Duality Transformations in the Ten-Dimensional Action and New Superstring Theories

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Abstract

Type II superstring theory with mixed Dirichlet and Neumann boundary conditions admit antisymmetric tensors with varying degrees in the spectrum. We show that there exists a family of dual supergravity lagrangians to the $N = 2$ type IIA action in ten dimensions. The duality transformations and the resulting actions are constructed explicitely.
At present there are many conjectured dualities between superstring theories as well as between superstring and super-p-brane theories in various dimensions [1-2]. As there is no known way of consistently quantizing the p-branes giving a massless spectrum, such conjectures could not be tested. Fortunately there has been a recent development where it is enough to consider open strings with fixed end points and Dirichlet boundary conditions in the presence of closed strings [3]. The Ramond-Ramond sectors (R-R) of the string Hilbert space contain vertex operators of the form \( \mathcal{O} \Gamma^{\mu_1} \cdot \cdot \cdot \Gamma^{\mu_n} Q F_{\mu_1} \cdot \cdot \cdot \mu_n \) where \( F \) is an \( n \)-form field strength. There is no analysis as yet predicting which mixtures of boundary conditions for the type II superstring with open strings having Dirichlet boundary conditions, are consistent and what the resulting spectrum is. It should be possible to make this study, but we shall attempt to answer this question by studying the duality transformations that could be carried on the supergravity action of type IIA in ten dimensions. Such a procedure has proven its effectiveness in the study of the \( N = 1 \) supergravity action in ten dimensions [4] where it was shown that an alternative formulation with the six-form replacing the two-form is possible [5]. This even led to simplifications in deriving the coupling to the super Yang-Mills sector [5] and to conjecturing the existence of super five-branes [2].

The massless spectrum of the type IIA superstring in ten-dimensions is easy to state. In the NS-NS sector we have the graviton the antisymmetric two-form and the dilaton. The R-R sector contains an abelian vector (a one-form) and a three form. The NS-R sectors contain left-handed and right-handed Majorana-Weyl gravitinos and spinors. This theory was shown to coincide with the dimensionally reduced eleven-dimensional supergravity action [6] not only for massless states [7] but for the massive ones as well [1]. Up to date, the formulation of the eleven dimensional action is unique and there is no known consistent modification of it. The alternative formulation with a five-form although conjectured to exist by super five-branes in eleven dimensions is inconsistent as a field theory [8]. What prevents carrying a duality transformation on the eleven-dimensional action is the existence of a Chern-Simons term in the action involving the three-form. It is not known how to apply a duality transformation to an action where not only the field strength appear but the gauge fields as well. Having mentioned that the type IIA supergravity action is obtained by a simple dimensional reduction from the eleven-dimensional theory would seem to indicate that this form of the theory is unique. This, however, is not the case as the reduction of the Chern-Simons form to ten dimensions could be manipulated in few interesting ways. The key observation is that a three form in eleven-dimensions becomes a three-form and a two-form in ten-dimensions. One can always write this.
term in such a way that the action could be expressed in terms of the field strength of one of these forms (but not both) thus allowing a duality transformation on that field to be performed. But this is not the whole story. The one-form which is also present in the theory comes from the metric of the eleven-dimensional theory. The surprising thing is that one can find certain combination of fields such that the action could be reexpressed in terms of two of the one, two or three forms. This would allow for a family of duality transformations to be performed. We shall show explicitly these transformations and that they are consistent with supersymmetry.

