -structure defines in the same way a section ω^3 in $\Lambda_*^3(M^7) = R(M^7) \times_{GL(7)} \Lambda_*^3(R^7) = P_{G_2} \times_{G_{2(2)}^*} \Lambda_*^3(R^7)$ by $\omega^3 : P \longrightarrow \Lambda_*^3(R^7); \omega^3(p) = \omega_0^3$. On the other hand the spinor ψ defines a $(2,1)$ -tensor field $A = A_\psi$ (see equation (6)) on M^7 and $\omega^3 = g_{4,3}(\cdot, A(\cdot, \cdot))$ holds.

Vice versa, suppose we are given a 3-form ω^3 in $\Lambda_*^3(M^7)$ then M^7 admits a $G_{2(2)}^*$ -structure P consisting of all frames relative to those ω^3 equals ω_0^3 . Secondly, given a pseudo-Riemannian metric $g_{4,3}$, a space and time orientation, a $Spin^+(4,3)$ -structure and a spinor ψ of length -1 on M^7 then M^7 admits a $G_{2(2)}^*$ -structure P consisting of all frames relative to those ψ equals ψ_0 .

Now we turn to geometrical $G_{2(2)}^*$ -structures.

Definition 3.4 Let $P \subset R(M^7)$ be a topological $G_{2(2)}^*$ -structure on M^7 and $g_{4,3}$ the associated Riemannian metric with Hodge operator $*$. P is said to be geometrical if one of the following equivalent conditions is satisfied.

- (i) ∇ reduces to P .
- (ii) The holonomy group $Hol(M^7, g)$ of M^7 is contained in $G_{2(2)}^*$.
- (iii) The associated 3-form ω^3 is parallel, i.e. $\nabla \omega^3 = 0$.
- (iv) $d\omega^3 = 0, \quad d * \omega^3 = 0$.
- (v) The associated spinor field ψ is parallel, i.e. $\nabla \psi = 0$.

For a proof of (iii) $\iff$ (iv) see [3], [5], [6].

Now we can generalize the condition $\nabla \psi = 0$ and obtain the notion of a nearly parallel $G_{2(2)}^*$ -structure.

Definition 3.5 Let $P \subset R(M^7)$ be a topological $G_{2(2)}^*$ -structure on M^7 and $g_{4,3}$ the associated Riemannian metric with Hodge operator $*$. P is said to be nearly parallel if one of the following equivalent conditions is satisfied.

(i) The associated spinor ψ is a Killing spinor with Killing number λ .

(ii) The associated tensor A satisfies

$$(\nabla_Z A)(Y, X) = 2\lambda\{g_{4,3}(Y, Z)X - g_{4,3}(X, Z)Y + A(Z, A(Y, X))\}.$$

(iii) The associated 3-form ω^3 satisfies

$$\nabla_Z \omega^3 = -2\lambda(Z \lrcorner * \omega^3).$$

(iv) The associated 3-form ω^3 satisfies

$$d * \omega^3 = 0, \quad d\omega^3 = -8\lambda * \omega^3.$$

For a proof of (iii) $\iff$ (iv) see [4].

3.2 Examples of homogeneous spaces with Killing spinors

This section is devoted to the construction of first examples of pseudo-Riemannian manifolds of signature (4,3) with Killing spinors. We use the twistor spaces $Z^-(\mathbb{R}P^{1,0})$ and $Z^-(\mathbb{C}P^{2,0})$ of the negativ definit elliptic spaces $\mathbb{R}P^{4,0}$ and $\mathbb{C}P^{2,0}$. One obtains $\mathbb{R}P^{1,0}$ and $\mathbb{C}P^{2,0}$ from the real and the complex hyperbolic space on replacing the metric by its negative. $Z^-(\mathbb{R}P^{4,0})$ is the homogeneous space $Sp(1,1)/U(1) \times Sp(1) = SO^+(4,1)/U(2)$ and $Z^-(\mathbb{C}P^{2,0})$ equals $U(2,1)/U(1) \times U(1) \times U(1)$. There exist two homogeneous Einstein metrics of signature (4,2) on each of these spaces one of both being Kählerian. These Kählerian metrics are used to construct S^1 -bundles over $Z^-(\mathbb{R}P^{1,0})$ and $Z^-(\mathbb{C}P^{2,0})$ admitting three linearly independent Killing spinors. One can obtain a further Einstein metric with one Killing spinor on each of these bundles by squashing the metric on the S^3 -fibres over $\mathbb{R}P^{4,0}$ and $\mathbb{C}P^{2,0}$, respectively. The other examples are warped products with six-dimensional pseudo-Riemannian manifolds of signature (4,2) with Killing spinors, in particular with the second Einstein metric on the twistor spaces.

