We study the amplitude and spectral tilt of density perturbations in the simplest hybrid inflation models. We give an exact expression for the amplitude of quantum fluctuations on all scales in the limit where we can neglect the backreaction on the metric. This is a very good approximation for values of the inflaton field well below the Planck scale and our results remain valid far from the usual massless limit. We confirm that the primordial density spectrum in this model has a constant spectral index $n > 1$ over all observable scales. For the small values of the tilt ($n < 1.4$) required by observations, the results remain close to those obtained using the quasi-massless approximation.

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Inflation provides a compelling explanation for many aspects of the observed universe, but it is the spectrum of primordial density perturbations that provides the best hope of distinguishing between different possible models of inflation. Vacuum fluctuations of the inflaton field during inflation can be swept up to astrophysical scales by the rapid expansion. The spectrum of perturbations on super-horizon scales are usually computed using the Bunch-Davies vacuum for a massless field [1] for scales within the horizon and matching to an approximate homogeneous solution outside that horizon. Exact solutions to the equations of motion for linear perturbations about the homogeneous field are only known in a handful of special cases [2–4].

One particular model of inflation that has received special attention recently is hybrid inflation [5–7]. It is possible to find models of this kind in supersymmetric [7–9] and some supergravity [7,10] particle physics models. The possibility that particle physics may give a workable model of inflation is very attractive. In particular, the mass scales present in the model appear naturally in hidden-sector supersymmetry breaking [11]. These models have a very rich low energy phenomenology, as well as important cosmological and astrophysical implications, some of which were explored in Ref. [12]. For certain values of the parameters, the model may have a second stage of inflation, which may lead to the production of cosmologically interesting black holes and topological defects. As part of our calculation of the density perturbations present in these two-stage models of hybrid inflation, we found solutions to the equations of motion for linear perturbations of both scalar fields when their backreaction on the metric can be neglected [12].

In this paper we show that our result for a single field evolving during hybrid inflation also applies when there is only one stage of inflation, and is an (almost) exact solution for a wide range of parameters in the hybrid inflation scenario, well beyond the usual slow-roll approximation. In fact, corrections to the amplitude and tilt of curvature perturbations produced during inflation can become large away from the quasi-massless limit. In some models of hybrid inflation it is possible to have a significant positive tilt; however, the maximum value of the tilt allowed by observations is $n \lesssim 1.4$, where deviations from the quasi-massless results remain small.

The simplest realization of chaotic hybrid inflation is provided by the potential [5]

$$V(\phi, \psi) = \left( M^2 - \frac{\sqrt{\lambda}}{2} \phi^2 \right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} \gamma \phi^2 \psi^2 .$$

(1)

The equations of motion for the homogeneous fields are then

$$\ddot{\phi} + 3H \dot{\phi} = -(m^2 + \gamma \psi^2) \phi ,$$

(2)

$$\ddot{\psi} + 3H \dot{\psi} = (2 \sqrt{\lambda} M^2 - \gamma \phi^2 - \lambda \psi^2) \psi ,$$

(3)

subject to the Friedmann constraint

$$H^2 = \frac{8\pi}{3 M_P^2} \left[ V(\phi, \psi) + \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \dot{\psi}^2 \right].$$

(4)

The potential has a local minimum with respect to the field $\psi$ at $\psi = 0$ for $\phi^2 > \phi_c^2$, where

$$\phi_c^2 = \frac{2 \sqrt{\lambda}}{\gamma} M^2 .$$

(5)

Thus inflation can occur while $\psi = 0$ and the potential simply reduces to

$$V(\phi) = M^4 + \frac{1}{2} m^2 \phi^2 .$$

(6)

while $\phi$ rolls down the potential towards $\phi_c$. For a wide range of parameters the constant term always dominates on scales of interest and the Hubble expansion can be taken to be de Sitter expansion with $H = H_0 \equiv \sqrt{8\pi/3} M^2 / M_P$. It is then useful to write the bare masses of the two fields $\phi$ and $\psi$ relative to the Hubble scale as
where a prime denotes a derivative with respect to $N$ and $\alpha > 9/4$ the solution describes damped oscillations about $\phi = 0$, while for $\alpha < 9/4$ the asymptotic solution is $\phi = \phi_c \exp(rN)$ where $r \equiv r_+ > 0$, which approaches the slow-roll solution $\phi = \phi_c \exp(\alpha N/3)$ for $\alpha \ll 1$.