Non-chiral $N = 2$ supergravity in ten dimensions was obtained [7] by trivially reducing the eleven dimensional theory [6]. The action is expressed in terms of the bosonic fields $A_{\mu \nu}$ (or $A_2$), $A_{\mu \nu \rho}$ (or $A_3$), $B_\mu$ (or $B$), $\phi$ and the vielbein $e^a_\mu$. Because of the presence of the one-form, two-form and three-form we will denote this formulation by $(1,2,3)$. The fermionic fields are the gravitino $\psi_\mu$ and the spinor $\lambda$ both of which are Majorana spinors. The action is given by [7]

$$I = \int d^{10}x e^{-\frac{1}{4\kappa^2} R(\omega(e))} - \frac{i}{2} \bar{\psi}_\mu \Gamma^{\mu \nu} D_\nu \psi_\rho - \frac{1}{48} e^{\kappa \phi} F'_{\mu \nu \rho \sigma} F'^{\mu \nu \rho \sigma} + \frac{1}{12} e^{-2\kappa \phi} F_{\mu \nu \rho} F_{\mu \nu \rho} - \frac{1}{4} e^{3\kappa \phi} G_{\mu \nu} G^{\mu \nu} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi$$

$$+ \frac{i}{2} \bar{\lambda} \Gamma^{\mu} D_\mu \lambda - \frac{i\kappa}{\sqrt{2}} \bar{\lambda} \Gamma^{11} \Gamma^{\mu} \Gamma^{\nu} \psi_\nu \partial_\mu \phi$$

$$+ \frac{\kappa}{8(12)^2} e^{-\epsilon^{\mu_1 \cdots \mu_{10}} F_{\mu_1 \cdots \mu_4} F_{\mu_5 \cdots \mu_8} A_{\mu_9 \mu_{10}}}$$

$$+ \frac{\kappa}{96} e^{\frac{\kappa}{3} \phi} (\bar{\psi}_\mu \Gamma^{\mu \nu \alpha \beta \gamma \delta} \psi_\nu + 12 \bar{\psi}^{\alpha} \Gamma^{\beta \gamma} \psi^{\delta} + \frac{1}{\sqrt{2}} \bar{\lambda} \Gamma^{\mu \nu \alpha \beta \gamma \delta} \psi_\mu + \frac{3}{4} \bar{\lambda} \Gamma^{\alpha \beta \gamma \delta} \lambda) F'_{\alpha \beta \gamma \delta}$$

$$- \frac{\kappa}{24} e^{-\frac{\kappa}{3} \phi} (\bar{\psi}_\mu \Gamma^{11} \Gamma^{\mu \nu \alpha \beta} \psi_\nu - 6 \bar{\psi}^{\alpha} \Gamma^{11} \Gamma^{\beta} \psi^{\gamma} - \sqrt{2} \bar{\lambda} \Gamma^{\mu \nu \alpha \beta} \psi_\mu) F_{\alpha \beta \gamma}$$

$$- \frac{i\kappa}{8} e^{\frac{3\kappa}{4} \phi} (\bar{\psi}_\mu \Gamma^{11} \Gamma^{\mu \nu \alpha \beta} \psi_\nu + 2 \bar{\psi}^{\alpha} \Gamma^{11} \Gamma^{\beta} \psi^{\beta} + \frac{3}{\sqrt{2}} \bar{\lambda} \Gamma^{\mu \nu \alpha \beta} \psi_\mu + \frac{5}{4} \bar{\lambda} \Gamma^{11} \Gamma^{\alpha \beta} \lambda) G_{\alpha \beta}$$

$$+ \text{quartic fermionic terms}$$

(1)

where $G_{\mu \nu}$, $F_{\mu \nu \rho}$ and $F'_{\mu \nu \rho \sigma}$ are field strengths of $B_\mu$, $A_{\mu \nu}$ and $A_{\mu \nu \rho}$ respectively. Because of the eleven dimensional origin of this theory one has the modified field strength $F'_{\mu \nu \rho \sigma}$ where

$$G_{\mu \nu} = 2 \partial_{[\mu} B_{\nu]}$$

$$F_{\mu \nu \rho} = 3 \partial_{[\mu} A_{\nu \rho]}$$

(2)

$$F'_{\mu \nu \rho \sigma} = 4 \partial_{[\mu} A_{\nu \rho \sigma]} + 2 B_{[\mu} F_{\nu \rho \sigma]}$$