3.2.1 The squashed 7-sphere

Consider the sphere $S^{4,3} = Sp(1,1)/Sp(1)$ where

$$Sp(1,1) = \{A \in \mathbb{H}(2) \mid {}^t A \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \bar{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\}$$

and the imbedding $Sp(1) \hookrightarrow Sp(1,1)$ is given by

$$Sp(1) \ni C \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix} \in Sp(1,1) .$$

Let B be the $Ad(Sp(1))$ -invariant bilinear form on $\mathfrak{sp}(1,1)$ defined by $B(X,Y) = -\text{Re tr}XY$ for $X,Y \in \mathfrak{sp}(1,1)$. We decompose

$$\mathfrak{sp}(1,1) = \mathfrak{sp}(1) \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$$

where

$$\begin{aligned} \mathfrak{m}_1 = \text{span} \left\{ \bar{e}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \bar{e}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \bar{e}_3 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \right. \\ \left. \bar{e}_4 = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix} \right\} \end{aligned}$$

$$\mathfrak{m}_2 = \text{span} \left\{ \bar{e}_5 = \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, \bar{e}_6 = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \bar{e}_7 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \right\} .$$

Note that $\mathfrak{m}_1 = \mathbb{H}, \mathfrak{m}_2 = \text{im}\mathbb{H}$. We have $[\mathfrak{sp}(1), \mathfrak{m}_1] = \mathfrak{m}_1$ and $[\mathfrak{sp}(1), \mathfrak{m}_2] = 0$. Now we rescale B in direction $\mathfrak{m}_2$ and get a bilinear form

$$B_t = B|_{\mathfrak{m}_1 \times \mathfrak{m}_1} \oplus 2tB|_{\mathfrak{m}_2 \times \mathfrak{m}_2}$$

on $\mathfrak{m}_1 \oplus \mathfrak{m}_2$. Then

$$e_i = \frac{1}{\sqrt{2}}\bar{e}_i (i = 1, \dots, 4), \quad e_j = \frac{1}{\sqrt{2t}}\bar{e}_j (j = 5, 6, 7)$$

is a pseudo-orthonormal basis of $\mathfrak{m}_1 \oplus \mathfrak{m}_2$. Identifying $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ with the tangent space of $Sp(1,1)/Sp(1)$ at $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot Sp(1,1)$ we obtain a left invariant metric g_t on $S^{4,3}$. The isotropy representation $\alpha : Sp(1) \longrightarrow SO(\mathfrak{m}_1 \oplus \mathfrak{m}_2)$

assigns to an element $q \in Sp(1)$ that linear map which is the multiplication by q on $\mathfrak{m}_1$ and the identity on $\mathfrak{m}_2$. Therefore α is the standard imbedding

$$\alpha : Sp(1) \hookrightarrow SO(4) \subset SO(4, 3) .$$

Since $Spin(4) = Sp(1) \times Sp(1)$,

$$\begin{aligned} \tilde{\alpha} : Sp(1) &\hookrightarrow Sp(1) \times Sp(1) \subset Spin^+(4, 3) \\ q &\longmapsto (q, 1) \end{aligned}$$

defines a lift of α . From equation (4) we conclude that $\tilde{\alpha}(q)$ acts trivially on $\text{span}\{\psi_5, \psi_6, \psi_7, \psi_8\}$ for any $q \in Sp(1)$. $\tilde{\alpha}$ defines a spin structure

$$Q = Sp(2, 1) \times_{\tilde{\alpha}} Spin(4, 3)$$

and the associated spinor bundle

$$S = Sp(2, 1) \times_{\tilde{\alpha}} \Delta_{4,3} .$$

Sections of S are identified with functions $\psi : Sp(2, 1) \rightarrow \Delta_{4,3}$ satisfying the equation $\psi(gh) = h^{-1}\psi(g)$ for all $g \in Sp(2, 1)$ and $h \in Sp(1)$. In particular constant functions $\psi(g) = \psi_0 \in \text{span}\{\psi_5, \psi_6, \psi_7, \psi_8\}$, $g \in Sp(2, 1)$ define spinors on $S^{4,3}$.

Using Wang's theorem we describe the connection on S induced by the Levi-Civita connection of $(S^{4,3}, g_t)$. It is given by

$$\begin{aligned} \tilde{\Lambda} : \mathfrak{m}_1 \oplus \mathfrak{m}_2 &\longmapsto \mathfrak{spin}(4, 3) \\ \tilde{\Lambda}(e_1) &= \frac{\sqrt{2t}}{4}(-e_2e_7 - e_3e_5 - e_4e_6) \\ \tilde{\Lambda}(e_2) &= \frac{\sqrt{2t}}{4}(e_1e_7 + e_3e_6 - e_4e_5) \\ \tilde{\Lambda}(e_3) &= \frac{\sqrt{2t}}{4}(e_1e_5 - e_2e_6 + e_4e_7) \\ \tilde{\Lambda}(e_4) &= \frac{\sqrt{2t}}{4}(e_1e_6 + e_2e_5 - e_3e_7) \\ \tilde{\Lambda}(e_5) &= \frac{t-1}{2\sqrt{2t}}(-e_1e_3 - e_2e_4) + \frac{1}{2\sqrt{2t}}e_6e_7 \\ \tilde{\Lambda}(e_6) &= \frac{t-1}{2\sqrt{2t}}(-e_1e_4 + e_2e_3) - \frac{1}{2\sqrt{2t}}e_5e_7 \\ \tilde{\Lambda}(e_7) &= \frac{t-1}{2\sqrt{2t}}(-e_1e_2 + e_3e_4) + \frac{1}{2\sqrt{2t}}e_5e_6 \end{aligned}$$