Note that the condition for neglecting the field's backreaction, Eq. (9), then becomes

$$4\pi r \phi^2 \ll M_P^2.$$  

Thus our analysis remains valid even for $\alpha > 1$ as long as we consider values of the field $\phi$ much less than the Planck scale.

In the quasi-massless approximation, the amplitude of quantum fluctuations of the field at horizon crossing ($k = aH$) is taken to be $H/2\pi$. However, if the mass of the $\phi$ field is not necessarily much smaller than the Planck scale, corrections to this massless field result could be large.

On the other hand, for small $\phi \ll M_P$ we can neglect the gravitational backreaction of the field, see Eq. (12). The equation of motion for linear perturbations in $\phi$ can then be written as

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} + \left(\frac{k^2}{a^2} + \alpha H^2\right)\delta\phi = 0.  \tag{13}$$

Note that when $\psi = 0$, the evolution of $\delta\psi$ and $\delta\phi$ decouple. We can write this equation, in terms of the canonically quantized field $[13]$ $u \equiv a\delta\phi$, as

$$u'' + \left(\frac{k^2 - 2 - \alpha}{\eta^2}\right)u_k = 0,  \tag{14}$$

where primes denote derivatives with respect to conformal time, $\eta = -1/aH$, and we have chosen $\eta = -1$ when $\phi = \phi_c$.

Since the mass of the $\phi$ field is constant, we can write an exact expression for the quantum fluctuations, $[1,3]$

$$u_k(\eta) = \frac{\sqrt{\pi}}{2\sqrt{k}} e^{i(r-1)\pi/2} (-k\eta)^{1/2} H^{(1)}_{3/2-r}(-k\eta),  \tag{15}$$

where $r = r_+$ is defined in Eq. (11). This has been normalized to have the correct flat-space behavior

$$u_k \to \frac{e^{ik\eta}}{\sqrt{2k}},  \tag{16}$$

as $-k\eta \to \infty$, while as $\phi \to 0$, and $-k\eta \to 0$, we find

$$u_k(\eta) = \frac{C(r)}{\sqrt{2k}} e^{i(r-1)\pi/2} (-k\eta)^{-1},  \tag{17}$$

where

$$C(r) = 2^{-r} \frac{\Gamma(3/2 - r)}{\Gamma(3/2)},  \tag{18}$$

as shown in Fig. 1.

*Note that $N$ is defined here as a positive quantity which decreases to zero at the end of inflation.
We will write the power spectrum of any quantity $A$ as $\mathcal{P}_A \equiv (k^3/2\pi^2)\langle |A|^2 \rangle$. Equation (17) thus gives a scale-invariant spectrum of the growing-mode perturbations at horizon crossing, $k\eta_* = -1$,
\[ \mathcal{P}_{\delta \phi}^{1/2} = C(r) \frac{H}{2\pi} . \] (19)
Note that the coefficient $C(r)$ gives a constant correction (independent of scale) to the usual amplitude of perturbations (obtained in the slow-roll limit where $C(0) = 1$). Near $\alpha = 9/4$, or $r = 3/2$, the coefficient $C(r)$ becomes large, see Fig. 1, giving a significant amplification of the perturbations compared to the usual slow-roll approximation.

A similar expression was obtained in Ref. [3] for natural inflation in the small angle approximation, where the potential is approximated given by Eq. (6) with $m^2 \to -m^2$ and thus $\alpha \to -\alpha$ and $r \to -3/2 + \sqrt{9/4 + |\alpha|} > 0$. In natural inflation the validity of both the small angle approximation and neglecting the backreaction, Eq. (9), become worse and must eventually fail as the field approaches the end of inflation, although this will only affect small scales. In our case the validity of the approximation improves as the field evolves towards the end of inflation.