As can be seen by compactifying the eleven-dimensional theory working in a flat frame [5], we can write the field strength $F'$ in terms of a modified potential $A_3$,
where

\[ A'_{\mu \nu \rho} = A_{\mu \nu \rho} - 6B_{[\mu}A_{\nu \rho]} \]

\[ F'_{\mu \nu \rho \sigma} = 4(\partial_{[\mu}A'_{\nu \rho \sigma]} + 3G_{[\mu \nu}A_{\rho \sigma]}) \]  \hspace{1cm} (3)

These identities will play a vital role in allowing for duality transformations. The supersymmetry transformations are given by

\[ \delta e^a_{\mu} = -i\epsilon \Gamma^a \psi_{\mu} \]

\[ \delta \psi_{\mu} = D_{\mu}(\omega) - \frac{1}{32}e^{3\phi} \left( \Gamma^a_{\mu \nu \rho} - 14\delta^a_{\mu} \Gamma^\rho \right) \Gamma^{11} \epsilon G_{\nu \rho} \]

\[ + \frac{i}{48}e^{-\kappa \phi} \left( \Gamma^a_{\mu \nu \rho \sigma} - 9\delta^a_{\mu} \Gamma^{\rho \sigma} \right) \Gamma^{11} \epsilon F'_{\nu \rho \sigma} + \cdots \]

\[ \delta B_{\mu} = \frac{i}{2}e^{-\frac{3}{2} \kappa \phi} \left( \overline{\psi}_{\mu} \Gamma^{11} \epsilon - \frac{\sqrt{2}}{4} \overline{\lambda} \Gamma_{\mu} \epsilon \right) \]

\[ \delta A_{\mu \nu} = e^{\kappa \phi} \left( \overline{\psi}_{[\mu} \Gamma_{\nu]} \Gamma^{11} \epsilon - \frac{1}{2\sqrt{2}} \overline{\lambda} \Gamma_{\mu \nu} \epsilon \right) \]

\[ \delta A_{\mu \nu \rho} = -\frac{3}{2}e^{-\frac{\kappa \phi}{2}} \left( \overline{\psi}_{[\mu} \Gamma_{\nu \rho]} \epsilon - \frac{1}{6\sqrt{2}} \overline{\lambda} \Gamma^{11} \Gamma_{\mu \nu \rho} \epsilon \right) \]

\[ + 6e^{\kappa \phi} B_{[\mu} \left( \overline{\psi}_{\nu] \Gamma_{\rho]} \Gamma^{11} \epsilon - \frac{1}{2\sqrt{2}} \overline{\lambda} \Gamma_{\nu \rho} \epsilon \right) \]

\[ \delta \lambda = \frac{1}{\sqrt{2}}D_{\mu} \phi(\Gamma^\mu \Gamma^{11} \epsilon) + \frac{3}{8\sqrt{2}}e^{\frac{3}{2} \phi} \Gamma^{\mu \nu} \epsilon G_{\mu \nu} \]

\[ + \frac{i}{12\sqrt{2}}e^{-\kappa \phi} \Gamma^{\mu \nu \rho} \epsilon F'_{\mu \nu \rho} + \cdots \]

\[ \delta \phi = \frac{i}{\sqrt{2}} \overline{\lambda} \Gamma^{11} \epsilon \]  \hspace{1cm} (4)

