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in Quantum Einstein Gravity**

By

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RESOLUTION OF THE BRS-SINGLET-PAIR PROBLEM IN QUANTUM EINSTEIN GRAVITY

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ABSTRACT

The possible existence of BRS singlet pairs causes a trouble for the proof of unitarity based on the BRS symmetry especially in quantum gravity. In quantum Einstein gravity, this difficulty is resolved in the framework of its covariant operator formalism by generalizing the proof based on both BRS and anti-BRS symmetries.

1. Introduction

The BRS symmetry[1] is very important for establishing the unitarity of the physical S-matrix in quantum gauge theories, quantum gravity, etc. In the Heisenberg picture, the unitarity problem is formulated as follows[2, 3].

Let Q_B be the BRS charge. Then the physical subspace $\mathcal{V}_{\text{phys}}^{\text{KO}}$ is defined by the totality of the states $|\text{phys}\rangle$ which are annihilated by Q_B . Since $Q_B^2 = 0$, there exist BRS singlets and BRS doublets only. A BRS doublet consists of a *parent* state $|p\rangle$ and its *daughter* state $|d\rangle = Q_B|p\rangle \neq 0$. Postulating the asymptotic completeness,* one can then show that any physical state is a superposition of singlet states and daughter states, which are mutually orthogonal. Symbolically, we may write

$$|\text{phys}\rangle = \sum_n \alpha_n |s\rangle_n + \sum_m \beta_m |d\rangle_m, \quad (1.1)$$

where $|s\rangle$ denotes a BRS singlet state and α_n and β_m are c-number coefficients. We then have

$$\langle \text{phys}' | \text{phys} \rangle = \sum_{n, n'} \alpha_n'^* \alpha_n \langle s | s \rangle_n. \quad (1.2)$$

This is the theorem due to Kugo and Ojima[2].

In order to establish the unitarity from (1.2), we further need two propositions described below. Let Q_C be the FP-ghost number charge. Consider one-particle (asymptotic) states. If $Q_C|s\rangle_n = 0$, $|s\rangle_n$ is called a pure singlet state and has either positive norm or negative norm. As a physical requirement, we must postulate that *there are no pure singlets of negative norm*. This postulate is, of course, supported by perturbation theory, provided that no artificial modification of the theory is made, such as introducing higher derivative terms into the Lagrangian density.

* As is well known, asymptotic completeness means that the Fock space of asymptotic states coincides with the total state-vector space.

If $Q_C|s\rangle_n \neq 0$, $|s\rangle_n$ has zero norm. Let

$$iQ_C|s\rangle_n = N_n|s\rangle_n, \quad (1.3)$$

where N_n is a nonzero integer. Then there always exists a state $|s^c\rangle_n$ such that

$$iQ_C|s^c\rangle_n = -N_n|s^c\rangle_n, \quad (1.4)$$

$${}_n\langle s^c|s\rangle_n \neq 0. \quad (1.5)$$

A pair of state $\{|s\rangle, |s^c\rangle\}$ is called a BRS singlet pair. Because of (1.5), a linear combination of $|s\rangle_n$ and $|s^c\rangle_n$ can always have negative norm. Hence we need the proposition that *there are no BRS singlet pairs*.

If the above two propositions are all right, (1.2) guarantees the unitarity of the physical S-matrix.

The problem is how to avoid the trouble caused by BRS singlet pairs. The nonexistence of BRS singlet pairs is supported by perturbation theory in gauge theories but *not necessarily in quantum gravity*, as has been uncovered recently by Abe and Nakanishi[4].

One may suppose that BRS singlet pairs could be excluded by adding an extra condition

$$Q_C|\text{phys}\rangle = 0. \quad (1.6)$$

Unfortunately, this is *not* the case[2, 3]. From one-particle states $|s\rangle$ and $|s^c\rangle$, we can construct a negative-norm *two-particle* state satisfying (1.6).

So far, there are two papers which attacked the BRS-singlet-pair problem. Nakanishi[5] directly showed the nonappearance of BRS singlet pairs in gauge theories. Bonora, Pasti and Tonin[6] proposed a resolution by adding a new subsidiary condition

$$\bar{Q}_B|\text{phys}\rangle = 0 \quad (1.7)$$

in gauge theories, where $\bar{Q}_B$ denotes the anti-BRS charge. Unfortunately, neither reasonings are completely free of a subtle question caused by the topology of the indefinite-metric Hilbert space[3], which is a prohibitively difficult mathematical problem.

The purpose of the present paper is to propose a satisfactory resolution of the BRS-singlet-pair problem in quantum Einstein gravity by using its covariant operator formalism[7, 3]. Our line of thought is based on the work of Bonora, Pasti and Tonin, but our approach is more general. Furthermore, we have succeeded in avoiding the subtle question stated above.