Quantum fluctuations of the scalar field are responsible for curvature perturbations on comoving hypersurfaces, which can be evaluated as the change in the time (or number of $e$-folds) it takes to end inflation. In the case of adiabatic perturbations, e.g. single-field inflation driven by the field $\phi$, the amplitude of the curvature perturbation on comoving hypersurfaces remains fixed on super-horizon scales, so it can be calculated as $\delta N = [H\delta \phi]/\dot{\phi}$, at horizon crossing, where $\delta \phi_*$ is given by Eq. (19).

At large values of $\phi \gg \phi_*$ the $\psi$ field has a large positive mass-squared and remains fixed at $\psi = 0$. The amplitude of $\psi$ fluctuations crossing outside the horizon are negligible. Thus we need only consider adiabatic fluctuations $\delta \phi_*$ along the trajectory, giving an amplitude of perturbations
\[ \mathcal{P}_{\delta N}^{1/2} = \frac{C(r) H}{2\pi r \dot{\phi}_*} . \] (20)

The power spectrum of curvature perturbations on comoving hypersurfaces, $\mathcal{R} = \delta N$, is then given by
\[ \mathcal{P}_R(N) = \frac{C(r)^2 \gamma}{4\pi^2 r^2 \beta} e^{-2\tau N} , \] (21)
where $N$ is the number of $e$-folds to the end of inflation, which occurs at $\phi = \phi_*$. If these curvature perturbations are responsible for the observed temperature anisotropies in the microwave background, Eq. (21) gives a constraint on the parameters of the model. The low multipoles of the angular power spectrum measured by COBE [14] gives a value $\mathcal{P}_R(N_{\text{CMB}}) \simeq 3 \times 10^{-9}$, on the scale of our current horizon.

Because the comoving scale at horizon crossing is just proportional to the scale factor, $k \propto e^{-N}$, the scale dependence of the power spectrum in Eq. (21) readily gives the tilt of the spectrum as
\[ n - 1 = \frac{d \ln \mathcal{P}_R}{d \ln k} = 2r , \] (22)
which becomes $2\alpha/3$ in the slow-roll limit. Note that the tilt is always positive, giving a “blue” perturbation spectrum [5–7,15,16].

This is one of the few cases in inflationary cosmology where we have an (almost) exact expression for the amplitude of curvature perturbations. The only approximation we have made is to assume that the energy density remains constant, see Eq. (9), so that we can neglect the backreaction on the metric. Using Eqs. (12) and (20), we see that this will be true as long as
\[ \frac{8C(r)^2}{3r} \frac{M^4}{M_p^4} \ll \mathcal{P}_R . \] (23)

This constraint is shown in Fig. 2. Expanding $C(r)$ given in Eq. (18), it follows that the allowed range of $r$, for $M^4/M_p^4 \ll \mathcal{P}_R$, is
\[ r \gg \frac{8}{3\mathcal{P}_R} \frac{M^4}{M_p^4} . \] (24)

For $\mathcal{P}_R \simeq 3 \times 10^{-9}$ and $M \ll 10^{16}$ GeV the right-hand-sides of these constraints are so small that $r$ is effectively left as a free parameter between zero and $3/2$, corresponding to $\alpha$ in the range $0 < \alpha < 9/4$. Thus in principle for this model one could have any value of the tilt in the range $1 < n < 4$.

Observations of the microwave background impose a direct constraint on the spectral index of the curvature perturbations. Present limits from the lowest multipoles of the microwave sky give $n = 1.2 \pm 0.3$ [14] at the 1σ level, corresponding to $\alpha \leq 0.75$. This implies that the size of the correction coefficient in Eq. (21) lies in the range $0.86 \leq C \leq 1$. A precise measurement of $n$, which would be possible with the next generation of satellite experiments [17], would give a much tighter constraint on $\alpha$.