Another important piece is the Chern-Simons term which can be written in terms of differential forms as \( \int A_2 \wedge dA_3 \wedge dA_3 \) where \( A_2 \) and \( A_3 \) stand for the two- and three-forms: \( A_2 = A_{\mu \nu} dx^\mu \wedge dx^\nu \), and \( A_3 = A_{\mu \nu \rho} dx^\mu \wedge dx^\nu \wedge dx^\rho \). This can be reexpressed in such a way that \( A_{\mu \nu} \) appears only through its field strength \( F_{\mu \nu} \). We derive this by using

\[ A_2 \wedge dA_3 \wedge dA_3 = d(A_2 \wedge A_3 \wedge dA_3) - dA_2 \wedge A_3 \wedge dA_3 \]  \hspace{1cm} (5)

and discarding the surface term after integration. Next, although the field \( B_{\mu} \) does not appear in the Chern-Simons term, it appears explicitly in the field strength \( F'_{\mu \nu \rho \sigma} \) in eq (2). If equation (3) is used instead of (2), then \( B_{\mu} \) appears only through its field strength \( G_{\mu \nu} \) but then the Chern-Simons term must be expressed in terms
of $A'_{\mu\nu\rho}$. It is not difficult to show that

$$A_2 \wedge dA_3 \wedge dA_3 = A_2 \wedge dA'_3 \wedge dA'_3 + 6A_2 \wedge A_2 \wedge dB \wedge dA'_3$$

$$+ 12A_2 \wedge A_2 \wedge A_2 \wedge dB \wedge dB$$

$$+ 6d(A_2 \wedge A_2 \wedge B \wedge (dA'_3 + 4A_2 \wedge dB))$$  \hspace{1cm} (6)

Discarding the surface term, we see that the action (1) is expressible in terms of $A_2$, $dA'_3$ and $dB$. From all of these considerations it is very suggestive that we can apply duality transformations to the following fields $(A_6, A_2)$, or $(A_3, A_2)$ or $(B, A_7)$. We now consider these transformations one at a time.

To obtain the dual theory where the two-form is replaced with a six-form, we add to the action (1) the term

$$\frac{1}{3!6!} \int A_6 \wedge dF_3$$  \hspace{1cm} (7)

where $A_6 = A_{\mu_1 \cdots \mu_6} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_6}$ is a six-form and $F_3$ is a three-form, $F_3 = F_{\mu\nu\rho} dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho}$, which in (1) is not assumed now to be a field strength. The equation of motion of $A_6$ forces $F_3$, locally, to be $dA_2$. Integrating by parts and discarding the surface term, eq (7) can be rewritten in the form

$$\frac{1}{3!6!} \int F_3 \wedge dA_6$$  \hspace{1cm} (8)

Since $F_3$ appears in the action (1) and (8) at most quadratically, we can perform the $F_3$ gaussian integration to obtain the dual version as a function of $A_6$. Therefore, the action in the form (1) plus (7) can give either one of the two dual actions, depending on what is integrated first, $A_6$ or $F_3$. The supersymmetry transformations of the combined action can be found as follows [9]. The supersymmetry transformations of $F_3$ are taken to be identical to those of $d\delta A_2$ as given in eq (2) (without identifying $F_3$ with $dA_2$), then the action (1) will be invariant except for one term proportional to $dF_3$ which does not vanish now because the Bianchi identity is no longer available. The non-invariant term will be cancelled by the transformation of the new term (7) which is also proportional to $\int \delta A_6 \wedge dF_3$. This determines $\delta A_6$ to be given by

$$\delta A_{\mu_1 \cdots \mu_6} = -3ie^{-\kappa \phi} (\bar{\epsilon} \Gamma_{[\mu_1 \cdots \mu_5} \psi_{\mu_6]} - \frac{i}{6\sqrt{2}} \bar{\epsilon} \Gamma_{\mu_1 \cdots \mu_6} \Gamma^{11} \lambda)$$  \hspace{1cm} (9)

and explicitely shows that the action (1) plus (7) admits a duality transformation between the two-form and the six-form. The duality transformation is at the level of the action and not only the equations of motion. As the field $F_{\mu\nu\rho}$ appears at
most quadratically, doing the gaussian integration for $F_{\mu\nu\rho}$, or solving its equation of motion and substituting back into the action, are equivalent. The equation of motion gives

$$M_{\alpha\beta\gamma}^{\mu\nu\rho} F_{\mu\nu\rho} = X_{\alpha\beta\gamma}$$  \hspace{1cm} (10)