The present paper is organized as follows. In Sec.2, we sketch the asymptotic states of two-dimensional quantum gravity so as to present an example of the BRS singlet pair. In Sec.3, the work of Bonora, Pasti and Tonin is reviewed. In Sec.4, we extend their theory to the form convenient for our purpose. In Sec.5, we apply this result to the covariant operator formalism of quantum Einstein gravity and resolve the BRS-singlet-pair problem in this framework. The final section is devoted to discussion.

2. Two-dimensional quantum gravity

In this section, we review an explicit example of the BRS singlet pair, which has recently been found in the covariant operator formalism of two-dimensional quantum gravity[4].

The covariant operator formalism of quantum Einstein gravity[7, 3] is based on the Lagrangian density

$$\mathcal{L}_{\text{qEG}} = \frac{1}{2\kappa} \sqrt{(-1)^{D-1}} g R - \tilde{g}^{\mu\nu} (\partial_\mu b_\nu + i \partial_\mu \bar{c}_\rho \cdot \partial_\nu c^\rho) + \mathcal{L}_{\text{matter}}. \quad (2.1)$$

in D dimensions. Here κ is the Einstein gravitational constant, $g_{\mu\nu}$ the gravitational field, $g = \det g_{\mu\nu}$, $\tilde{g}^{\mu\nu} = \sqrt{(-1)^{D-1}} g g^{\mu\nu}$, R the scalar curvature, b_ρ the gravitational B-field, and c^σ and $\bar{c}_\tau$ the gravitational FP ghost and antighost, respectively.

As is well known, the first term of (2.1), i.e., the Einstein-Hilbert Lagrangian density, becomes a total divergence when $D = 2$. Hence, in the two-dimensional case, the Lagrangian density reduces to

$$\mathcal{L}_{2\text{DQG}} = -\tilde{g}^{\mu\nu}(\partial_\mu b_\nu + i\partial_\mu \bar{c}_\rho \cdot \partial_\mu c^\rho) + \mathcal{L}_{\text{matter}}. \quad (2.2)$$

Since $\det \tilde{g}^{\mu\nu} = -1$ for $D = 2$, $\tilde{g}^{\mu\nu}$ has only two degrees of freedom. Accordingly, if $\mathcal{L}_{\text{matter}}$ is expressible only in terms of $\tilde{g}^{\mu\nu}$ and the matter fields, it is possible to canonically quantize (2.2). Then we can explicitly construct the exact solution in closed form.

According to the exact solution, asymptotic states coincide with those which are derived naively by perturbation theory, apart from a subtlety caused by infrared pathology.* The asymptotic fields are defined by

$$\begin{aligned} \tilde{g}^{\mu\nu} &\rightarrow \eta^{\mu\nu} - 2\psi^{\mu\nu}, & b_\rho &\rightarrow \beta_\rho, \\ c^\sigma &\rightarrow \gamma^\sigma, & \bar{c}_\tau &\rightarrow \bar{\gamma}_\tau \end{aligned} \quad (2.3)$$

together with those of matter fields. Of course, $\psi^\lambda{}_\lambda = 0$. The asymptotic-field Lagrangian density is given by

$$\mathcal{L}_{2\text{DQG}}^{\text{as}} = -2\partial_\mu \psi^{\mu\nu} \cdot \beta_\nu - i\partial^\mu \bar{\gamma}_\rho \cdot \partial_\mu \gamma^\rho \quad (2.4)$$

apart from the matter part.

The asymptotic-field equations are

$$\partial^\mu \psi_{\mu\nu} = 0, \quad (2.5)$$

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu - \eta_{\mu\nu} \partial^\lambda \beta_\lambda = 0, \quad (2.6)$$

$$\partial^\lambda \partial_\lambda \bar{\gamma}_\rho = \partial^\lambda \partial_\lambda \gamma_\rho = 0. \quad (2.7)$$

* In the two-dimensional massless theories, infrared divergences are so serious that massless poles can be induced through momentum integration. Since such poles are infinitely many, we neglect the possible asymptotic states corresponding to them.

Let δ be the BRS transformation. The BRS symmetry is linearly represented in terms of asymptotic fields as follows:

$$\delta\psi_{\mu\nu} = -\frac{1}{2}(\partial_\mu\gamma_\nu + \partial_\nu\gamma_\mu - \eta_{\mu\nu}\partial^\lambda\gamma_\lambda), \quad (2.8)$$

$$\delta\bar{\gamma}_\rho = i\beta_\rho, \quad (2.9)$$

$$\delta\beta_\rho = \delta\gamma_\rho = 0. \quad (2.10)$$

Setting $\psi_{00} = \psi_{11} \equiv \psi$ and $\psi_{01} = \psi_{10} \equiv \tilde{\psi}$, we rewrite (2.5) as $\partial_0\psi = \partial_1\tilde{\psi}$ and $\partial_1\psi = \partial_0\tilde{\psi}$; and (2.8) as

$$\delta\psi = -\frac{1}{2}(\partial_0\gamma_0 + \partial_1\gamma_1). \quad (2.11)$$

Likewise, (2.6) is rewritten as $\partial_0\beta_0 + \partial_1\beta_1 = 0$ and $\partial_1\beta_0 + \partial_0\beta_1 = 0$.

