Decoherence and Localization in Quantum Two-Level Systems

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Abstract

We study and compare the decoherent histories approach, the environment-induced decoherence and the localization properties of the solutions to the stochastic Schrödinger equation in quantum jump simulation and quantum state diffusion approaches, for a quantum two-level system model. We show, in particular, that there is a close connection between the decoherent histories and the quantum jump simulation, complementing a connection with the quantum state diffusion approach noted earlier by Diósi, Gisin, Halliwell and Percival. In the case of the decoherent histories analysis, the degree of approximate decoherence is discussed in detail.

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I. INTRODUCTION

Two primary paradigms – the environment-induced decoherence approach, proposed by Zurek [1–3], and the consistent histories approach by Griffiths [4] and later by Omnès [5] and by Gell-Mann and Hartle [6–8] have been recently developed to solve the fundamental issues in quantum theory, especially, quantum measurement problems and the transition from quantum to classical. The environment-induced decoherence emphasizes the division between the system and its environment. The interaction of the system with its environment is responsible for the decay of the quantum coherence of the system. The decoherent histories approach is designed to provide the most general descriptions for a closed system by using the concept of history – a sequence of events at a succession of times. Both approaches are applicable to open quantum systems.

Another set of viable theories within the framework of quantum mechanics are the various unravellings of master equation as stochastic Schrödinger equations for the single member of the ensemble. Among others, the quantum state diffusion and the quantum jump simulation approaches have been extensively studied in recent years (e.g., see [9–12]). As phenomenological theories, these stochastic approaches are not only of theoretical interest but also of practical value.

The open quantum system provides a unified framework to exhibit the properties of the various approaches we have mentioned above. The master equation which describes the evolution of the open quantum system plays a central role in the investigations into the decay of quantum coherence due to the interaction with a much larger environment. However, it does not tell us how an individual member of an ensemble evolves in a dissipative environment. The unravelling of master equation as stochastic Schrödinger equation could provide such a description within its domain of applicability. Corresponding to the decoherence process in the density operator formalism, in stochastic Schrödinger equation approaches, the solution to the stochastic Schrödinger equation often possesses a very remarkable property – the solution tends to localize at some special states after a localization time scale. For quantum
diffusion approach, this localization property has been justified in many different situations [13–18].

It is worth emphasizing that a key point in these approaches is the mutual influence between the system of interests and its environment. This mutual influence is the common sources of the many different phenomena such as dissipation, fluctuation, decoherence, localization, etc.

Analysis of decoherence and localization properties is usually rather involved. The entanglement of the complicated mathematics and the subtle conceptual issues often tends to make the detail scrutiny of the basic concepts impossible. The attractiveness of two-level system model is that, perhaps it is one of the simplest yet physical meaningful models.

The purpose of this paper is to employ a widely used two-level system model as a unified framework to examine the dynamics of the open quantum system by the decoherent histories, the environment-induced decoherence and the stochastic Schrödinger equations. We mainly consider the decoherence process and the localization process. The various time scales concerning these processes are discussed. We provide a detailed analysis of the degree of decoherence which is important for a real physical process.

The plan of this paper is as follows. In Section II, we briefly present our two-level system model and its basic properties. We study the consistent histories approach, the degree of decoherence in Sections III and IV, respectively. We study the unravelling of master equation and the localization properties of the solutions to the stochastic Schrödinger equations in both quantum jump simulations and quantum state diffusions in Section V. We make some remarks in Section VI. In Appendix we present a proof of Theorem in Section IV.
II. THE MODEL

A fundamental building block in quantum theory is the two-level system. We consider a two-level atom system, which is radiatively damped by its interaction with the many modes of a radiation field in thermal equilibrium at temperature $T$. The upper level and lower level are denoted by $|2\rangle$ and $|1\rangle$, respectively. A widely used master equation for the two-level atom takes standard Lindblad form (in the Schrödinger picture) (e.g., see [10,11]):

$$
\dot{\rho} = -\frac{i}{\hbar}[H, \rho] + \frac{\gamma}{2}(\pi + 1)(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a)
+ \frac{\gamma}{2}\pi(2a^\dagger \rho a - aa^\dagger \rho - \rho aa^\dagger). 
$$

(2.1)

Here, the Hamiltonian of the atom in the absence of the environment is given by

$$
H = \frac{\hbar \omega}{2}\sigma_z. 
$$

(2.2)

The Lindblad operators, which model the effects of the environment in this situation, are

$$
L_1 = \sqrt{\gamma(\pi + 1)}a, \quad L_2 = \sqrt{\gamma\pi a^\dagger}. 
$$

(2.3)

The transition rate from $|2\rangle \rightarrow |1\rangle$ is described by the term proportional to $(\gamma/2)(\bar{n} + 1)$, and the transition rate from $|1\rangle \rightarrow |2\rangle$ is described by the term proportional to $(\gamma/2)\bar{n}$. We use $\sigma_x, \sigma_y$ and $\sigma_z$ to denote Pauli matrices and $a, a^\dagger$ atomic lowering and raising operators, which are defined in the usual way

$$
\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} 
$$

(2.4)

and

$$
a = \frac{1}{2}(\sigma_x - i\sigma_y), \quad a^\dagger = \frac{1}{2}(\sigma_x + i\sigma_y). 
$$

(2.5)

The master equation (2.1) in the basis $|2\rangle, |1\rangle$ may be written as
\[ \dot{\rho}_{11} = -\gamma(\pi + 1)\rho_{11} + \gamma\pi\rho_{22}, \quad (2.6) \]
\[ \dot{\rho}_{22} = \gamma(\pi + 1)\rho_{11} - \gamma\pi\rho_{22}, \quad (2.7) \]
\[ \dot{\rho}_{12} = -\left[\frac{\gamma}{2}(2\pi + 1) + i\omega\right]\rho_{12}, \quad (2.8) \]
\[ \dot{\rho}_{21} = -\left[\frac{\gamma}{2}(2\pi + 1) - i\omega\right]\rho_{21}. \quad (2.9) \]

The general solutions to Eqs. (2.6)-(2.9) are as follows

\[ \rho_{11}(t) = \frac{\pi}{\pi + 1}B_1 + B_2 e^{-\gamma(2\pi + 1)t}, \quad (2.10) \]
\[ \rho_{22}(t) = B_1 - B_2 e^{-\gamma(2\pi + 1)t}, \quad (2.11) \]
\[ \rho_{12}(t) = B_3 e^{-\left[\frac{\gamma}{2}(2\pi + 1) + i\omega\right]t}, \quad (2.12) \]
\[ \rho_{21}(t) = B_4 e^{-\left[\frac{\gamma}{2}(2\pi + 1) - i\omega\right]t}, \quad (2.13) \]

where \( B_i (i = 1, 2, 3, 4) \) are arbitrary constants which can be easily determined once the initial condition is given. For the initial density matrix with \( \text{Tr}(\rho_0) = 1 \) we easily get

\[ B_1 = \frac{\pi + 1}{2\pi + 1} \quad (2.14) \]
\[ B_2 = \rho_{11}(0) - \frac{\pi}{2\pi + 1} \quad (2.15) \]
\[ B_3 = \rho_{12}(0) \quad (2.16) \]
\[ B_4 = \rho_{21}(0) \quad (2.17) \]