We have now to check for which choice of t and λ do exist common solutions $\psi_0 \in \text{span}\{\psi_5, \psi_6, \psi_7, \psi_8\}$ of

$$\tilde{\Lambda}(c_i)\psi_0 - \lambda c_i\psi_0 = 0, \quad i = 1, \dots, 7.$$

There are exactly two possibilities. In case $t = 1$, $\lambda = -\frac{1}{2\sqrt{2}}$ we obtain three linear independent Killing spinors (ψ_6, ψ_7, ψ_8) on the standard sphere. In case $t = \frac{1}{5}$, $\lambda = \frac{3}{2\sqrt{10}}$ we get a further Einstein metric on $S^{4,3}$ together with one Killing spinor (ψ_5) .

3.2.2 The space $\tilde{N}(1,1)$

Consider now the homogeneous space $\tilde{N}(1,1) = SU(2,1)/S^1$ where the imbedding of S^1 into $SU(2,1)$ is given by

$$\begin{aligned} S^1 &\hookrightarrow SU(2,1) \\ \theta &\longmapsto \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-2i\theta} \end{pmatrix} \end{aligned}$$

Denote by $\mathfrak{k}$ the Lie algebra of S^1 contained in $\mathfrak{su}(2,1)$. Let B be the $Ad(S^1)$ -invariant bilinear form on $\mathfrak{su}(2,1)$ defined by $B(X,Y) = -\text{Re tr}XY$ for $X, Y \in \mathfrak{su}(2,1)$. Now we decompose $\mathfrak{su}(2,1)$ into

$$\mathfrak{su}(2,1) = \mathfrak{k} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$$

where

$$\begin{aligned} \mathfrak{m}_1 = \text{span} \quad & \left\{ \bar{e}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \bar{e}_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \right. \\ & \left. \bar{e}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \bar{e}_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \right\} \\ \mathfrak{m}_2 = \text{span} \quad & \left\{ \bar{e}_5 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{e}_6 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \right. \\ & \left. \bar{e}_7 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}. \end{aligned}$$

We have $[\mathfrak{k}, \mathfrak{m}_1] = \mathfrak{m}_1$ and $[\mathfrak{k}, \mathfrak{m}_2] = 0$. Now we continue as in the case of the sphere. We rescale B and obtain

$$B_t = B|_{\mathfrak{m}_1 \times \mathfrak{m}_1} \oplus 2tB|_{\mathfrak{m}_2 \times \mathfrak{m}_2}$$

on $\mathfrak{m}_1 \oplus \mathfrak{m}_2$, which yields a left invariant metric g_t on $\tilde{N}(1, 1)$. Then

$$e_i = \frac{1}{\sqrt{2}}\bar{e}_i (i = 1, \dots, 4), \quad e_j = \frac{1}{2\sqrt{t}}\bar{e}_j (j = 5, 6, 7)$$

is a pseudo-orthonormal basis of $\mathfrak{m}_1 \oplus \mathfrak{m}_2$. With respect to this basis the isotropy representation $\alpha : S^1 \rightarrow SO(\mathfrak{m}_1 \oplus \mathfrak{m}_2)$ is given by

$$\alpha(\theta) = \begin{pmatrix} \cos 3\theta & \sin 3\theta & 0 & 0 \\ -\sin 3\theta & \cos 3\theta & 0 & 0 \\ 0 & 0 & \cos 3\theta & \sin 3\theta \\ 0 & 0 & -\sin 3\theta & \cos 3\theta \end{pmatrix} \oplus id|_{\mathfrak{m}_2}.$$

We can lift α into $Spin^+(4, 3)$ and obtain

$$\begin{aligned} \tilde{\alpha} : S^1 &\hookrightarrow Spin^+(4, 3) \\ \theta &\longmapsto (\cos \frac{3}{2}\theta + \sin \frac{3}{2}\theta e_1 e_2)(\cos \frac{3}{2}\theta + \sin \frac{3}{2}\theta e_3 e_4). \end{aligned}$$

Therefore $\tilde{N}(1, 1)$ admits a spin structure. In particular constant functions $\psi(g) = \psi_0 \in \text{span}\{\psi_5, \psi_6, \psi_7, \psi_8\}$, $g \in SU(2, 1)$ define sections in the spinor bundle S of $\tilde{N}(1, 1)$. The connection on S with respect to g_t is given by