where the tensors $M_{\alpha\beta\gamma}^{\mu\nu\rho}$ and $X_{\alpha\beta\gamma}$ are given by

$$M_{\alpha\beta\gamma}^{\mu\nu\rho} = \left( \frac{1}{3!} e^{-2\kappa\phi} \left( 1 - e^{3\kappa\phi} B_\sigma B^\sigma \right) \delta_{\alpha\beta\gamma}^{\mu\nu\rho} + \frac{1}{4} e^{\kappa\phi} B_{[\alpha} \delta_{\beta\gamma]}^{\mu\nu\rho} B^\mu \right)$$  \hspace{1cm} (11)

$$X_{\alpha\beta\gamma} = -\frac{1}{216} \epsilon_{\alpha\beta\gamma}^{\mu_1 \cdots \mu_7} \left( \frac{1}{7!} F_{\mu_1 \cdots \mu_7} + A_{\mu_1 \mu_2 \mu_3} \partial_{\mu_4} A_{\mu_5 \mu_6 \mu_7} \right) + \frac{k}{24} e^{-\frac{3}{2}\kappa\phi} \left( \bar{\psi}_{\mu} \Gamma_{\alpha\beta\gamma}^\mu \psi_{\nu} - 6 \bar{\psi}_{[\alpha} \Gamma_{\beta\gamma]}^\mu \Gamma_{\gamma\nu]} - \sqrt{2} \lambda_{\mu} \Gamma_{\alpha\beta\gamma}^\mu \psi_{\mu} \right) - \frac{k}{12} e^{\frac{3}{2}\kappa\phi} \left( \bar{\psi}_{\mu} \Gamma_{\alpha\beta\gamma}^\mu \psi_{\nu} + 12 \bar{\psi}_{[\alpha} \Gamma_{\beta\gamma]} \psi_{\nu} \right) + \frac{1}{\sqrt{2}} \lambda_{\mu} \Gamma_{\alpha\beta\gamma}^\mu \psi_{\mu} + \frac{3}{4} \lambda_{\rho} \Gamma_{\alpha\beta\gamma}^\rho \right) B_{\rho}$$  \hspace{1cm} (12)

and we have denoted $F_{\mu_1 \cdots \mu_7} = 7 \partial_{[\mu_1} A_{\mu_2 \cdots \mu_7]}$.

Solving equation (10) for $F_{\mu\nu\rho}$ gives

$$F_{\mu\nu\rho} = M^{-1\alpha\beta\gamma}_{\mu\nu\rho} X_{\alpha\beta\gamma}$$  \hspace{1cm} (13)

where the tensor $M^{-1\alpha\beta\gamma}_{\mu\nu\rho}$ is the inverse of $M_{\alpha\beta\gamma}^{\mu\nu\rho}$.

$$M^{-1\alpha\beta\gamma}_{\mu\nu\rho} M_{\alpha\beta\gamma}^{\kappa\lambda\sigma} = \frac{1}{3!} \delta_{\alpha\beta\gamma}^{\mu\nu\rho}$$  \hspace{1cm} (14)

The explicit form of $M^{-1}$ is

$$M^{-1\alpha\beta\gamma}_{\mu\nu\rho} = \frac{6e^{2\kappa\phi}}{1 - e^{3\kappa\phi} B_\sigma B^\sigma} \left( \frac{1}{3!} \delta_{\alpha\beta\gamma}^{\mu\nu\rho} - \frac{3}{2} e^{\kappa\phi} B_{[\alpha} \delta_{\beta\gamma]}^{\mu\nu\rho} B^\mu \right)$$  \hspace{1cm} (15)

Therefore to obtain the dual action from (1) plus (7), we discard all the $F_{\mu\nu\rho}$ contributions and replace them with

$$-\frac{1}{2} X_{\alpha\beta\gamma} M^{-1\alpha\beta\gamma}_{\mu\nu\rho} X_{\mu\nu\rho}$$  \hspace{1cm} (16)