In the basis \( |2\rangle, |1\rangle \) the off-diagonal elements tend to zero as \( t \) goes to infinite. The density operator \( \rho \) tends to the stationary density operator \( \rho_s \):

\[ \rho \rightarrow \rho_s = \begin{bmatrix} \frac{\pi}{2\pi + 1} & 0 \\ 0 & \frac{\pi + 1}{2\pi + 1} \end{bmatrix}. \quad (2.18) \]

This is an elementary example of environment-induced decoherence. Diagonalization occurs in the basis \( |2\rangle, |1\rangle \). From equations (2.8) and (2.9), it follows that the decoherence time scale \( t_D \) is given by

\[ t_D \sim \frac{1}{\gamma(2\pi + 1)}. \quad (2.19) \]
It is easily shown that master equation (2.1) is invariant under unitary transformations of the Lindblad operators:

\[
a \mapsto UaU^\dagger, \quad a^\dagger \mapsto Ua^\dagger U^\dagger,
\]

where \(U\) is a unitary matrix. Correspondingly, the density operator \(\rho\) transforms in the same way:

\[
\rho \mapsto U\rho U^\dagger.
\]

Thus, when \(t \to \infty\)

\[
\rho \longrightarrow U\rho_s U^\dagger.
\]

Generally, the density matrix \(U\rho_s U^\dagger\) is no longer diagonal. This indicates that environment-induced decoherence does not occur in other bases.

**III. CONSISTENT HISTORIES IN THE TWO-LEVEL SYSTEM MODEL**

The decoherent histories approach [4–8] offers a sensible way to assign probabilities to a sequence of properties of a quantum system without referring to the measurements or to a classical domain.

A history is defined in general as a sequence of properties of a closed system occurring at different times, which is denoted as

\[
C_\alpha = P^{(n)}_{\alpha_n}(t_n), \ldots, P^{(1)}_{\alpha_1}(t_1),
\]

where \(P^{(i)}_{\alpha_i}(t_i)\) are the projection operators in the Heisenberg picture at times \(t_i\):

\[
P^{(i)}_{\alpha_i}(t_i) = e^{\frac{i}{\hbar}(t_i-t_0)H}P^{(i)}_{\alpha_i}e^{-\frac{i}{\hbar}(t_i-t_0)H},
\]

here, \(H\) is the Hamiltonian of the closed system. These projection operators satisfy exhaustive and exclusive conditions:
\[
\sum_{\alpha_i} P^{(i)}_{\alpha_i}(t_i) = I, \quad P^{(i)}_{\alpha_i}(t_i) P^{(i)}_{\beta_i}(t_i) = \delta_{\alpha_i\beta_i} P^{(i)}_{\alpha_i}(t_i).
\]

(3.3)

The superscript \((i)\) labels the set of projections used at time \(t_i\) and \(\alpha_i\) denotes the particular alternative.

A natural way to assign the probability to a history is

\[
p(C_\alpha) = \text{Tr}(C_\alpha \rho(t_0) C^\dagger_\alpha)
= \text{Tr} \left( P^{(n)}_{\alpha_n}(t_n) ... P^{(1)}_{\alpha_1}(t_1) \rho(t_0) P^{(1)}_{\alpha_1}(t_1) ... P^{(n)}_{\alpha_n}(t_n) \right).
\]

(3.4)

However, one finds that (3.4) generally does not satisfy the usual probability sum rules. The necessary and sufficient condition to guarantee that the probability sum rules hold is that the real part of the decoherence functional \(D(\alpha, \alpha')\) vanishes,

\[
\text{Re} D(\alpha, \alpha') = \text{Re} \text{Tr}(C_\alpha \rho(t_0) C^\dagger_{\alpha'}) = 0
\]

(3.5)

for any two different histories \(C_\alpha\) and \(C_{\alpha'}\). The sets of histories satisfying (3.5) are said to be consistent (or weakly decoherent). Physical mechanisms causing (3.5) to be satisfied typically lead also to the stronger condition

\[
D(\alpha, \alpha') = 0, \quad \forall \alpha \neq \alpha'
\]

(3.6)

which is called medium decoherence [7]. (In this paper, we simply refer it as decoherence)

Although the decoherent histories approach was primary designed for a closed system, the approach is of particular importance for the open system which may be regarded as a subsystem of a large closed system. For the open quantum system, a natural coarse-graining is to focus only on the properties of the distinguished system whilst ignoring the environment. In this case, a natural selection of projections at each time is of form \(P_\alpha \otimes I^E\), where \(P_\alpha\) is a projection onto the distinguished subsystem and \(I^E\) denotes the identity projection on the environment. In the Markovian regime, the decoherence functional could be constructed entirely in terms of the reduced density matrix of the system [19]:

\[
D[\alpha, \alpha'] = \text{Tr} \left( P^{(n)}_{\alpha_n} K^{t_n}_{(n-1)} P^{(n-1)}_{\alpha_{n-1}} ... K^{t_1}_{(1)} P^{(1)}_{\alpha_1} K^{t_1}_{t_0} [\rho(t_0)] P^{(1)}_{\alpha_1} ... P^{(n-1)}_{\alpha_{n-1}} P^{(n)}_{\alpha_n} \right),
\]

(3.7)
where the trace is taken over the distinguished system only. The quantity \( K_t^{t_k}_{k-1}[\cdot] \) is the reduced density operator propagator: \( \rho_t = K_0^t[\rho_0] \).

In what follows we shall make a detailed analysis of the decoherent histories in the two-level model described by the master equation (2.1) which depicts a Markovian process.

First, let us consider the projection operators represented by

\[ P_1 = |1\rangle\langle 1| \quad \text{and} \quad P_2 = |2\rangle\langle 2|. \]  

(3.8)

Obviously, \( \{P_i, i = 1, 2\} \) form a set of complete and exclusive projection operators. Physically, \( P_1 \) may represent that the atom emits a photon whereas \( P_2 \) may represent that the atom absorbs a photon. Then the decoherence functional at two time points is given by

\[ D[\alpha, \alpha'] = \delta_{\alpha_2 \alpha'_2} \text{Tr} \left( P_{\alpha_2}^2 K^{t_2}_{t_1} \left[ P_{\alpha_1}^1 K^{t_1}_{0} [\rho_0] P_{\alpha'_1}^1 \right] \right). \]  

(3.9)

It is easily shown that, for any 2 \( \times \) 2 matrix \( A \), the matrix \( P_i A P_j (i \neq j) \) is an upper (or a lower) triangle matrix. From equations (2.8),(2.9) we know that \( K^{t_{i+1}}_{t_i}[\cdot] \) propagates the matrix with zero diagonal elements into the matrix with zero diagonal elements. So, for any initial density matrix \( \rho_0 \), the trace in Eq. (3.9) is exactly zero for any different pairs of histories \( (\alpha \neq \alpha') \) and for any interval \( t_2 - t_1 \). This demonstrates that the set of histories consisting of projectors (3.8) are exactly decoherent. The generalization to \( n \) time points is straightforward. The exact decoherence for any time interval is slightly surprising. (The density matrix, by contrast, only becomes exactly diagonal as \( t \to \infty \)). This exactness is due to the simplicity of the model and we do not expect it to be a generic feature.