$$\begin{aligned} \tilde{\Delta} : \mathfrak{m}_1 \oplus \mathfrak{m}_2 &\longmapsto \mathfrak{spin}(4, 3) \\ \tilde{\Delta}(e_1) &= \frac{\sqrt{t}}{4}(-e_2 e_7 + e_3 e_5 - e_4 e_6) \\ \tilde{\Delta}(e_2) &= \frac{\sqrt{t}}{4}(e_1 e_7 + e_3 e_6 + e_4 e_5) \\ \tilde{\Delta}(e_3) &= \frac{\sqrt{t}}{4}(-e_1 e_5 - e_2 e_6 + e_4 e_7) \\ \tilde{\Delta}(e_4) &= \frac{\sqrt{t}}{4}(e_1 e_6 - e_2 e_5 - e_3 e_7) \\ \tilde{\Delta}(e_5) &= \frac{t-1}{4\sqrt{t}}(e_1 e_3 + e_2 e_4) - \frac{1}{4\sqrt{t}}e_6 e_7 \\ \tilde{\Delta}(e_6) &= \frac{t-1}{4\sqrt{t}}(-e_1 e_4 + e_2 e_3) + \frac{1}{4\sqrt{t}}e_5 e_7 \\ \tilde{\Delta}(e_7) &= \frac{t-1}{4\sqrt{t}}(-e_1 e_2 + e_3 e_4) - \frac{1}{4\sqrt{t}}e_5 e_6. \end{aligned}$$

Again we have to check whether there are common solutions $\psi_0 \in \text{span}\{\psi_5, \psi_6, \psi_7, \psi_8\}$ of

$$\tilde{\Lambda}(e_i)\psi_0 - \lambda e_i\psi_0 = 0, \quad i = 1, \dots, 7.$$

Again there are two possible choices of t . In case $t = 1$ we obtain the standard metric together with three linear independent Killing spinors (ψ_5, ψ_6, ψ_8) with Killing number $\lambda = \frac{1}{4}$. For $t = \frac{1}{5}$ we obtain a further Einstein metric on $\tilde{N}(1, 1)$ together with one Killing spinor (ψ_7) with Killing number $\lambda = -\frac{3}{4\sqrt{5}}$.

3.3 Warped products with Killing spinors

3.3.1 pseudo-Riemannian manifolds of signature (4,2)

Consider first $\mathbb{R}^6 = \text{span}\{e_1, \dots, e_6\} \subset \mathbb{R}^7$ with pseudo-Euclidean product $g_{4,2} = g_{4,3}|_{\mathbb{R}^7}$. We may restrict the real $Spin(4,3)$ -representation to $Spin(4,2)$ and obtain the unique irreducible real representation $\Delta_{4,2}$ of $Spin(4,2)$. The connected component $Spin^+(4,2)$ of $1 \in Spin(4,2)$ acts transitively on $S^{4,3}$ and $H^{3,4}$. Actually, the proof of Proposition 2.1 remains valid.

The multiplication of spinors by the volume form of $(\mathbb{R}^6, g_{4,2})$ yields a complex structure on $\Delta_{4,2}$. In fact, let $X_1, \dots, X_6$ be a positively oriented pseudo-orthonormal basis of $(\mathbb{R}^6, g_{4,2})$. Then we define J^Δ by $J^\Delta(\psi) = X_1 \cdot \dots \cdot X_6 \psi$. J^Δ does not depend on the choice of the pseudo-orthonormal basis. We have $J^\Delta = -I \otimes I \otimes \varepsilon$ with respect to the standard basis $\psi_1, \dots, \psi_8$. Furthermore J^Δ has the following properties.

1. $(J^\Delta)^2 = -1$
2. $X \cdot J^\Delta(\psi) = -J^\Delta(X \cdot \psi)$ for any $X \in \mathbb{R}^6$
3. Besides $\langle X \cdot \psi, \psi \rangle_\Delta = 0$ we also have $\langle X \cdot \psi, J^\Delta(\psi) \rangle_\Delta = 0$.

Therefore the map

$$\begin{aligned} \mathbb{R}^6 &\longrightarrow \{\psi, J^\Delta(\psi)\}^\perp \subset \mathbb{R}^8 \\ X &\longmapsto X \cdot \psi \end{aligned}$$

is an isomorphism for any spinor $\psi \in \Delta_{4,2}$ with $\langle \psi, \psi \rangle_\Delta \neq 0$. In particular, we obtain a complex structure J_ψ of $\mathbb{R}^6$ defined by

$$J_\psi(X) \cdot \psi := J^\Delta(X \cdot \psi) \text{ for any } X \in \mathbb{R}^6.$$

For instance,

$$J_{\psi_1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

with respect to $e_1, \dots, e_6$.

Now let $(F^{4,2}, h)$ be a pseudo-Riemannian manifold of signature $(4,2)$. J^Δ defines a complex structure J^S on the spinorbundle S^F of $F^{4,2}$. We have $\nabla J^S = 0$. Furthermore any nowhere vanishing nor isotropic section $\psi \in \Gamma(S^F)$ defines a complex structure J_ψ on $F^{4,2}$. Assume now that $F^{4,2}$ admits a Killing spinor $\varphi \neq 0$ with Killing number λ . Then $J^S(\varphi)$ is a Killingspinor with Killingnumber $-\lambda$, since

$$\nabla_X(J^S(\varphi)) = J^S(\nabla_X \varphi) = J^S(\lambda X \cdot \varphi) = -\lambda X \cdot J^S(\varphi).$$

If φ is non-isotropic then we can define J_φ which is nearly-Kachlerian.

Next we discuss two examples of such manifolds with Killing spinors.