The action in (16) is a non-polynomial function of $B_\mu$. It is an interesting question to find whether some field redefinitions involving the dilaton can change the dependence to a polynomial one.
To find the \( N = 2 \) supergravity action where the three-form is replaced with a five-form we proceed as before. First, we write the action (1) in such a way that the three-form appears only through its field strength. We use eq (3) for \( F'_{\mu\nu\rho\sigma} \), and write it as \( F_{\mu\nu\rho\sigma} + 12G_{[\mu}\chi_{\rho\sigma]} \). Then we assume that \( F_{\mu\nu\rho\sigma} \) is an independent field and not the field strength of \( A'_{\mu\nu\rho} \), and add the following term to the action:

\[
\frac{1}{4!5!} \int A_5 \wedge dF_4 \tag{17}
\]

where \( A_5 = A_{\mu_1 \ldots \mu_5} dx^{\mu_1} \ldots dx^{\mu_5} \). The \( A_5 \) equation implies, locally, that \( F_4 = dA_3' \) and this gives again the action (1). If, however, we integrate eq (17) by parts, and then do the gaussian integration of \( F_{\mu\nu\rho\sigma} \) we will be left with an action in terms of the dual field strength \( F_{\mu_1 \ldots \mu_6} \). To restore the supersymmetry invariance after adding (17) to the action (1) we assume that \( \delta F_{\mu\nu\rho\sigma} = 4 \partial_{[\mu} \delta A_{\nu\rho\sigma]} \), then the extra terms that spoil the invariance of the action (1) are cancelled by those arising from the non-invariance of the term (17). This is achieved by taking

\[
\delta A_{\mu_1 \ldots \mu_5} = \frac{5}{2} i e^{\frac{\kappa \phi}{2}} \epsilon \Gamma_{\mu_1 \ldots \mu_4} \psi_{\mu_5} \tag{18}
\]

The sum of the actions (1) and (17) gives both dual actions depending on the order of integration and is invariant under the new supersymmetry transformations.

The gaussian integration of \( F_{\mu\nu\rho\sigma} \) gives

\[
\frac{1}{2} X_{\mu\nu\rho\sigma} M^{-1} X_{\alpha\beta\gamma\delta} \tag{19}
\]

where \( X_{\mu\nu\rho\sigma} \) is defined by

\[
X_{\mu\nu\rho\sigma} = \frac{\kappa}{4!} \epsilon_{\mu\nu\rho\sigma} \mu_{\nu_{1} \ldots \mu_{6}} \left( \frac{1}{6} F_{\mu_{1} \ldots \mu_{6}} + \frac{1}{16} A_{\mu_{1} \mu_{2}} A_{\mu_{3} \mu_{4}} A_{\mu_{5} \mu_{6}} \right) + \frac{\kappa}{96} e^{\frac{\kappa \phi}{2}} \left( \psi_{\mu} \Gamma^{\alpha}_{\mu \rho \sigma} \psi_{\beta} + 12 \overline{\psi}_{[\mu} \Gamma_{\nu \rho \sigma] \psi_{\beta]} + \frac{1}{\sqrt{2}} \overline{\lambda} \Gamma^{\alpha}_{\mu \rho \sigma} \psi_{\alpha} + \frac{3}{4} \overline{\lambda} \Gamma_{\mu \rho \sigma} \lambda \right)
- \frac{1}{2} e^{\frac{\kappa \phi}{2}} G_{[\mu \nu \rho \sigma]} \tag{20}
\]

and the matrix \( M^{-1} \) is the inverse of

\[
M^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} = \left( \frac{1}{4!} \right)^2 \left( e^{\kappa \phi} \delta_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta} - \kappa \epsilon_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta} \lambda \right) \tag{21}
\]

defined by

\[
M^{-1} X_{\mu\nu\rho\sigma} M^{\kappa\lambda\tau\eta}_{\alpha\beta\gamma\delta} = \frac{1}{4!} \delta_{\mu\nu\rho\sigma}^{\kappa\lambda\tau\eta} \tag{22}
\]
The explicit expression of $M^{-1}$ is too long to give here. The field strength $F_6$ is given by

$$F_{\mu_1 \cdots \mu_6} = 6 \partial_{[\mu_1} A_{\mu_2 \cdots \mu_6]}$$

(23)