Now, we consider one-particle states in momentum space. Of course, we have $|p_1| = p_0$, and therefore, $\tilde{\psi} = \epsilon(p_1)\psi$ and $\beta_1 + \epsilon(p_1)\beta_0 = 0$. Accordingly, the independent one-particle states are as follows:

$$|\psi(p_1)\rangle, |\gamma_+(p_1)\rangle, |\bar{\gamma}_-(p_1)\rangle, |\beta_-(p_1)\rangle; \quad (2.12)$$

$$|\gamma_-(p_1)\rangle, |\bar{\gamma}_+(p_1)\rangle, \quad (2.13)$$

where

$$\varphi_\pm(p_1) \equiv \varphi_0(p_1) \pm \epsilon(p_1)\varphi_1(p_1) \quad \text{for } \varphi = \gamma, \bar{\gamma}, \beta. \quad (2.14)$$

Evidently, the first four states form a BRS doublet pair, which is usually called “quartet”, and the remaining two form a BRS singlet pair.

We thus encounter a BRS singlet pair explicitly in two-dimensional quantum gravity[4]. Hence it is an urgent problem to resolve the trouble due to the BRS singlet pair in quantum gravity, though quantum Einstein gravity involves no BRS singlet pairs *perturbatively*.

3. Bonora-Pasti-Tonin theory

In this section, we review the work of Bonora, Pasti and Tonin[6], who proposed a method for expelling BRS singlet pairs from the physical subspace.

They considered the quantum nonabelian gauge theory which has both BRS and anti-BRS symmetries. BRS charge Q_B , anti-BRS charge $\bar{Q}_B$ and FP-ghost number charge Q_C form the so-called extended BRS algebra:

$$Q_B^2 = \bar{Q}_B^2 = \{Q_B, \bar{Q}_B\} = 0, \quad (3.1)$$

$$i[Q_C, Q_B] = Q_B, \quad i[Q_C, \bar{Q}_B] = -\bar{Q}_B. \quad (3.2)$$

Their crucial observation is that the FP-ghost number current J_C^μ can be expressed as

$$J_C^\mu = \{Q_B, \bar{C}^a A^{\mu a}\} + \{\bar{Q}_B, C^a A^{\mu a}\}, \quad (3.3)$$

where A_μ^a , C^a and $\bar{C}^a$ denote the gauge field, FP ghost and FP anti-ghost, respectively. Although $C^a A^{\mu a}$ and $\bar{C}^a A^{\mu a}$ are *nonconserved* currents, one may derive from (3.3) that

$$Q_C = \{Q_B, \bar{\mathcal{R}}\} + \{\bar{Q}_B, \mathcal{R}\}, \quad (3.4)$$

where $\mathcal{R}$ and $\bar{\mathcal{R}}$ are some x^0 -dependent quantities.

A new physical subspace $\mathcal{V}_{\text{phys}}^{\text{BPT}} \subset \mathcal{V}_{\text{phys}}^{\text{KO}}$ is defined by the totality of the states $|\text{phys}\rangle$ annihilated both by Q_B and by $\bar{Q}_B$:

$$Q_B|\text{phys}\rangle = \bar{Q}_B|\text{phys}\rangle = 0. \quad (3.5)$$

Then, from (3.4), we have

$$Q_C|\text{phys}\rangle = Q_B\bar{\mathcal{R}}|\text{phys}\rangle + \bar{Q}_B\mathcal{R}|\text{phys}\rangle, \quad (3.6)$$

whence

$$\langle \text{phys}' | Q_C | \text{phys} \rangle = 0. \quad (3.7)$$

Accordingly, no BRS singlet pairs belong to $\mathcal{V}_{\text{phys}}^{\text{BPT}}$ because of (1.4) and (1.5).