Next, consider more general projection operators which correspond to the projection to any direction. With any direction denoted by a unit vector \( n = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \phi) \), we associate a vector \( |n\rangle \) which belongs to the Hilbert space of the two-level system,

\[ |n\rangle = \cos \frac{\theta}{2} |1\rangle - e^{-i\phi} \sin \frac{\theta}{2} |2\rangle. \]  

(3.10)

Then one can define the following projection operators on the Hilbert space of the system:

\[ P_+ = |n\rangle\langle n|, \quad P_- = |n'\rangle\langle n'|, \]  

(3.11)
where \( |n'\rangle \) is the orthogonal complimentary of \( |n\rangle \):

\[
|n'\rangle = e^{i\phi} \sin \frac{\theta}{2} |1\rangle + \cos \frac{\theta}{2} |2\rangle.
\]  

(3.12)

We shall show that a set of histories consisting of the projection operators \( P_+ \) and \( P_- \) are approximately decoherent. To this end, first, note that for any \( 2 \times 2 \) matrix \( A \),

\[
\text{Tr} (P_+ A P_-) = \text{Tr} (P_- A P_+) = 0.
\]  

(3.13)

Hence, from (2.10)–(2.13), it can be seen that, after the propagation of \( K^{t_{i+1}} \), all of the diagonal elements of matrix \( K^{t_2} \) contain an exponential damping factor

\[
\text{Damping factor} = e^{-\gamma(2\pi+1)t}.
\]  

(3.14)

Thus, we conclude

\[
D[\alpha, \alpha'] \approx 0, \quad \forall \alpha \neq \alpha'.
\]  

(3.15)

This proves that the set of histories consisting of \( P_+, P_- \) are approximately decoherent if time interval between \( t_k \) and \( t_{k+1} \) is larger than the characteristic time scale,

\[
t_{\text{decoherence}} \sim \frac{1}{\gamma(2\pi+1)}.
\]  

(3.16)

This is an expected result. We will give a more detailed estimate of the degree of decoherence in the next section.

Note that \( t_{\text{decoherence}} \) decreases as the coupling \( \gamma \) is made stronger. To see the influence of the temperature of the bath, a simple expression for \( \pi \) may be chosen as

\[
\pi = \frac{1}{e^{\hbar\omega/k_B T} - 1}.
\]  

(3.17)

From this, it is easy to see that the decoherence is more effective if the temperature of bath, \( T \), increases. Conversely, decreasing temperature will make system spend more time to decohere. The maximum decoherence time scale is \( 1/\gamma \) which corresponds to zero temperature of bath.
In summary, we find exact decoherence of histories characterized by the projections onto \(|1\rangle\) and \(|2\rangle\), and approximate decoherence in any other basis.

Finally, we examine the probabilities for two times histories consisting of projections (3.8). These are given by

\[
p(1, 2) = \text{Tr} \left( P_{1,K_{t_1}^2} P_{2,K_{t_2}^2} \rho_0 \right) = \frac{\pi + 1}{2\pi + 1} (1 - \delta) \rho_{11}(t_1),
\]

(3.18)

\[
p(2, 1) = \text{Tr} \left( P_{2,K_{t_1}^2} P_{1,K_{t_2}^2} \rho_0 \right) = \frac{\pi}{2\pi + 1} (1 - \delta) \rho_{22}(t_1),
\]

(3.19)

\[
p(1, 1) = \text{Tr} \left( P_{1,K_{t_1}^2} P_{1,K_{t_2}^2} \rho_0 \right) = \frac{1}{2\pi + 1} \left[ \pi + (\pi + 1)\delta \right] \rho_{11}(t_1),
\]

(3.20)

\[
p(2, 2) = \text{Tr} \left( P_{2,K_{t_1}^2} P_{2,K_{t_2}^2} \rho_0 \right) = \frac{1}{2\pi + 1} \left[ (\pi + 1) + \pi\delta \right] \rho_{22}(t_1).
\]

(3.21)

Where

\[
\rho_{11}(t_1) = \frac{\pi}{2\pi + 1} + \left[ \rho_{11}(0) - \frac{\pi}{2\pi + 1} \right] \delta,
\]

(3.22)

\[
\rho_{22}(t_1) = \frac{\pi + 1}{2\pi + 1} - \left[ \rho_{11}(0) - \frac{\pi}{2\pi + 1} \right] \delta,
\]

(3.23)

and \( \delta = \exp\{-\gamma(2\pi + 1)\Delta t\} (\Delta t = t_i - t_{i-1}, i = 1, 2) \). As for the \( n \) times histories, the calculation for the elementary probabilities will be straightforward. For instance,

\[
p(1, 1, \cdots 1) = \text{Tr} \left( P_{1,K_{t_1}^n} \cdots P_{1,K_{t_1}^2} P_{1,K_{t_1}^2} \rho_0 \right) = \frac{1}{(2\pi + 1)^{n-1}} \left[ \pi + (\pi + 1)\delta \right]^{n-1} \rho_{11}(t_1).
\]

(3.24)

Similarly, one may calculate the transition probabilities, etc.

**IV. DEGREE OF DECOHERENCE**

Physically, one would not expect the decoherence takes place exactly. Therefore the investigation of the approximate decoherence is of importance. In practical problems, one
can, at best, only expect that probability sum rules are satisfied up to order $\epsilon$, for some constant $\epsilon < 1$. Namely, the interference terms do not have to be exactly zero, but small than probabilities by a factor of $\epsilon$. One simple inequality which turns out to be very useful to the study of the degree of decoherence is [20,21]:

$$|D[\alpha, \alpha']|^2 \leq \epsilon^2 D[\alpha, \alpha]D[\alpha', \alpha'] .$$

(4.1)

We say that a system decoheres to order $\epsilon$ if the decoherence functional satisfies (4.1). As shown in [20], such a condition implies that the most probability sum rules will then be satisfied to order $\epsilon$.

Based on this two-level model we will study the degree of decoherence in some detail. To begin with, we establish the following trace inequality which is useful to our studies of the approximate decoherence.

**Theorem:** Suppose that $M$ and $N$ are two $n \times n$ positive definite matrices. Let $P$ and $Q$ be two $n \times n$ Hermitian matrices satisfying

$$QP = PQ = 0. \quad (4.2)$$

Then

$$|\text{Tr} (MPNQ)|^2 \leq \epsilon^2 \text{Tr} (MPNP) \text{Tr} (MQNQ) \quad (4.3)$$

where $\epsilon = \min\{\epsilon^M, \epsilon^N\}$, here $\epsilon^M = (\lambda_{\max}^M - \lambda_{\min}^M)/(\lambda_{\max}^M + \lambda_{\min}^M)$, $\epsilon^N = (\lambda_{\max}^N - \lambda_{\min}^N)/(\lambda_{\max}^N + \lambda_{\min}^N)$, $\lambda_{\max}^M, \lambda_{\min}^M$ and $\lambda_{\max}^N, \lambda_{\min}^N$ are the maximal and the minimal eigenvalues of $M$ and $N$, respectively.