3.3.2 $U(2,1)/U(1) \times U(1) \times U(1)$

Consider the homogeneous space $U(2,1)/U(1) \times U(1) \times U(1)$ where the imbedding of $U(1) \times U(1) \times U(1)$ into $U(2,1)$ is given by

$$\begin{aligned} U(1) \times U(1) \times U(1) &\hookrightarrow U(2,1) \\ (e^{it}, e^{is}, e^{ir}) &\longmapsto \text{diag}(e^{it}, e^{is}, e^{ir}) \end{aligned}$$

Denote by $\mathfrak{k}$ the Lie algebra of $U(1) \times U(1) \times U(1)$ contained in $\mathfrak{u}(2,1)$. Let B be the $Ad(U(1) \times U(1) \times U(1))$ -invariant bilinear form on $\mathfrak{u}(2,1)$ defined by $B(X, Y) = -\text{Re tr}XY$ for $X, Y \in \mathfrak{su}(2,1)$. We decompose $\mathfrak{u}(2,1)$ into

$$\mathfrak{u}(2,1) = \mathfrak{k} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$$

where

$$\begin{aligned} \mathfrak{m}_1 = \text{span} \quad \{ \quad \bar{e}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \bar{e}_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \\ \bar{e}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \bar{e}_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \} \end{aligned}$$

$$\mathfrak{m}_2 = \text{span} \left\{ \bar{e}_5 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \bar{e}_6 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

We rescale B and obtain

$$B_t = B|_{\mathfrak{m}_1 \times \mathfrak{m}_1} \oplus 2tB|_{\mathfrak{m}_2 \times \mathfrak{m}_2}$$

on $\mathfrak{m}_1 \oplus \mathfrak{m}_2$, which yields a left invariant metric g_t on $U(2,1)/U(1) \times U(1) \times U(1)$. Then

$$e_i = \frac{1}{\sqrt{2}}\bar{e}_i (i = 1, \dots, 4), \quad e_j = \frac{1}{2\sqrt{t}}\bar{e}_j (j = 5, 6)$$

is a pseudo-orthonormal basis of $\mathfrak{m}_1 \oplus \mathfrak{m}_2$. With respect to this basis the isotropy representation $\alpha : U(1) \times U(1) \times U(1) \longrightarrow SO(\mathfrak{m}_1 \oplus \mathfrak{m}_2)$ is given by

$$\alpha(e^{it}, e^{is}, e^{ir}) = \begin{pmatrix} D_{r-t} & 0 & 0 \\ 0 & D_{r-s} & 0 \\ 0 & 0 & D_{s-t} \end{pmatrix}$$

where D_θ denotes the rotation of the plane about θ . We can lift α into $Spin(4,2)^+$ and obtain

$$\begin{aligned} \tilde{\alpha} : U(1) \times U(1) \times U(1) &\hookrightarrow Spin^+(4,2) \\ \tilde{\alpha}(e^{it}, e^{is}, e^{ir}) &= (\cos \frac{r-t}{2} - \sin \frac{r-t}{2} e_1 e_2)(\cos \frac{r-s}{2} - \sin \frac{r-s}{2} e_3 e_4) \\ &\quad \cdot (\cos \frac{s-t}{2} + \sin \frac{s-t}{2} e_5 e_6) \end{aligned}$$

Therefore $U(2,1)/U(1) \times U(1) \times U(1)$ admits a spin structure. In particular constant functions $\psi(g) = \psi_0 \in \text{span}\{\psi_7, \psi_8\}, g \in U(2,1)$ define sections in the spinor bundle S of $U(2,1)/U(1) \times U(1) \times U(1)$. The connection on S with respect to g_t is given by

$$\begin{aligned} \tilde{\Lambda} : \mathfrak{m}_1 \oplus \mathfrak{m}_2 &\longmapsto \mathfrak{spin}(4,3) \\ \tilde{\Lambda}(e_1) &= \frac{\sqrt{t}}{4}(e_3 e_5 - e_4 e_6) \\ \tilde{\Lambda}(e_2) &= \frac{\sqrt{t}}{4}(+e_3 e_6 + e_4 e_5) \\ \tilde{\Lambda}(e_3) &= \frac{\sqrt{t}}{4}(-e_1 e_5 - e_2 e_6) \end{aligned}$$

$$\begin{aligned}
\tilde{\Lambda}(e_4) &= \frac{\sqrt{t}}{4}(e_1e_6 - e_2e_5) \\
\tilde{\Lambda}(e_5) &= \frac{t-1}{4\sqrt{t}}(e_1e_3 + e_2e_4) \\
\tilde{\Lambda}(e_6) &= \frac{t-1}{4\sqrt{t}}(-e_1e_4 + e_2e_3) .
\end{aligned}$$

If $t = \frac{1}{2}$ we obtain an Einstein metric with one Killingspinor for each of the values $\lambda = \pm \frac{1}{2\sqrt{2}}$ (ψ_8 and ψ_7 , respectively).