Of course, the above fact alone does not guarantee the unitarity of the physical S-matrix. As remarked in the Introduction, we must also expel the zero-FP-ghost-number states made from BRS singlet pairs. To show this, Bonora, Pasti and Tonin analyzed the possible representations of the extended BRS algebra in asymptotic states. According to them, the only possible representations are as follows:

- (i) *Singlet.* $\{ |f\rangle \}$ such that $Q_B|f\rangle = \bar{Q}_B|f\rangle = 0$, whence $Q_C|f\rangle = 0$.
- (ii) *Quadruplet.* It can be expressed as $\{ |f\rangle, Q_B|f\rangle, \bar{Q}_B|f\rangle, Q_B\bar{Q}_B|f\rangle \}$.
- (iii) *Symmetrical a-chain* ($r = 0, 1, 2, \dots$). It consists of $2r + 1$ parents $\{ |f_r\rangle, |f_{r-1}\rangle, \dots, |f_{-r}\rangle \}$ and $2r + 2$ daughters $\{ Q_B|f_r\rangle, \bar{Q}_B|f_r\rangle = Q_B|f_{r-1}\rangle, \dots, \bar{Q}_B|f_{-r+1}\rangle = Q_B|f_{-r}\rangle, \bar{Q}_B|f_{-r}\rangle \}$, where $Q_C|f_0\rangle = 0$ and $Q_B\bar{Q}_B|f_r\rangle = Q_B\bar{Q}_B|f_{-r}\rangle = 0$.
- (iv) *Symmetrical b-chain* ($r = 0, 1, 2, \dots$). It consists of $2r + 2$ parents $\{ |f_{r+\frac{1}{2}}\rangle, \dots, |f_{-r-\frac{1}{2}}\rangle \}$ and $2r + 1$ daughters $\{ \bar{Q}_B|f_{r+\frac{1}{2}}\rangle = Q_B|f_{r-\frac{1}{2}}\rangle, \dots, \bar{Q}_B|f_{-r+\frac{1}{2}}\rangle = Q_B|f_{-r-\frac{1}{2}}\rangle \}$, where $Q_C\bar{Q}_B|f_{\frac{1}{2}}\rangle = 0$ and $Q_B|f_{r+\frac{1}{2}}\rangle = \bar{Q}_B|f_{-r-\frac{1}{2}}\rangle = 0$.

Two states, which are not necessarily distinct, are called *conjugate* if they have a nonvanishing inner product as in (1.5). The state-vector space is decomposed into subspaces in such a way that no two states belonging to different ones are conjugate. The conjugate of a singlet is that singlet itself. The conjugate of a quadruplet is a (same or different) quadruplet. The conjugate of a symmetrical a-chain is a symmetrical b-chain, and *vice versa*.

The states satisfying (3.5) are singlets, the $Q_B\bar{Q}_B|f\rangle$ states of quadruplets, and the daughters of symmetrical a-chains and symmetrical b-chains. The physical subspace

$\mathcal{V}_{\text{phys}}^{\text{BPT}}$, therefore, contains no states conjugate to them except for singlets. Thus no BRS singlet pairs can produce nonvanishing inner product in $\mathcal{V}_{\text{phys}}^{\text{BPT}}$.

For better understanding, we present a simple example. Although the model discussed in Sec.2 is not a gauge theory, the above representation theory of the extended BRS algebra is applicable to it. Indeed, its one-particle states are classified into a symmetrical a-chain of $r = 0$ and a symmetrical b-chain of $r = 0$:

$$\{ |\psi(p_1)\rangle; |\gamma_+(p_1)\rangle, |\bar{\gamma}_+(p_1)\rangle \}, \quad (3.8)$$

$$\{ |\gamma_-(p_1)\rangle, |\bar{\gamma}_-(p_1)\rangle; |\beta_-(p_1)\rangle \}. \quad (3.9)$$

Since the states satisfying (3.5) are $|\gamma_+(p_1)\rangle, |\bar{\gamma}_+(p_1)\rangle$ and $|\beta_-(p_1)\rangle$, $\mathcal{V}_{\text{phys}}^{\text{BPT}}$ contains no BRS singlet pair.

In short, the essence of the Bonora-Pasti-Tonin approach is *not* to prove the nonexistence of BRS singlet pairs *but* to expel them from the physical subspace.

Finally, we make a comment on the subtlety of (3.4). Because of nonconservation of the currents, $\bar{C}^a A_\mu{}^a$ and $C^a A_\mu{}^a$, both ‘charges’, $\bar{\mathcal{R}}$ and $\mathcal{R}$, are *time dependent* quantities *even if* they should be well defined. Since the notions of asymptotic fields and asymptotic states involve the time limiting procedure $\lim_{x^0 \rightarrow \mp\infty}$ in an essential way, it is very difficult to control the limiting behavior of such time-dependent quantities as $\bar{\mathcal{R}}$ and $\mathcal{R}$. For this reason, (3.4) may not allow us to get a definite conclusion.

4. General formulation of BRS-type subsidiary conditions

We extend the Bonora-Pasti-Tonin theory to the case in which there are a number of BRS-like symmetries. Since it is unpractical to construct all representations of such a large algebra, we here adopt a more direct way.