**Remark:** In fact, the condition that both $M$ and $N$ are the positive definite matrices could be generalized to that one is positive definite, say $M$, while another $N$ is positive semidefinite. In this case, $\epsilon = \epsilon^M$. It is hoped that the above theorem is also useful in some other cases. The theorem is proved in the Appendix.

For a general initial state represented by $\rho_0$ (pure or mixed state), the decoherence functional of two time points may be written as
\[ D[\alpha, \alpha'] = \text{Tr} \left( P_\pm K_{i_1}^{t_2} \left[ P_- K_{i_1}^{t_1} [\rho_0] P_+ \right] \right). \]  

(4.4)

We now write

\[ A = K_{i_1}^{t_1} [\rho_0], \]  

(4.5)

\[ B = \tilde{K}_{i_1}^{t_2} [P_\pm]. \]  

(4.6)

Then Eq. (4.4) may be rewritten, in the new notation, as

\[ D[\alpha, \alpha'] = \text{Tr} (BP_- AP_+). \]  

(4.7)

Note that \( \tilde{K} \) in (4.6) is the super-propagator for the projection operators

\[ P(t) = \tilde{K}_0^t [P(0)]. \]  

(4.8)

The evolution equation for the projection operators is given by

\[ \dot{P} = \frac{i}{\hbar} [H, P] \]

\[ + \frac{\gamma}{2}(\pi + 1)(2a^\dagger Pa - a^\dagger aP - Pa^\dagger a) \]

\[ + \frac{\gamma}{2}\pi(2aPa^\dagger - aa^\dagger P - aa^\dagger P), \]  

(4.9)

where \( H, a, a^\dagger \) are defined as before. Note that the evolution equation for the projection operator \( P \) is different from that for the density operator \( \rho \) (2.1). This reflects the difference between the Schrödinger and Heisenberg pictures in the density operator formalism. The explicit form of Eq. (4.9) may be written

\[ \dot{P}_{11} = \gamma(\pi + 1)(P_{22} - P_{11}), \]  

(4.10)

\[ \dot{P}_{22} = \gamma\pi(P_{11} - P_{22}), \]  

(4.11)

\[ \dot{P}_{12} = -\frac{\gamma}{2}(2\pi + 1) - i\omega]P_{12}, \]  

(4.12)

\[ \dot{P}_{21} = -\frac{\gamma}{2}(2\pi + 1) + i\omega]P_{21}. \]  

(4.13)

The general solutions to the above equations are
\[ P_{11}(t) = C_1 + C_2 e^{-\gamma(2\pi + 1)t}, \quad (4.14) \]
\[ P_{22}(t) = C_1 - \frac{n}{n+1} C_2 e^{-\gamma(2\pi + 1)t}, \quad (4.15) \]
\[ P_{12}(t) = C_3 e^{-\frac{[2(2\pi + 1)-i\omega]}{2}t}, \quad (4.16) \]
\[ P_{21}(t) = C_4 e^{-\frac{[2(2\pi + 1)+i\omega]}{2}t}, \quad (4.17) \]

where \( C_i \) \((i = 1, 2, 3, 4)\) are arbitrary constants. For given initial values, these constants can be expressed as

\[ C_1 = \frac{n}{2\pi + 1} P_{11}(0) + \frac{n + 1}{2\pi + 1} P_{22}(0), \quad (4.18) \]
\[ C_2 = \frac{n + 1}{2\pi + 1} (P_{11}(0) - P_{22}(0)), \quad (4.19) \]
\[ C_3 = P_{12}(0), \quad (4.20) \]
\[ C_4 = P_{21}(0). \quad (4.21) \]

From the definitions \((4.5)\) and \((4.6)\), it is easy to see that in general, both \( A \) and \( B \) could be positive definite matrices, and since \( P_- \) and \( P_+ \) are projection operators, so the condition \((4.2)\) is automatically satisfied. Using the theorem above, we immediately arrive at

\[ |\text{Tr} (BP_+AP_-)|^2 \leq \epsilon^2 \text{Tr} (BP_+AP_+ \text{Tr} (BP_-AP_-)). \quad (4.22) \]

That is,

\[ |D[\alpha, \alpha']|^2 \leq \epsilon^2 D[\alpha, \alpha] D[\alpha', \alpha'] \quad (4.23) \]

where \( \epsilon = \min\{\epsilon^A, \epsilon^B\} \),

\[ \epsilon^A = |\lambda_1^A - \lambda_2^A|, \quad (4.24) \]
\[ \epsilon^B = \frac{|\lambda_1^B - \lambda_2^B|}{\lambda_1^B + \lambda_2^B}. \quad (4.25) \]

Here \( \lambda_i^A \) \((i = 1, 2)\) and \( \lambda_i^B \) \((i = 1, 2)\) are two eigenvalues of \( A \) and \( B \), respectively. (Note that \( \lambda_1^A + \lambda_2^A = 1 \) From Eqs. \((4.24)\) and \((4.25)\), it is easily seen that the degree of decoherence may depend on both the projection operators we use and the initial state of the system. This is also an expected result. For the two-level system, \( \epsilon^A \) and \( \epsilon^B \) can be calculated exactly.
Consider, first, the eigenvalues of $A$. Since Eq. (2.1) preserves the trace, $\epsilon^A$ can be written as

$$\epsilon^A = \sqrt{1 - 4\lambda_1^A \lambda_2^A} = \sqrt{1 - 4 \det A}.$$  \hspace{1cm} (4.26)$$

The determinant of $A$ can be explicitly evaluated from the general solutions (2.10, 2.11, 2.12, 2.13),

$$\det A = \frac{n(n + 1)}{(2n + 1)^2} + \left[ \frac{n}{2n + 1} \rho_{22}(0) + \frac{n + 1}{2n + 1} \rho_{11}(0) - \frac{2n(n + 1)}{(2n + 1)^2} - \rho_{12}(0) \rho_{21}(0) \right] \delta$$

$$+ \left[ \rho_{11}(0) - \frac{n}{2n + 1} \right] \left[ \rho_{22}(0) - \frac{n + 1}{2n + 1} \right] \delta^2,$$  \hspace{1cm} (4.27)

where $\delta = \exp\{-\gamma(2n + 1)t_1\}$. In order that a set of histories are decoherent, one expects that $\delta$ should be small.

Similarly, $\epsilon^B$ can be expressed as

$$\epsilon^B = \sqrt{1 - \frac{4 \det B}{(\text{Tr}B)^2}}.$$ \hspace{1cm} (4.28)

From (4.14)–(4.17), $\text{Tr}B$ and $\det B$ can be easily obtained:

$$\text{Tr}B = 2C_1 + \frac{1}{n + 1} C_2 \delta_1,$$ \hspace{1cm} (4.29)

$$\det B = C_1^2 + \left[ \frac{1}{n + 1} C_1 C_2 - C_3 C_4 \right] \delta_1 - \frac{n}{n + 1} C_2^2 \delta_1^2,$$ \hspace{1cm} (4.30)

where $\delta_1 = \exp\{-\gamma(2n + 1)(t_2 - t_1)\}$. The above discussions show explicitly how the degree of decoherence is related to the projection operators, the initial states and the temperature of bath, as well as the time spacing interval, in accordance with our general expectations.