Therefore, to obtain the dual action we discard all the terms containing $F_{\mu \nu \rho \sigma}$ and replace them with (19). This completes the derivation of the dual action where the three-form is replaced by the five-form.

By writing the $N = 2$ supergravity action IIA in such a way that the one-form $B$ appears only through its field strength required a redefinition of the three-form. The procedure of obtaining the action where the one-form is replaced with the seven-form is the same as before. We first manipulate the action (1) so that the field $B_\mu$ appears only through its field strength $G_{\mu \nu}$ then we assume that $G_{\mu \nu}$ is an independent field and add a term to the action (1) of the form:

$$\frac{1}{2!7!} \int A_7 \wedge dG$$

(24)

where we have defined the seven-form $A_7 = A_{\mu_1 \cdots \mu_7} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_7}$ Integrating the $A_7$ field out implies the constraint $dG = 0$ whose solution, locally, is $G_{\mu \nu} = 2 \partial_{[\mu} B_{\nu]}$ and this takes us back to the action (1). Integrating the action (24) by parts and discarding the surface term we obtain

$$\frac{1}{2!7!} \int dA_7 \wedge G$$

(25)

The field $F'_{\mu \nu \rho \sigma}$ in the action (1) is taken to be of the form (3) and the Chern-Simons term is rearranged to be given by (5). Then the full action is at most quadratic in $G_{\mu \nu}$ and the gaussian integration can be performed. This will give the dual action expressed in terms of the field strength of $A_7$. The non-invariance of (1) under the supersymmetry transformations due to the removal of the identification $G = dB$ is cancelled by the variation of (24) provided one identifies the variation of $G$ with

$$\delta G_{\mu \nu} = 2 \partial_{[\mu} \delta B_{\nu]}$$

(26)

and the variation of $A_7$ with

$$\delta A_{\mu_1 \cdots \mu_7} = e^{\frac{2}{3} \kappa \phi} \left( -\frac{7}{2} \bar{\epsilon} \Gamma_{[\mu_1 \cdots \mu_6} \psi_{\mu_7]} + \frac{\sqrt{2}}{8} \bar{\epsilon} \Gamma_{\mu_1 \cdots \mu_7} \Gamma^{11} \lambda \right)$$

(27)

The gaussian integration of $G_{\mu \nu}$ gives

$$-\frac{1}{4} X_{\mu \nu} M^{-1}_{\alpha \beta} X^{\alpha \beta}$$

(28)
where $X_{\mu\nu}$ is defined by

$$X_{\mu\nu} = -\frac{1}{8} e^{\kappa \phi} A^{\rho\sigma} (4 \partial_{(\mu} A_{\nu\rho\sigma)})$$

$$+ \frac{3 \kappa}{16} \left( \bar{\psi}_\alpha \Gamma^{\alpha\beta}_{\mu\nu\rho\sigma} \psi_\beta + 12 \bar{\psi}_{[\mu} \Gamma_{\nu]\rho\sigma} \psi_\sigma \right) + \frac{1}{\sqrt{2}} \bar{\lambda} \Gamma^{\alpha}_{\mu\nu\rho\sigma} \psi_\alpha + \frac{3}{4} \bar{\lambda} \Gamma^{\mu\nu\rho\sigma} \Lambda$$