Our assumptions are as follows. There are several nilpotent hermitian charges $Q_j^{(r)}$ ($j = 1, \dots, N$; $r = 1, \dots, N_j$). The index j denotes a set of certain quantum

numbers such as the FP-ghost number, according to which the total state vector space is supposed to be decomposed into different sectors, mutually orthogonal for different j 's. So, we need not consider linear combinations of $Q_j^{(r)}$ with different j 's. We set

$$Q_j(a) \equiv \sum_{r=1}^{N_j} a_j^{(r)} Q_j^{(r)}, \quad (4.1)$$

where $a = \{a_j^{(r)}\}$ denotes a set of c -number constants. For any a , we have

$$[Q_j(a)]^2 = 0, \quad (4.2)$$

but the anticommutator $\{Q_j(a), Q_k(a')\}$ ($j \neq k$) need not necessarily vanish. All charges suffer neither from spontaneous breakdown nor anomaly, and asymptotic completeness is satisfied, so that they are linearly represented in terms of asymptotic states.

The physical subspace $\mathcal{V}_{\text{phys}} = \{|\text{phys}\rangle\}$ is defined by the subsidiary conditions

$$Q_j(a)|\text{phys}\rangle = 0 \quad (j = 1, \dots, N) \quad (4.3)$$

for any a .

A state $|d\rangle$ is called a *descendant* if there exists j , a and a state $|f\rangle$ such that

$$|d\rangle = Q_j(a)|f\rangle. \quad (4.4)$$

Non-descendant states are classified into singlets and ancestors. A *singlet* state $|s\rangle$ is characterized by the additional condition that

$$Q_j(a)|s\rangle = 0 \quad (4.5)$$

for any j and any a . A non-descendant state $|p\rangle$ for which (4.5) does not hold is called an *ancestor*.

The above classification is complete in the sense of "classical logic" but not in the sense of "quantum logic", because, for example, $|p\rangle + |s\rangle$ is again an ancestor. We have to make more detailed specification in terms of asymptotic states.

Since each $Q_j(a)$ is a BRS-like charge, we can define the totality, $\mathcal{H}_j(a)$, of the singlet states with respect to each particular $Q_j(a)$. Here we know that $\mathcal{H}_j(a)$ is a subspace having its orthogonal complement[2]. Let

$$\mathcal{H} \equiv \bigcap_{j, a^{(j)}} \mathcal{H}_j(a^{(j)}). \quad (4.6)$$

Then, by definition, $\mathcal{H}$ consists of singlet states only and has its orthogonal complement, which is denoted by $\mathcal{H}^\perp$.

The subspace $\mathcal{H}^\perp$ is spanned by multiplets. Since j is a set of good quantum numbers, each multiplet consists of a certain number of ancestors and their descendants which are obtained by the action of one or several of $Q_1(a^{(1)}), \dots, Q_N(a^{(N)})$ on the ancestors. We need not know the detailed structure of multiplets. What we need are the facts that $\mathcal{H}^\perp$ is spanned by ancestors and their descendants and that any nontrivial linear combination of the ancestors cannot be expressed by descendants.

From the above consideration, we see that $\mathcal{V}_{\text{phys}}$ is spanned by singlet states and descendants. No ancestors are contained in $\mathcal{V}_{\text{phys}}$. That is, as in (1.1), we have

$$|\text{phys}\rangle = \sum_n \alpha_n |s\rangle_n + \sum_m \beta_m |d\rangle_m. \quad (4.7)$$

Then, we obtain a generalization of the theorem presented in (1.2):

$$\langle \text{phys}' | \text{phys} \rangle = \sum_{n, n'} \alpha_{n'}^* \alpha_n \langle s | s \rangle_n. \quad (4.8)$$

The proof of (4.8) is very simple. Using (4.7), we have

$$\begin{aligned} \langle \text{phys}' | \text{phys} \rangle &= \sum_{n, n'} \alpha_{n'}^* \alpha_n \langle s | s \rangle_n \\ &+ \sum_{n', m} \alpha_{n'}^* \beta_m \langle s | d \rangle_m + \sum_{m'} \beta_{m'}^* \beta_m \langle d | \text{phys} \rangle. \end{aligned} \quad (4.9)$$

Substituting (4.4), we see that both second term and third one vanish because of (4.5) and (4.3). (q.e.d.)

The above theorem should be supplemented by the following proposition: *No many-particle singlet states can be formed from one-particle multiplets*, that is, the totality $\mathcal{H}$ of singlets has a Fock space structure. This proposition is very crucial because *if there are no one-particle BRS singlet pairs in $\mathcal{H}$, then we can safely assert that there are no many-particle states composed of them in $\mathcal{H}^*$* .