In the long time limit, the density matrix will tend to the stationary density matrix. Then we may get a much simpler expression for $\epsilon^A$:

$$\epsilon^A \sim \frac{1}{2n + 1}.$$ \hspace{1cm} (4.31)

As mentioned before, for the decoherent histories, $\delta$ and $\delta_1$ should be small. If we only keep the terms up to the first order of $\delta_1$, then $\epsilon^B$ becomes
\[ \epsilon^B \sim \left[ \frac{C_3 C_4 \delta_1}{C_1^2 + \frac{1}{\pi + 1} C_1 C_2 \delta_1} \right]^{1/2}. \]  

(4.32)

The expression for \( \epsilon^A \) can be obtained from Eq. (4.27).

It is seen from the above expressions that the degree of decoherence improves as the bath temperature increases. We also see that the projections with the smaller off-diagonal elements will give a better degree of decoherence. For a given system with the initial state, then the matter for investigation is to determine which histories, i.e., which string of projections, will lead to the decoherence condition being satisfied. Therefore, we see that \( \epsilon^B \) serves as the main criterion for the degree of decoherence.

It is also of interest to compute the von Neumann entropy of \( \rho(t) \) [3,22]. We find that the von Neumann entropy supplies a restriction on the degree of the decoherence of histories. It exhibits a tension between the predictability of the quantum state and the degree of decoherence.

The von Neumann entropy provides a convenient measure of the loss of predictability:

\[ S = -\text{Tr} (\rho \ln \rho), \]  

(4.33)

By definition, the more predictable state (pure state) may have less increase of the entropy in a fixed time period. This characterization process of predictability is called the “predictability sieve” (coined by Zurek) which has been studied recently in quantum Brownian motion model by using the linear entropy [3,22,23]. We will see that two-level system provides a very simple model to exhibit the relationship between the degree of decoherence and entropy by directly using the von Neumann entropy.

For the purpose of the evaluating the entropy, we choose a special basis in which \( \rho \) is diagonal. Let \( \lambda_1 \) and \( \lambda_2 \) be the eigenvalues of \( \rho \), then Eq. (4.33) reduces to

\[ S = -\sum_{i=1}^{2} \lambda_i \ln \lambda_i. \]  

(4.34)

Obviously, \( \lambda_1 \) and \( \lambda_2 \) can be expressed as

\[ \lambda_1 = \frac{1 + \epsilon^A}{2}, \quad \lambda_2 = \frac{1 - \epsilon^A}{2}. \]  

(4.35)
Hence, Eq. (4.34) can be rewritten as
\[
S = - \left[ \frac{1 + \epsilon^A}{2} \ln \left( \frac{1 + \epsilon^A}{2} \right) + \frac{1 - \epsilon^A}{2} \ln \left( \frac{1 - \epsilon^A}{2} \right) \right].
\] (4.36)

It can be shown that \( S(\epsilon^A) \) is a monotonically decreasing function of \( \epsilon^A \). Here, we find that there is a tension between the predictability and the degree of decoherence. That is, the initial density matrix which leads to less entropy production will give worse degree of decoherence, namely, \( \epsilon^A \) would be not small (Here we are not considering \( \epsilon^B \)). This tension between the predictability and the degree of the decoherence is a physically expected result. To obtain the higher degree of the decoherence one would expect that the environment has stronger influence on the system of interest, such as increasing the temperature of the bath. Then the predictability of the state, correspondingly, decreases.

As an example, it is easy to see that the pure states \( |1\rangle \) and \( |2\rangle \) will lead to the largest entropy production. However, we have seen that the histories consisting of the projection onto these two levels give the best degree of decoherence (In this case \( \epsilon^B \) is zero). In fact, it has been shown that the degree of decoherence is related to both the initial density matrix and projection operators used in the histories.

V. UNRAVELLING OF MASTER EQUATION

The master equation provides an ensemble description of a quantum system. The unravelling of master equation as the stochastic Schrödinger equation for the state vector has provided many insights into the foundation of quantum theory, especially in quantum measurement and the useful tools to study various practical problems in the quantum optics (e.g., see [15,16]). In this section we will study the localization in the two different unravellings of the master equation – quantum jump simulation and quantum state diffusion approaches. The former use the discrete random variables whereas the latter use the continuous random variables.
A. Quantum Jump Simulation

The stochastic Schrödinger equation used in the quantum jump simulation describes a single quantum process. The system undergoes a smooth evolution until a jump takes place. This jump process is characterised by the real random variables which only take the discrete values. For the master equation (2.1), the stochastic Schrödinger equation takes the following form:

\[ \frac{d|\psi\rangle}{dt} = -\frac{i}{\hbar}H|\psi\rangle dt + \sum_{i=1}^{2} \left( \frac{L_i}{\sqrt{\langle N_i \rangle}} - 1 \right) |\psi\rangle dW_i + \sum_{i=1}^{2} \left( \frac{\langle N_i \rangle}{2} - \frac{N_i}{2} \right) dt. \]  

(5.1)

Here \( L_1 = \sqrt{\gamma(n+1)}a, L_2 = \sqrt{\gamma}a^\dagger \) are the Lindblad operators representing the influence of the environment and \( N_i = L_i^\dagger L_i \) \((i = 1, 2)\). \( \langle N_i \rangle = \langle \psi|N_i|\psi \rangle \) represents quantum average and \( M \) represents the ensemble average. The real random variables \( dW_i \) \((i = 1, 2)\) satisfy

\[ dW_i dW_j = \delta_{ij} dW_i, \]  

(5.2)

\[ M(dW_i) = \langle N_i \rangle dt \quad (i = 1, 2). \]  

(5.3)

Under condition (5.2), it is easy to see that \( dW_i \) only take two values: 0 and 1. The master equation (2.1) can be recovered from the stochastic Schrödinger equation (5.1) in the sense that if \( |\psi\rangle \) is the solution to Eq. (5.1) then \( \rho = M|\psi\rangle\langle \psi | \) satisfies master equation (2.1).

In what follows we shall discuss the the ‘localization’ properties of the single jump trajectories. Here, by ‘localization’ we mean that the quantum state vector generated by the stochastic Schrödinger equation will converge to some fixed states in the mean square . The physical meanings of this localization will become clear from the later discussions.

More precisely, let \( A \) be an operator (not necessarily Hermitian), then we define the quantum mean square deviation as

\[ \sigma(A, A) = \langle A^\dagger A \rangle - \langle A^\dagger \rangle \langle A \rangle. \]  

(5.4)

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If the solution of the stochastic Schrödinger equation (5.1) satisfies

\[ M \frac{d}{dt} \sigma(A, A) \leq 0, \quad (5.5) \]

namely, the dispersion of the operator \( A \) tends to decrease as time evolves. Then we say that the solution localizes at the eigenstates of the operator \( A \) (\( A \) is sometimes called the collapse operator).