3.3.3 $SO^+(4, 1)/U(2)$

with canonical imbedding

$$U(2) \hookrightarrow SO(4) \hookrightarrow \begin{pmatrix} SO(4) & 0 \\ 0 & 1 \end{pmatrix} \hookrightarrow SO^+(4, 1).$$

Denote by D_{ij} the 5×5 -matrix consisting of a single 1 in the i -th row and j -th column, and zeros elsewhere. We set $E_{ij} = -D_{ij} + D_{ji}$ and $A_{ij} = D_{ij} + D_{ji}$. Then we have

$$\mathfrak{u}(2) = \text{span}\{E_{12}, E_{34}, E_{13} + E_{24}, E_{14} - E_{23}\}.$$

Let B be the $Ad(U(2))$ -invariant bilinear form on $\mathfrak{so}(4, 1)$ defined by $B(X, Y) = -\frac{1}{2}\text{Re tr}XY$ for $X, Y \in \mathfrak{so}(4, 1)$. We decompose $\mathfrak{so}(4, 1)$ into

$$\mathfrak{so}(4, 1) = \mathfrak{u}(2) \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$$

where

$$\mathfrak{m}_1 = \text{span}\{A_{15}, A_{25}, A_{35}, A_{45}\}, \mathfrak{m}_2 = \text{span}\{-E_{13} + E_{24}, -E_{14} - E_{23}\}.$$

We rescale B and obtain

$$B_t = B|_{\mathfrak{m}_1 \times \mathfrak{m}_1} \oplus tB|_{\mathfrak{m}_2 \times \mathfrak{m}_2}$$

on $\mathfrak{m}_1 \oplus \mathfrak{m}_2$, which yields a left invariant metric g_t on $SO^+(4, 1)/U(2)$. Then

$$\begin{aligned}
e_1 &= A_{15}, e_2 = A_{25}, e_3 = A_{35}, e_4 = A_{45} \\
e_5 &= \frac{1}{\sqrt{2t}}(E_{13} + E_{24}), e_6 = \frac{1}{\sqrt{2t}}(-E_{14} - E_{23})
\end{aligned}$$

is a pseudo-orthonormal basis of $\mathfrak{m}_1 \oplus \mathfrak{m}_2$. With respect to this basis the isotropy representation $\alpha : U(2) \longrightarrow SO(\mathfrak{m}_1 \oplus \mathfrak{m}_2)$ is given by

$$\begin{aligned} \alpha : U(2) &\longrightarrow SO(4, 2) \\ h &\longmapsto (h, \text{deth}) \text{ , where } \text{deth} \in S^1 = SO(2). \end{aligned}$$

Considering this map at the level of homotopy groups we see that we can lift α into $Spin^+(4, 2)$. Let $\tilde{\alpha}$ denote this lift. One calculates

$$\begin{aligned} \tilde{\alpha}_*(E_{12}) &= \frac{1}{2}(-e_1e_2 + e_5e_6) \\ \tilde{\alpha}_*(E_{34}) &= \frac{1}{2}(-e_1e_2 + e_5e_6) \\ \tilde{\alpha}_*(E_{13} + E_{24}) &= \frac{1}{2}(-e_1e_3 - e_2e_4) \\ \tilde{\alpha}_*(E_{14} - E_{23}) &= \frac{1}{2}(-e_1e_4 + e_2e_3) . \end{aligned}$$

Consequently, constant functions $\psi(g) = \psi_0 \in \text{span}\{\psi_3, \psi_4\}$, $g \in SO^+(4, 1)$ define sections in the spinor bundle S of $SO^+(4, 1)/U(2)$. The connection on S with respect to g_t is given by

$$\begin{aligned} \tilde{\Lambda} : \mathfrak{m}_1 \oplus \mathfrak{m}_2 &\longmapsto \mathfrak{spin}(4, 3) \\ \tilde{\Lambda}(e_1) &= \frac{1}{4}\sqrt{\frac{t}{2}}(-e_3e_5 - e_4e_6) \\ \tilde{\Lambda}(e_2) &= \frac{1}{4}\sqrt{\frac{t}{2}}(-e_3e_6 + e_4e_5) \\ \tilde{\Lambda}(e_3) &= \frac{1}{4}\sqrt{\frac{t}{2}}(e_1e_5 + e_2e_6) \\ \tilde{\Lambda}(e_4) &= \frac{1}{4}\sqrt{\frac{t}{2}}(e_1e_6 - e_2e_5) \\ \tilde{\Lambda}(e_5) &= \frac{t-2}{2\sqrt{2t}}(-e_1e_3 + e_2e_4) \\ \tilde{\Lambda}(e_6) &= \frac{t-2}{2\sqrt{2t}}(-e_1e_4 - e_2e_3) . \end{aligned}$$

If $t = 1$ we obtain an Einstein metric with one Killing spinor for each of the values $\lambda = \pm \frac{1}{2\sqrt{2}}$ (ψ_4 and ψ_3 , respectively).