The proof goes as follows. Let $\mathcal{D}_a$ and $\mathcal{D}_b$ be two arbitrary irreducible representations of the algebra formed by $\{ Q_1(a^{(1)}), \dots, Q_N(a^{(N)}) \}$. Then what to be shown is that no singlet representation is contained in the product representation $\mathcal{D}_a \otimes \mathcal{D}_b$ unless both of $\mathcal{D}_a$ and $\mathcal{D}_b$ are singlet ones.

We employ reduction to absurdity. Suppose that a singlet state $|s\rangle$ is formed as a product state in the following way:

$$|s\rangle = \sum_m |\hat{f}_m\rangle \otimes |\hat{g}_m\rangle, \quad (4.10)$$

where $|\hat{f}_1\rangle, |\hat{f}_2\rangle, \dots$ are linearly independent and also $|\hat{g}_1\rangle, |\hat{g}_2\rangle, \dots$ are so. Since $|s\rangle$ is a singlet, we have

$$Q_j(a)|s\rangle = 0 \quad (4.11)$$

for any j and a .

Given any particular $Q_j(a)$, it is convenient to rewrite (4.10) into a more detailed form as

$$\begin{aligned} |s\rangle = & \sum_m |f'_m\rangle \otimes |g'_m\rangle + \sum_m |f_m\rangle \otimes |g_m^{(0)}\rangle \\ & + \sum_m |f_m^{(0)}\rangle \otimes |g_m\rangle + \sum_m |f_m^{(0)}\rangle \otimes |g_m^{(0)}\rangle, \end{aligned} \quad (4.12)$$

* This should be contrasted with the situation found in [8]. For explaining color confinement, one can take in a *positive* way the lack of Fock space structure for *many-particle color-singlet states* in the relation to colored quark and gluon one-particle states. Of course, this classification between color singlets and multiplets is done with respect to the *color charge*, which should not be confused with the one based on *nilpotent charges* in the present context.

where the state having a superscript (0) denotes the one annihilated by $Q_j(a)$. In the form of (4.12), we can assume, without loss of generality, that $Q_j(a)|f'_1\rangle$, $Q_j(a)|f'_2\rangle$, $\dots$ are linearly independent, and likewise for $\{Q_j(a)|g'_m\rangle\}$, $\{Q_j(a)|f_m\rangle\}$ and $\{Q_j(a)|g_m\rangle\}$.

Now, from (4.11) and (4.12), we have

$$\begin{aligned} & \sum_m [Q_j(a)|f'_m\rangle] \otimes |g'_m\rangle + \sum_m \epsilon'_m |f'_m\rangle \otimes [Q_j(a)|g'_m\rangle] \\ & + \sum_m [Q_j(a)|f_m\rangle] \otimes |g_m^{(0)}\rangle + \sum_m \epsilon_m^{(0)} |f_m^{(0)}\rangle \otimes [Q_j(a)|g_m\rangle] = 0, \end{aligned} \quad (4.13)$$

where ϵ'_m and $\epsilon_m^{(0)}$ are some sign factors. Hence, the action of $1 \otimes Q_j(a)$ on (4.13) yields

$$\sum_m [Q_j(a)|f'_m\rangle] \otimes [Q_j(a)|g'_m\rangle] = 0. \quad (4.14)$$

We take inner products for the second factors of (4.14) with an arbitrary state. Then the linear independence of the first factors implies that all those inner products vanish. Hence we see that $Q_j(a)|g'_m\rangle = 0$ for each m . The linear independence of $\{Q_j(a)|g'_m\rangle\}$, therefore, implies that $|g'_m\rangle \neq 0$ for no m . That is, the first term of (4.12) must be absent. Then (4.13) reduces to

$$\sum_m [Q_j(a)|f_m\rangle] \otimes |g_m^{(0)}\rangle = - \sum_m \epsilon_m^{(0)} |f_m^{(0)}\rangle \otimes [Q_j(a)|g_m\rangle]. \quad (4.15)$$

Because of the linear independence of $\{Q_j(a)|f_m\rangle\}$ and of $\{Q_j(a)|g_m\rangle\}$, there must exist one-to-one correspondence between the terms of both sides of (4.15). Accordingly, by adjusting c -number coefficients appropriately, we have

$$Q_j(a)|f_m\rangle = |f_m^{(0)}\rangle, \quad (4.16)$$

$$Q_j(a)|g_m\rangle = \epsilon_m |g_m^{(0)}\rangle, \quad (\epsilon_m \equiv -\epsilon_m^{(0)}). \quad (4.17)$$