For the stochastic Schrödinger equation for the quantum jump simulation in two-level system, the collapse operator is \( \sigma_z \). Then quantum mean square deviation in this case is

\[ (\Delta \sigma_z)^2 = 1 - \langle \sigma_z \rangle^2. \quad (5.6) \]

In order to prove the localization, we should first derive the evolution equation of the expectation value of \( \sigma_z \) by using the following formula:

\[ d\langle A \rangle = \langle \psi | A | d\psi \rangle + \langle d\psi | A | \psi \rangle + \langle d\psi | A | d\psi \rangle, \quad (5.7) \]

where \( A \) is an operator. From (5.1), it is straightforward to arrive at the following equation

\[ d\langle \sigma_z \rangle = (1 - \langle \sigma_z \rangle)dW_1 - (1 + \langle \sigma_z \rangle)dW_2 \]

\[ + [\langle \sigma_z \rangle(N_1 + N_2) + \langle N_1 - N_2 \rangle] dt. \quad (5.8) \]

Notice that

\[ d(\Delta \sigma_z)^2 = -2\langle \sigma_z \rangle d\langle \sigma_z \rangle - (d\langle \sigma_z \rangle)^2. \quad (5.9) \]

Then, inserting Eq. (5.8) into the above equation, taking the ensemble means and remembering (5.3), we obtain

\[ M \frac{d}{dt} (\Delta \sigma_z)^2 = -\frac{1}{2} \gamma(\pi + 1)(1 - \langle \sigma_z \rangle)^2(1 + \langle \sigma_z \rangle) \]

\[ -\frac{1}{2} \gamma \pi (1 + \langle \sigma_z \rangle)^2(1 - \langle \sigma_z \rangle). \quad (5.10) \]

The right-hand side of Eq. (5.10) is non-positive, and that it vanishes if and only if \( |\psi \rangle \) is \( |2\rangle \) or \( |1\rangle \). Hence we conclude that the solution to the stochastic Schrödinger equation (5.1)
will localize at $|2\rangle$ or $|1\rangle$ after a certain time. That is, any initial state (which will be a superposition of $|1\rangle$ and $|2\rangle$) will tend to a solution in which the atom undergoes stochastic jumps between $|1\rangle$ and $|2\rangle$.

Let us now estimate this localization time. From (5.10), a few manipulations directly give

$$M \frac{d}{dt} (\Delta \sigma_z)^2 \leq -\gamma (2\pi + 1)(\Delta \sigma_z)^2.$$  \hfill (5.11)

So the localization rate $t_{\text{localization}}$ is

$$t_{\text{localization}} \sim \frac{1}{\gamma (2\pi + 1)},$$ \hfill (5.12)

which agrees with the decoherence time scale (2.19).

In order to see the meanings of the localization of the quantum jump process, let us compute the evolution of the populations of the two levels:

$$d|\langle 1|\psi\rangle|^2 = \frac{1}{2}(\langle \sigma_z \rangle - 1) dW_1 + \frac{1}{2}(1 + \langle \sigma_z \rangle) dW_2$$
$$- \frac{1}{2} \left[ \langle \sigma_z \rangle \langle N_1 + N_2 \rangle + \langle N_1 - N_2 \rangle \right] dt,$$ \hfill (5.13)

$$d|\langle 2|\psi\rangle|^2 = \frac{1}{2}(1 - \langle \sigma_z \rangle) dW_1 - \frac{1}{2}(1 + \langle \sigma_z \rangle) dW_2$$
$$+ \frac{1}{2} \left[ \langle \sigma_z \rangle \langle N_1 + N_2 \rangle + \langle N_1 - N_2 \rangle \right] dt.$$ \hfill (5.14)

Then it follows that from the above equations,

$$|\langle 1|\psi\rangle|^2 \longrightarrow \frac{\pi + 1}{2\pi + 1},$$ \hfill (5.15)
$$|\langle 2|\psi\rangle|^2 \longrightarrow \frac{\pi}{2\pi + 1}.$$ \hfill (5.16)

That is to say, due to the influence of the environment to the system, after a localization time scale (5.12), the average populations of the first and second levels will become constant. This may assist to understand the meanings of the localization in the quantum jump simulation.

In some sense, that the localization in quantum jump simulation chooses the basis $|1\rangle$, $|2\rangle$ appears to be natural, since they correspond to the trajectories that would actually observed...
in an individual experiment. As expected, the set of histories consisting of projection onto
the basis give the best degree of decoherence. In addition, we have seen that density matrix
become diagonal in this basis. Here, we have demonstrated a close connection between the
different approaches.

B. Quantum state diffusion

In this subsection, we will illustrate the localization process in another unravelling of the
master equation – the quantum state diffusion approach, which was introduced by Gisin and
Percival [9] to describe the quantum open system by using a stochastic Schrödinger equation
( which is often called the Langevin-Ito stochastic differential equation) for the normalized
pure state vector of an individual system of the ensemble. Similar to the quantum jump
simulation, a solution of the Langevin-Ito equation for the diffusion of a pure quantum state
in state space represents a single member of an ensemble whose density operator satisfies
the corresponding master equation.

Generally, if the master equation takes the standard Lindblad form:
\[
\dot{\rho} = -\frac{i}{\hbar}[H, \rho] + \sum_i (L_i \rho L_i^\dagger - \frac{1}{2} L_i^\dagger L_i \rho - \frac{1}{2} \rho L_i L_i^\dagger) \tag{5.17}
\]
Then, correspondingly, the Langevin-Ito stochastic equation can be written as
\[
|d\psi\rangle = -\frac{i}{\hbar} H|\psi\rangle dt + \sum_i \left( (\langle L_i^\dagger L_i \rangle - \frac{1}{2} L_i^\dagger L_i - \frac{1}{2} \langle L_i^\dagger \rangle \langle L_i \rangle ) |\psi\rangle dt + \sum_i (L_i - \langle L_i \rangle ) |\psi\rangle d\xi_i. \tag{5.18}
\]
where \( H \) is a Hamiltonian (of the open system) and \( L_i \) are Lindblad operators, as before,
\( \langle L_i \rangle = \langle \psi | L_i |\psi \rangle \). The complex Wiener processes \( d\xi_i \) satisfy
\[
M(d\xi_i) = 0, M(d\xi_i d\xi_j) = 0, M(d\xi_i^* d\xi_j) = \delta_{ij} dt, \tag{5.19}
\]
where \( M \) denotes a mean over the ensemble. Quantum state diffusion reproduces the master
equation in the mean:
\[ \rho = M |\psi\rangle \langle \psi|, \quad (5.20) \]

where \(|\psi\rangle\) satisfy the quantum state diffusion equation (5.18), then it can be shown that \(\rho\) satisfies the master equation (5.17).

In order to show the localization properties of the Langevin-Ito equation we now consider the simplest case which is assumed that bath temperature is zero (\(\bar{n} = 0\)). In this case the master equation (2.1) reduces to

\[
\dot{\rho} = -\frac{i}{\hbar} [H, \rho] + \frac{\gamma}{2} (2a \rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a), \quad (5.21)
\]

Then the corresponding Langevin-Ito equation is given by

\[
|d\psi\rangle = -\frac{i}{\hbar} H |\psi\rangle dt + \frac{\gamma}{2} (2\langle a^\dagger \rangle a - a^\dagger a - \langle a^\dagger \rangle \langle a\rangle) |\psi\rangle dt + \sqrt{\gamma} (a - \langle a\rangle) |\psi\rangle d\xi, \quad (5.22)
\]

where \(d\xi\) is the complex Wiener process satisfying

\[
M(d\xi) = 0, \ M(d\xi d\xi) = 0, \ M(d\xi^* d\xi) = dt, \quad (5.23)
\]

where \(M\) denotes a mean over probability distribution.