3.3.4 Construction of warped products with Killing spinors

Let $(F^{4,2}, h)$ be a pseudo-Riemannian spin manifold of signature $(4,2)$ with spin structure Q_F and spinor bundle S_F . Furthermore let $I = (a, b) \subseteq \mathbb{R}$ be an open interval and $\sigma \in C^\infty(I, (0, \infty))$ be a smooth positive function. We consider the warped product

$$(M^{4,3}, g) := F^{4,2} \times_\sigma I := (F^{4,2} \times I, \sigma(t)h \oplus dt^2).$$

Denote by $\pi : F^{4,2} \times I \longrightarrow F^{4,2}$ the projection. Let $\tilde{Q}$ be that spin structure of $(M^{4,3}, g)$ whose $Spin(n-1)$ -reduction with respect to $\xi = \frac{\partial}{\partial t}$ restricted to any fibre $F^{4,2} \times \{t\}$ yields that spin structure of $(F^{4,2}, \sigma(t)h)$ which is conformally equivalent to the spin structure Q_F of $(F^{4,2}, h)$. The spinor bundle S of $(M^{4,3}, g)$ can be identified with the bundle π^*S_F by

$$\begin{aligned} \pi^*S_F &\xrightarrow{\sim} S = \tilde{Q} \times_{Spin(4,3)} \Delta_{4,3} \\ \psi = [q, u(x, t)] &\longmapsto \tilde{\psi} = [\tilde{q}, u(x, t)] \end{aligned}$$

where $\tilde{q}$ denotes that element of $\tilde{Q}_{(x,t)}$ which corresponds to $q \in (Q_F)_x$ relative to the conformal equivalence of Q_F and $\tilde{Q}|_{F^{4,2} \times \{t\}}$. For a section $\psi \in \Gamma(\pi^*S_F)$ we denote by $\psi_t \in \Gamma(S_F)$ the spinor field $\psi_t(x) := \psi(x, t)$. Furthermore, for a vector field X on $F^{4,2}$ let $\tilde{X}$ be the vector field $\tilde{X}(x, t) := \sigma(t)^{-\frac{1}{2}}X(x)$ on $M^{4,3}$. Then the following formulae for the Clifford multiplication and the spinor derivative hold.

$$\tilde{X}(x, t) \cdot \tilde{\psi}(x, t) = \widetilde{X(x) \cdot \psi_t(x)} \quad (16)$$

$$\xi \cdot \tilde{\psi} = -\widetilde{J^S \psi} \quad (17)$$

$$\nabla_{\tilde{X}} \tilde{\psi} = \sigma(t)^{-\frac{1}{2}} \widetilde{\nabla_X \psi_t} - \frac{1}{4} \sigma^{-1} \sigma' \tilde{X} \cdot \xi \cdot \tilde{\psi} \quad (18)$$

$$\nabla_\xi \tilde{\psi} = \widetilde{\frac{\partial}{\partial t} \psi} \quad (19)$$

Theorem 3.6 *Let now φ^+ and $\varphi^- := J^S(\varphi^+)$ be Killing spinors on $F^{4,2}$ with Killing numbers λ and $-\lambda$, respectively. We may assume $\lambda > 0$. Denote by ψ^+ and ψ^- the sections $\psi^+(x, t) = \cos(\lambda t)\varphi^+(x) - \sin(\lambda t)\varphi^-(x)$ and $\psi^-(x, t) = \sin(\lambda t)\varphi^+(x) - \cos(\lambda t)\varphi^-(x)$ of π^*S_F . Then $\tilde{\psi}^+$ and $\tilde{\psi}^-$ are Killing spinors on $F^{4,2} \times_{\cos^2(2\lambda t)} (-\frac{\pi}{4\lambda}, \frac{\pi}{4\lambda})$ with Killing numbers λ and $-\lambda$, respectively.*

Proof. Direct calculations using (16) - (19).

References

- [1] H. Baum, Th. Friedrich, R. Grunewald, I. Kath, *Twistors and Killing Spinors on Riemannian Manifolds*, Teubner-Verlag, Stuttgart-Leipzig 1991.
- [2] R. L. Bryant, *Metrics with exceptional holonomy*, Ann. Math.126 (1987), 525-576.
- [3] R. L. Bryant, S. M. Salamon, *On the Construction of some Complete Metrics with Exceptional Holonomy*, Duke Math. Journ. vol 58 (1989), 829 - 850.
- [4] Th. Friedrich, I. Kath, A. Moroianu, U. Semmelmann, *On Nearly Parallel G_2 -Structures*, to appear in J. Geom. Phys.
- [5] A. Gray, *Vector Cross Products on Manifolds*, Trans. Amer. Math. Soc. 141 (1969), 465 - 504.
- [6] A. Gray, *Weak Holonomy Groups*, Math. Zeitschrift 123 (1971), 290 - 300.
- [7] N. Jacobson, *Exceptional Lie Algebras*, Marcel Dekker, New York 1971.
- [8] J. Tits, *Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen*, LNM 40, Springer-Verlag, Berlin Heidelberg-New York 1967.