Hence, (4.12) reduces to

$$|s\rangle = \sum_m \{\epsilon_m |f_m\rangle \otimes [Q_j(a)|g_m\rangle] + [Q_j(a)|f_m\rangle] \otimes |g_m\rangle\}$$

$$\begin{aligned}
& + \sum_m |f'_m{}^{(0)}\rangle \otimes |g'_m{}^{(0)}\rangle \\
& = Q_j(a) \left(\sum_m |f_m\rangle \otimes |g_m\rangle \right) + \sum_m |f'_m{}^{(0)}\rangle \otimes |g'_m{}^{(0)}\rangle. \tag{4.18}
\end{aligned}$$

Since the first term of (4.18) is a descendant, this part should be excluded. Thus we have only to consider the remainder

$$|s'\rangle = \sum_m |f'_m{}^{(0)}\rangle \otimes |g'_m{}^{(0)}\rangle. \tag{4.19}$$

This is true for any *particular* $Q_j(a)$.

Regarding (4.19) as (4.10), we *repeat* the above procedure $\sum_{j=1}^N N_j$ times. Then, we finally obtain

$$|\hat{s}\rangle = \sum_m |\hat{f}_m^{(0)}\rangle \otimes |\hat{g}_m^{(0)}\rangle, \tag{4.20}$$

where

$$Q_j(a) |\hat{f}_m^{(0)}\rangle = Q_j(a) |\hat{g}_m^{(0)}\rangle = 0 \tag{4.21}$$

for *all* $Q_j(a)$'s. By assumption, both $|\hat{f}_m^{(0)}\rangle$ and $|\hat{g}_m^{(0)}\rangle$ are simultaneously singlets for no m . That is, for each m , there exists *some* $Q_j(a)$ such that at least either

$$|\hat{f}_m^{(0)}\rangle = Q_j(a) |f_m\rangle \quad \text{or} \quad |\hat{g}_m^{(0)}\rangle = Q_j(a) |g_m\rangle \tag{4.22}$$

holds. Then, because of (4.21), $|\hat{f}_m^{(0)}\rangle \otimes |\hat{g}_m^{(0)}\rangle$ is a descendant.

Thus all terms of $|s\rangle$ are written in terms of descendants. This contradicts the assumption. (q.e.d.)

5. Application to quantum Einstein gravity

The covariant operator formalism of quantum Einstein gravity[7, 3] is quite beautifully formulated on the basis of the Lagrangian density (2.1). From it, the d'Alembert

equations

$$\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu X) = 0 \quad \text{for} \quad X = x^\lambda, b_\rho, c^\sigma, \bar{c}_\tau \quad (5.1)$$

follow. Accordingly, not only $\tilde{g}^{\mu\nu}\partial_\nu X$ but also

$$J^\mu(X, Y) \equiv \sigma \tilde{g}^{\mu\nu}(X\partial_\nu Y - \partial_\nu X \cdot Y) \quad (5.2)$$

are conserved currents, where σ is a certain phase factor. Hence we have conserved charges

$$M(X, Y) \equiv \int d^{D-1}x J^0(X(x), Y(x)), \quad (5.3)$$

where X and Y of the left-hand side should be understood as indices. In particular, $M(b_\mu, c^\nu)$, $M(b_\mu, \bar{c}_\nu)$ and $M(\bar{c}_\mu, c^\nu)$ are tensor extensions of BRS charge, anti-BRS charge and FP-ghost number charge, respectively.

The (anti)commutation relations between $M(X, Y)$ and $M(U, V)$ are known[7, 3]. What we need here are as follows:

$$\{M(b_\mu, c^\nu), M(x^\lambda, \bar{c}_\rho)\} = -\delta^\nu_\rho M(b_\mu, x^\lambda) - \delta^\lambda_\mu M(\bar{c}_\rho, c^\nu), \quad (5.4)$$

$$\{M(b_\mu, \bar{c}_\rho), M(x^\lambda, c^\nu)\} = \delta^\nu_\rho M(b_\mu, x^\lambda) - \delta^\lambda_\mu M(\bar{c}_\rho, c^\nu). \quad (5.5)$$

We therefore have

$$\begin{aligned} Q_C &\equiv M(\bar{c}_\nu, c^\nu) \\ &= -\frac{1}{2D}(\{M(b_\mu, c^\nu), M(x^\mu, \bar{c}_\nu)\} + \{M(b_\mu, \bar{c}_\nu), M(x^\mu, c^\nu)\}). \end{aligned} \quad (5.6)$$

This formula should be compared with (3.4). In contrast with the Bonora-Pasti-Tonin case, however, $M(x^\mu, \bar{c}_\nu)$ and $M(x^\mu, c^\nu)$ are *conserved* charges. Thus we do not encounter the subtlety discussed at the end of Sec.3.