The evolution of the quantum average of operators can be calculated by using the following formula:

\[
d\langle G \rangle = \frac{i}{\hbar} \langle [H, G] \rangle - \frac{1}{2} \sum_i \langle L_i^\dagger [L_i, G] + [G, L_i^\dagger] L_i \rangle dt + \sum_i (\sigma(G_i^\dagger, L_i) d\xi_i + \sigma(L_i, G) d\xi_i^*), \quad (5.24)
\]

where

\[
\sigma(A, B) = \langle A^\dagger B \rangle - \langle A^\dagger \rangle \langle B \rangle \quad (5.25)
\]
Using Eq. (5.24), it is straightforward to get the following equations:

\[
\begin{align*}
\frac{d\langle \sigma_x \rangle}{dt} &= -\frac{\omega}{\hbar} \langle \sigma_y \rangle - \frac{\gamma^2}{2} \langle \sigma_x \rangle \ dt \\
&\quad + \sqrt{\frac{\gamma}{2}} \left[ 1 + \langle \sigma_z \rangle - \langle \sigma_x \rangle^2 + i\langle \sigma_x \rangle \langle \sigma_y \rangle \right] d\xi \\
&\quad + \sqrt{\frac{\gamma}{2}} \left[ 1 + \langle \sigma_z \rangle - \langle \sigma_x \rangle^2 - i\langle \sigma_x \rangle \langle \sigma_y \rangle \right] d\xi^*, \quad (5.26)
\end{align*}
\]

\[
\begin{align*}
\frac{d\langle \sigma_y \rangle}{dt} &= -\frac{\omega}{\hbar} \langle \sigma_x \rangle - \frac{\gamma^2}{2} \langle \sigma_y \rangle \ dt \\
&\quad + \sqrt{\frac{\gamma}{2}} \left[ i(1 + \langle \sigma_z \rangle) - i\langle \sigma_y \rangle^2 - \langle \sigma_x \rangle \langle \sigma_y \rangle \right] d\xi \\
&\quad + \sqrt{\frac{\gamma}{2}} \left[ i(1 + \langle \sigma_z \rangle) - i\langle \sigma_y \rangle^2 + \langle \sigma_x \rangle \langle \sigma_y \rangle \right] d\xi^*, \quad (5.27)
\end{align*}
\]

\[
\begin{align*}
\frac{d\langle \sigma_z \rangle}{dt} &= -[\langle \sigma_z \rangle \gamma + \gamma] dt \\\n&\quad - \frac{\sqrt{\gamma}}{2} (1 + \langle \sigma_z \rangle) \langle \sigma_x \rangle - i\langle \sigma_y \rangle) d\xi \\
&\quad - \frac{\sqrt{\gamma}}{2} (1 + \langle \sigma_z \rangle) \langle \sigma_x \rangle + i\langle \sigma_y \rangle) d\xi^*. \quad (5.28)
\end{align*}
\]

Moreover, we need to calculate the higher order moments. For any Hermitian operator \( A \) we have from (5.4),

\[
\begin{align*}
d\langle A^2 \rangle &= d\langle A \rangle d\langle A \rangle - (d\langle A \rangle)^2 \\
&= d\langle A^2 \rangle - 2\langle A \rangle d\langle A \rangle - (d\langle A \rangle)^2. \quad (5.29)
\end{align*}
\]

Then we easily obtain

\[
\begin{align*}
M \frac{d}{dt} (\Delta \sigma_x)^2 &= 2\omega \langle \sigma_x \rangle \langle \sigma_y \rangle + \gamma \langle \sigma_x \rangle^2 \\
&\quad - \frac{\gamma}{2} (\Delta \sigma_x)^4 - \gamma (\Delta \sigma_x)^2 \langle \sigma_z \rangle \\
&\quad - \frac{\gamma}{2} \langle \sigma_z \rangle^2 - \frac{\gamma}{2} (\langle \sigma_x \rangle \langle \sigma_y \rangle)^2, \quad (5.30)
\end{align*}
\]

\[
\begin{align*}
M \frac{d}{dt} (\Delta \sigma_y)^2 &= -2\omega \langle \sigma_x \rangle \langle \sigma_y \rangle + \gamma \langle \sigma_y \rangle^2 \\
&\quad - \frac{\gamma}{2} (\Delta \sigma_y)^4 - \gamma (\Delta \sigma_y)^2 \langle \sigma_z \rangle \\
&\quad - \frac{\gamma}{2} \langle \sigma_z \rangle^2 - \frac{\gamma}{2} (\langle \sigma_x \rangle \langle \sigma_y \rangle)^2. \quad (5.31)
\end{align*}
\]

Now, we are in the position to consider the localization of solutions to Eq. (5.22). Using master equation or quantum trajectories approach, it is very easy to see that the atom will
soon collapse into the lower state $|1\rangle$ and keeps there forever. Here we shall demonstrate that any solution to Langevin-Ito equation (5.22) will localize at the lower state after a localization time. The collapse operator in this case is

$$A = \sigma_x + i\sigma_y.$$  (5.32)

Then by using (5.4) we get

$$\sigma(A, A) = (\Delta\sigma_x)^2 + (\Delta\sigma_y)^2 + 2\langle\sigma_z\rangle.$$  (5.33)

Hence we have

$$M \frac{d}{dt} \sigma(A, A) = \gamma \langle\sigma_x\rangle^2 + \gamma \langle\sigma_y\rangle^2 - \frac{\gamma}{2} (\Delta\sigma_x)^4 - \frac{\gamma}{2} (\Delta\sigma_y)^4 - \gamma (\Delta\sigma_x)^2 \langle\sigma_z\rangle - \gamma (\Delta\sigma_y)^2 \langle\sigma_z\rangle - \gamma \langle\sigma_z\rangle^2 - 2\gamma (\langle\sigma_z\rangle + 1) - \gamma (\langle\sigma_x\rangle \langle\sigma_y\rangle)^2.$$  (5.34)

In order to prove that the left-hand side of Eq. (5.34) is non-positive, let us denote

$$(\Delta\sigma_x)^2 = 1 + X,$$  (5.35)

$$(\Delta\sigma_y)^2 = 1 + Y,$$  (5.36)

$$\langle\sigma_z\rangle = -1 + Z.$$  (5.37)

Substituting equations (5.35),(5.36) and (5.37) into Eq. (5.34) we have

$$M \frac{d}{dt} \sigma(A^\dagger, A) = \gamma \left[ -R^2 - X - Y - 2Z - \frac{1}{2}(Y - Z)^2 - \frac{1}{2}(X - Z)^2 \right],$$  (5.38)

where $R = \langle\sigma_x\rangle \langle\sigma_y\rangle$. Note that

$$X + Y + 2Z = \sigma(A, A) \geq 0.$$  (5.39)

Then we show that

$$M \frac{d}{dt} \sigma(A^\dagger, A) \leq 0$$  (5.40)

and the equality holds if and only if
\[ X = Y = Z = 0. \quad (5.41) \]

That is, the average in the left hand sides of equations (5.35), (5.36), and (5.37) is taken over the ground state \( |1\rangle \). This proves that the solution to Eq. (5.22) will localize at the ground state when the evolution time is larger than the localization time.