$$- \frac{i \kappa}{8} e^{\frac{3}{2} \kappa \phi} \left( \bar{\psi}_\alpha \Gamma^{11\mu\nu} \psi_\beta + 2 \bar{\psi}_{[\mu} \Gamma^{11}_{\nu]\rho\sigma} \psi_\sigma \right) + \frac{3}{2} \bar{\lambda} \Gamma^{\alpha}_{\mu\nu} \psi_\beta + \frac{5}{4} \bar{\lambda} \Gamma^{11\mu\nu} \Lambda$$

$$+ \epsilon_{\mu_1 \ldots \mu_8} \left( \frac{1}{2!7!} \partial_{\mu_1} A_{\mu_2 \ldots \mu_8} - \frac{\kappa}{192} A_{\mu_1 \mu_2} \ldots A_{\mu_7 \mu_8} \right) \quad (29)$$

and the tensor $M^{-1\alpha\beta}_{\mu\nu}$ is the inverse of

$$M^{\alpha\beta}_{\mu\nu} = \frac{1}{4} e^{3\kappa \phi} (1 + \frac{3}{2} A_{\rho\sigma} A^{\rho\sigma}) \delta^{\alpha\beta}_{\mu\nu} + A_{\mu\nu} A^{\alpha\beta}$$

$$- 4 \delta^{[\alpha}_{(\mu} A_{\nu]\rho} A^{\beta \rho]} - \frac{\kappa}{96} \epsilon^{\alpha\beta}_{\mu_1 \ldots \mu_6} A_{\mu_1 \mu_2} A_{\mu_3 \mu_4} A_{\mu_5 \mu_6} \quad (30)$$

The inverse of $M$ is defined by:

$$M^{-1\alpha\beta}_{\mu\nu} M^{\rho\sigma}_{\alpha\beta} = \frac{1}{2!} \delta^{\rho\sigma}_{\mu\nu} \quad (31)$$

but again the explicit expression is too long to give here. Finally, $G_{\mu\nu}$ is related to its dual by the relation

$$G_{\mu\nu} = M^{-1\alpha\beta}_{\mu\nu} X_{\alpha\beta} \quad (32)$$

The dual action is obtained by discarding all the $G_{\mu\nu}$ contributions in (1) plus (25) and replacing them with (28). This completes the derivation of the dual action where the one-form is replaced with a seven-form.

Therefore, we have shown that the original formulation of $N = 2$ supergravity type IIA given in terms of a one-form, a two-form and a three-form (we denote this by (1,2,3)), admits three other dual formulations. In the first, the two-form is replaced with a six-form giving rise to a formulation in terms of a one-form, a six-form and a three-form (denoted by (1,6,3)). In the second the three-form is replaced with a five-form giving rise to a formulation in terms of (1,2,5) forms. Finally, in the third the one-form is replaced with a seven-form giving rise to the (7,2,3) formulation. It is easy to see that the (1,2,5) formulation depends on the three-form through its field strength suggesting that it is possible to find a duality transformation that takes the one-form to a seven-form. This will give the (7,2,5) formulation. This can also be reached by performing a duality transformation on the three-form in the (7,2,3) formulation as it appears only through its field strength. This also implies that the (7,2,5) formulation can be reached by applying a double duality transformation to the
one-form and three-form simultaneously. If we arrange the (1,2,3), (7,2,3), (7,2,5) and (1,2,5) formulations at the corners of a square in a clockwise fashion, then all adjacent vertices could be transformed to each other by a simple duality transformation, and the opposite edges by a double duality transformation. But it seems that the (1,6,3) formulation can only be connected to the (1,2,3) formulation as it depends on the one-form and three-form explicitly.

The fact that only certain combinations of field configurations are allowed seems to indicate some consistency conditions. It will be useful to deduce such conditions as some projections on physical states of the spectrum of the Dirichlet-branes [3]. It will also be useful to find out whether such conditions give only the theories mentioned here, or whether they allow for other combinations signalling the possibility of new theories in ten-dimensions and may be in eleven.

References