The physical subspace $\mathcal{V}_{\text{phys}}$ is now defined by the subsidiary conditions

$$M(b_\mu, c^\nu)|\text{phys}\rangle = M(b_\mu, \bar{c}_\nu)|\text{phys}\rangle = 0. \quad (5.7)$$

Then, corresponding to (3.6), we have

$$Q_C | \text{phys} \rangle = -\frac{1}{2D} [M(b_\mu, c^\nu) M(x^\mu, \bar{c}_\nu) + M(b_\mu, \bar{c}_\nu) M(x^\mu, c^\nu)] | \text{phys} \rangle, \quad (5.8)$$

whence

$$\langle \text{phys}' | Q_C | \text{phys} \rangle = 0. \quad (5.9)$$

Thus the BRS singlet pairs are absent at least in the one-particle sector.

Now, since

$$\{M(b_\mu, c^\nu), M(b_\lambda, c^\rho)\} = 0, \quad (5.10)$$

$$\{M(b_\mu, \bar{c}_\nu), M(b_\lambda, \bar{c}_\rho)\} = 0, \quad (5.11)$$

$$\{M(b_\mu, c^\nu), M(b_\lambda, \bar{c}_\rho)\} = -\delta^\nu_\rho M(b_\mu, b_\lambda), \quad (5.12)$$

$$i[Q_C, M(b_\mu, c^\nu)] = M(b_\mu, c^\nu), \quad (5.13)$$

$$i[Q_C, M(b_\mu, \bar{c}_\nu)] = -M(b_\mu, \bar{c}_\nu), \quad (5.14)$$

we can apply the theory developed in Sec.4 to the above formalism* ($N = 2$, $N_j = D^2$).

Hence, we see that no BRS singlet pairs are contained in $\mathcal{V}_{\text{phys}}$.

We have thus succeeded in establishing the unitarity of the physical S-matrix in the covariant operator formalism of quantum Einstein gravity. We have also learnt that the existence of tensor-type BRS and anti-BRS charges are not merely beautiful accessory of the theory but the property of practical importance.

6. Discussion

In this paper, we have successfully resolved the problem of BRS singlet pairs in the covariant operator formalism of quantum Einstein gravity by extending the idea of Bonora, Pasti and Tonin. Our result is significant especially in the following respects:

* No anomaly is encountered because dimensional regularization is possible.

- (1) It is more important to resolve the BRS-singlet-pair problem in quantum gravity than in quantum gauge theories because BRS singlet pairs are really present in two-dimensional quantum gravity.
- (2) Our theory is free of the subtlety encountered in the Bonora-Pasti-Tonin theory.
- (3) The so-called ghost-counting rule, which claims the equality between the number of bosonic ghosts and that of fermionic ones, is not necessarily required for guaranteeing the unitarity of the physical S-matrix.

For clarity, we emphasize that our proof is applicable only to quantum Einstein gravity but neither to gauge theories nor to supergravity, because the existence of the tensor extensions of BRS and anti-BRS charges is crucial.

Finally, we make some comments on the possible extension of our theory developed in Sec.4. The question which we want to address is the following: Why must all charges $Q_j^{(r)}$ be BRS-like?

If we consider a possibility of employing a charge Q which has a nonzero eigenvalue q ;

$$Q|f\rangle = q|f\rangle, \quad (6.1)$$

then $|f\rangle$ is a daughter of $|f\rangle$ itself. To avoid this situation, we should require for Q to be nilpotent:

$$Q^n = 0, \quad Q^{n-1} \neq 0. \quad (6.2)$$

For $n \geq 3$, however, Q cannot be regarded as a symmetry generator.

To show this, we consider the $n = 3$ case for definiteness. If Q is a symmetry generator, its action must be a derivation or an anti-derivation, that is, Q must obey the bosonic or fermionic Leibniz rule. We consider a product representation of two triplets $\{|f\rangle, Q|f\rangle, Q^2|f\rangle\}$ and $\{|g\rangle, Q|g\rangle, Q^2|g\rangle\}$. It must contain $3 \times 3 = 9$

linearly independent states. If Q obeys the Leibniz rule, however, this is *not* the case because

$$Q|f\rangle \otimes Q^2|g\rangle + Q^2|f\rangle \otimes Q|g\rangle = \frac{1}{3}Q^3(|f\rangle \otimes |g\rangle) = 0 \quad (6.3)$$

in the bosonic case and likewise in the fermionic case.

Thus we should have

$$Q^2 = 0. \quad (6.4)$$

In this case, Q must be fermionic. Indeed, if it is bosonic, we have

$$Q|f\rangle \otimes Q|g\rangle = \frac{1}{2}Q^2(|f\rangle \otimes |g\rangle) = 0. \quad (6.5)$$

In this way, we have led to the conclusion that all charges should be BRS-like.

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