Sfb 288 Preprints are available at:
<http://www-sfb288.math.tu-berlin.de>
List of most recent 50 sfb288 preprints

- 153 Große-Brauckmann; Polthier: Numerical Examples of Compact Constant Mean Curvature Surfaces
- 154 D. Ferus, F. Pedit: Isometric Immersions of Space Forms and Soliton Theory
- 155 J. E. Avron, R. Seiler, P. Zograf: Viscosity of Quantum Hall Fluids
- 156 T. Friedrich: Neue Invarianten der 4-dimensionalen Mannigfaltigkeiten
- 157 A. Fring: Braid Relations in Affine Toda Field Theory
- 158 A. Schmitt: Solutions of a 2-dimensional Skyrme-Model on $\text{Aff}(\mathbb{R})$
- 159 B. Nobbe: Classical motion in two-dimensional crystals
- 160 F. V. Andreev, A. V. Kitaev: Connection Formulas for Asymptotics of the Fifth Painlevé Transcendent on the Real Axis. I
- 161 A. Tralle: Rational Models of Solvmanifolds with Kahlerian Structures
- 162 Th. Friedrich, I. Kath, A. Moroianu, U. Semmelmann: On Nearly Parallel G₂-Structures
- 163 M. Karowski, R. Schrader: A lattice model of local algebras of observables and fields with braid group statistics
- 164 H. C. Hege, K. Polthier: Visualization and Mathematic
- 165 I. Krichever, A. Zabrodin: Spin generalisation of the Ruijsenaars-Schneider model, non-abelian 2D Toda chain and representations of Sklyanin algebra
- 166 V. Bazhanov, A. Bobenko, N. Reshetikhin: Quantum Discrete Sine-Gordon Model at Roots of 1: Integrable Quantum System on the Integrable Classical Background
- 167 U. Bunke, M. Olbrich: Fuchsian groups of the second kind and representations carried by the limit set
- 168 H. Ferguson, A. Gray, St. Markvorsen: Costa's Minimal Surface via Mathematica
- 169 J. Dorfmeister, I. McIntosh, F. Pedit, H. Wu: On the Meromorphic Potential for a Harmonic Surface in a k-Symmetric Space
- 170 M. Lüdde: Notes on generalised Magnus modules over the braid group
- 171 V. Bach, J. Fröhlich, I. M. Sigal: Mathematical Theory of Non-Relativistic Matter and Radiation
- 172 P. Contucci, A. Knauf: The Phase Transition of the Number-Theoretical Spin Chain
- 173 K. Mohnke: On Seiberg-Witten Equations on Symplectic 4-manifolds
- 174 D. Ferus, F. Pedit: Curved Flats in Symmetric Spaces
- 175 M. Hinze: On the Numerical Approximation and Computation of Minimal-Surface-Continua bounded by One-Parameter-Families of Polygonal Contours
- 176 U. Hertrich-Jeromin: On Conformally flat hypersurfaces, Curved flats and Cyclic systems
- 177 U. Bunke, M. Olbrich: Cohomological properties of the smooth globalization of a Harish-Chandra module

- 178 F. Nill, K. Szlachányi: Quantum Chains of Hopf Algebras with Quantum Double Cosymmetry
- 179 St. Meißner, B.-D. Dörfel: Ground state and low excitations of an integrable chain with alternating spins
- 180 Ch. Bär: Harmonic Spinors for Twisted Dirac Operators
- 181 F. Constantinescu, M. Luedde: The Alexander and Jones-invariants and the Burau module
- 182 A. Bobenko, U. Eitner, A. Kitaev: Harmonic Inverse Mean Curvature Surfaces and Painleve Equations
- 183 K.-D. Kirchberg: Kählerian E-spinors
- 184 J. Kellendonk: Integer Groups of Coinvariants Associated to Octagonal Tilings
- 185 V. Kostrykin, R. Schrader: Ionization of Atoms and Molecules by Short Strong Laser Pulses
- 186 M. J. Pflaum: A new concept of deformation quantization I. Normal order quantization on cotangent bundles
- 187 J. Brüning: The local index theorem without smoothness
- 188 U. Bunke, M. Olbrich: Cohomological properties of the canonical globalizations of Harish-Chandra modules
- 189 U. Bunke, M. Olbrich: Group cohomology and the singularities of the Selberg zeta function associated to a Kleinian group
- 190 H. Gollek: Deformations of isotropic curves in C^3 and minimal surfaces in R^3
- 191 F. Guerra, A. Knauf: Free Energy and Correlations of the Number-Theoretical Spin-Chain
- 192 H.-W. Wiesbrock: Symmetries and Modular Intersections of von-Neumann-Algebras
- 193 H.-W. Wiesbrock: Modular Intersections of von-Neumann-Algebras in Quantum Field Theory
- 194 H. Karcher, K. Polthier: Construction of Triply Periodic Minimal Surfaces
- 195 V. Bach, J. Poelchau: Accuracy of the Hartree-Fock Approximation for the Hubbard Model
- 196 F. V. Andreev, A. V. Kitaev: Connection Formulas for the asymptotics of the fifth Painleve transcendent on the real axis. II
- 197 J. Dorfmeister, G. Haak: On symmetries of constant mean curvature surfaces
- 198 Th. Friedrich: On Superminimal Surfaces
- 199 M. J. Pflaum: The normal symbol on Riemannian manifolds
- 200 M. U. Schmidt: On complex Bloch-spaces of periodic Schroedinger operators
- 201 Fring:
- 202 C. Binnenhei: On The Even CAR Algebra
- 203 I. Kath: $G^*(2)$ -Structures on pseudo-Riemannian manifolds