Finally, let us estimate the localization rate of the quantum state evolution. Using Eq. (5.38) and Eq. (5.39), we immediately obtain

\[
M \frac{d}{dt} \sigma(A, A) \leq -\gamma \sigma(A, A)^2. \quad (5.42)
\]

So the localization rate \( t_{\text{localization}} \) is

\[
t_{\text{localization}} \sim \frac{1}{\gamma}. \quad (5.43)
\]

Again we see that the above result is in agreement with the decoherence time scale (2.19).

VI. CONCLUDING REMARKS

In this paper, based on the two-level system models, we have studied in detail the decoherent histories approaches. We present an explicit analysis of the degree of decoherence and its relation to the von Neumann entropy. We have demonstrated the localization in both quantum jump simulations and quantum state diffusion approaches. Here we conclude with a few remarks.

Firstly, we have shown that the environment-induced decoherence, decoherent histories and the localization process in the stochastic Schrödinger equation approaches generally agree each other. We have shown that the set of histories consisting of the projection onto the basis in which the density operator tends to become diagonal give the better degree of decoherence. These results are in agreement with former studies on the quantum Brownian models [18] as well as the quantum optical models [24].

In this paper, we have shown that there are a number of sets of decoherent histories in this two levels model. Clearly, these decoherent histories are not equally important from
physical point of view. Among those, the most natural one is that which consist of the projections onto $|1\rangle$ and $|2\rangle$. We have proven that this set of histories give the best degree of decoherence. Note that the density matrix in the basis $|1\rangle$ and $|2\rangle$ will become diagonal after a typically short time. Moreover, we show that the solutions to the stochastic Schrödinger equation in the quantum jump simulation will localize at $|1\rangle$ or $|2\rangle$ after certain time which is basically same as the decoherence time.

It is known that the solution generated by stochastic Schrödinger equation (5.1) will randomly jump between the two levels $|1\rangle$ and $|2\rangle$. However, due to the influence of the bath, the average populations of the two levels will gradually become stable, and the localization process occurs. As we have shown, the set of histories consisting of the projection onto these two levels are perfectly decoherent. Here, we have seen that quantum jump simulation is entirely compatible with the history point of view. This is a very nice result. Similar results in the quantum state diffusion have been discussed before [17,18]

In addition, we have found that the environment-induced decoherence, decoherent histories and the localization process are more effective as the bath temperature increases. Physically, this is an expected result as the bath at a higher temperature would have stronger influence on the system. As expected, the time scales concerning both decoherence and localization are basically same.

Secondly, the approximate decoherence is of basic importance in practical physical process. By using this two-level system model we can clearly see what determines the degree of decoherence. For a given set of histories, the only adjustable parameters are the temperature of bath, the time-spacing interval and the initial state of the system. We also see a interesting relation between the degree of decoherence and the von Neumann entropy. This relation indicates that there is a tension between the predictability of state of system and the degree of decoherence. A similar tension has been discussed in a different situation based on quantum Brownian motion models [7].

Thirdly, it is important to notice that, as phenomenological theories, both quantum state diffusion and quantum jump simulation must be used under some conditions (e.g.
The comparison between different approaches therefore must be made in caution since the correspondence between them is by no means mathematically one-to-one correspondence. Rather, we emphasize that, underlying the open quantum system, the mutual influence between the system and its environment is the common theoretical base of all of those approaches and both decoherence and localization are nothing more than the different manifestations of a single entity.

Finally, let us note that the coarse-graining in our two-level system is made by using projection operators on the system whilst ignoring the environment. It would be interesting to consider the general $n$-dimensional model in which the effect of a further coarse-graining on the degree of decoherence can be discussed. Work towards to this aspect is in progress.

The environment-induced decoherence, decoherent histories as well as various stochastic Schrödinger equations have provided many important insights into the understanding of fundamental problems in quantum theory. The investigation into the similarity and difference between the different approaches is of importance. The more thorough studies in this aspect would be useful.

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APPENDIX A: PROOF OF THEOREM

In this Appendix, we shall give a proof of Theorem in Section III. Since both $M$ and $N$ are positive definite matrices, therefore, one of them, say, $N$ can be decomposed as
\[ N = S^\dagger S, \tag{A1} \]

where \( S \) is an \( n \times n \) matrix. After an arrangement, the right-hand side of Eq. (4.3) becomes

\[ |\text{Tr}(MPS^\dagger SQ)| = |\text{Tr}(SQMPS^\dagger)|. \tag{A2} \]

Suppose \( x_m \) are an orthonormal basis in \( n \) dimensional space \( V \). Then

\[ |\text{Tr}(MPNQ)| = |\sum_m x_m^T (SQMPS^\dagger) x_m|, \tag{A3} \]

where \( x_m^T \) is the transpose of \( x_m \). Now, we set

\[ y_m = QS^\dagger x_m, \tag{A4} \]
\[ z_m = PS^\dagger x_m. \tag{A5} \]

Then the trace in Eq. (A3) may be rewritten as

\[ \text{Tr}(MPNQ) = \sum_m y_m^T M z_m. \tag{A6} \]

Since \( y_m, z_m \) are orthogonal vectors and \( M \) is a positive definite matrix, then it is not difficult to arrive at the following inequality (see [25])

\[ |y_m^T M z_m| \leq \epsilon^M (y_m^T M y_m)^{1/2} (z_m^T M z_m)^{1/2}, \tag{A7} \]

where \( \epsilon^M = (\lambda_{\text{max}}^M - \lambda_{\text{min}}^M)/(\lambda_{\text{max}}^M + \lambda_{\text{min}}^M) \), \( \lambda_{\text{max}}^M \) and \( \lambda_{\text{min}}^M \) are the largest and the smallest eigenvalues of \( M \), respectively. Combining (A7) with Cauchy’s inequality,

\[ (\sum_m a_n b_n)^2 \leq \sum_m a_n^2 \sum_m b_n^2, \tag{A8} \]

then (A3) becomes

\[ |\text{Tr}(MPNQ)| = |\sum_m x_m^T (SQMPS^\dagger) x_m| \]
\[ \leq \sum_m |y_m^T M z_m| \]
\[ \leq \epsilon^M (\sum_m y_m^T M y_m)^{1/2} (\sum_m z_m^T M z_m)^{1/2}. \tag{A9} \]
It is easy to identify that

\[
\text{Tr} (MPNP) = \sum_m y_m^T M y_m, \quad (A10)
\]
\[
\text{Tr} (MQNQ) = \sum_m z_m^T M z_m. \quad (A11)
\]

This proves that

\[
|\text{Tr} (MPNP)| \leq \epsilon^M [\text{Tr} (MPNP)]^{1/2} [\text{Tr} (MQNQ)]^{1/2}. \quad (A12)
\]

Since \(M\) and \(N\) are in the completely symmetric position, so the similar result is true for \(\epsilon^N\). Then it completes the proof of the theorem. \(\square\)
REFERENCES