There is another important constraint on the allowed values of the tilt of the spectrum from small scales. The microwave background observations constrain the amplitude of perturbations to be small on large scales, but a positive tilt leads to an increasing amplitude of curvature perturbations on smaller scales. The maximum amplitude depends on both the tilt and the number of $e$-folds to the end of inflation. Large amplitude perturbations on small scales may lead to the production of primordial black holes [15]. The most conservative constraint is to require $\mathcal{P}_R(N) < 1$ on all scales. Combined with the
Note that the energy scale during inflation, \( M \), can take any value below \( 10^{16} \) GeV. For \( M = 10^{-8} M_P \simeq 10^{11} \) GeV, we have \( m \simeq 2 \) TeV and \( m_\phi \equiv \sqrt{2} \lambda^{1/4} M \simeq 30 \) TeV in Eq. (1). These are very natural parameters from the point of view of supersymmetry. The scalar fields could correspond to flat directions, e.g. moduli fields, which get mass corrections of order the gravitino mass \( m_{3/2} \simeq 1 \) TeV when supersymmetry is broken in the observable sector, see Ref. [11,12]. For these parameters, the tilt of the primordial spectrum of density perturbations becomes \( n = 1.4 \). Note also that the phase transition that triggers the end of inflation occurs rapidly for these parameters, and we are able to avoid a second stage of inflation and the danger of producing too many large black holes, as discussed in Ref. [12].

Note that in Ref. [7] an upper limit on \( n < 1.14 \) was obtained for values of the dimensionless couplings \( \gamma \) and \( \lambda \) not far from unity. On the other hand, we are able to obtain a larger tilt by considering arbitrary values of the couplings. For instance the value \( \beta = 100 \) given above corresponds to a very small coupling, \( \lambda \simeq 10^{-27} \).

However such flat potentials are quite natural in supersymmetric models, see Ref. [11,12]. The absolute upper limit \( n < 1.3 \) given in [7] comes from taking \( N_{\text{CMB}} = 60 \) in Eq. (25), which corresponds to \( M \sim 10^{10} \) GeV and \( T_{\text{rh}} \sim 10^{13} \) GeV in Eq. (26), much greater than the values considered above.

For completeness we note that the amplitude of the gravitational wave spectrum can also be computed,

\[
P_G = \frac{16}{\pi} \frac{H_0^2}{M_P^3} = \frac{128}{3} \frac{M^4}{M_P^4}.
\]

A small Hubble constant during inflation implies that the contribution of gravitational waves to the microwave background anisotropies will be small [22,6,7]. Indeed the validity of our curvature perturbation calculation, see Eq. (23), requires their contribution to be negligible.

In summary, we have shown that hybrid inflation may be added to a select group of inflationary models [2–4] for which the curvature perturbation spectrum may be calculated (almost) exactly. The spectral index of the power spectrum is \( n = 1 + 2r \) where \( r = 3/2 - \sqrt{9/4 - m^2/H^2} \), which approaches the quasi-massless approximation \( m_{\sigma} \simeq 1 + 2m^2/3H^2 \) for \( m < H \), see Fig. 3. The correction coefficient for the amplitude of perturbations for a given value of the field \( \phi \) at horizon crossing, relative to the quasi-massless result, is given by \( C(r) \) in Eq. (18), which is never far from unity for \( m < H \), see Fig. 1. Nonetheless, our calculation remains valid for a wide range of parameters, not necessarily close to the usual quasi-massless limit. In principle one can obtain large values for the tilt of the spectrum, but in practice the maximum value is constrained by observations.

It is worth emphasizing that these hybrid inflation models contain mass scales that are consistent with natural values present in particle physics models [5,7,11,12]. Perhaps it will be possible in the not-so-far future to test this models both by cosmological observations of the microwave background and in high energy particle physics experiments.

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Figure 1: The coefficient $C(r)$ in Eq. (18), giving the amplitude of perturbations relative to the massless field limit as a function of the parameter $r$.

Figure 2: The maximum value of $\log_{10}(M/M_P)$ given in Eq. (23) for $P_R = 3 \times 10^{-9}$ as a function of the parameter $r$. The allowed region is below the curve, which effectively means that, for $M \ll 10^{-3}M_P$, any value of $r$ in the range $(0, 3/2)$ is in principle possible.

Figure 3: The spectral index, $n$, of the curvature perturbations as a function of $m/H$. The dashed line shows the tilt obtained using the quasi-massless approximation, $n \simeq 1 + 2m^2/3H^2$. 

Figure captions

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