Stress-Energy Must be Singular on the Misner Space Horizon even for Automorphic Fields

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Abstract. We use the image sum method to reproduce Sushkov’s result that for a massless automorphic field on the initial globally hyperbolic region $IGH$ of Misner space, one can always find a special value of the automorphic parameter $\alpha$ such that the renormalized expectation value $\langle \alpha|T_{ab}|\alpha \rangle$ in the Sushkov state “$\langle \alpha|\cdot|\alpha \rangle$” (i.e. the automorphic generalization of the Hiscock-Konkowski state) vanishes. However, we shall prove by elementary methods that the conclusions of a recent general theorem of Kay-Radzikowski-Wald apply in this case. I.e. for any value of $\alpha$ and any neighbourhood $N$ of any point $b$ on the chronology horizon there exists at least one pair of non-null related points $(x, x') \in (N \cap IGH) \times (N \cap IGH)$ such that the renormalized two-point function of an automorphic field $G^\alpha_{\text{ren}}(x, x')$ in the Sushkov state is singular. In consequence $\langle \alpha|T_{ab}|\alpha \rangle$ (as well as other renormalized expectation values such as $\langle \alpha|\phi^2|\alpha \rangle$) is necessarily singular on the chronology horizon. We point out that a similar situation (i.e. singularity on the chronology horizon) holds for states on Gott space and Grant space.

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There has recently been interest in quantum field theory on spacetimes with a chronology horizon. (See [1, 2, 3] and references therein.) The reason for this interest is that these spacetimes (or rather those for which the chronology horizon is compactly generated [2]) are models for spacetimes in which time machines get manufactured. Hawking [2] has proposed the Chronology Protection Conjecture according to which the laws of physics (and in particular quantum effects) prevent such spacetimes from being physically realizable. In fact, there were arguments due to

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Hawking and others (see e.g. references in [3, 2]) that, at least in the case of simple model quantum field theories on such spacetimes [2] and for a wide range of states on the initial globally hyperbolic region, the (expectation value of the renormalized) stress-energy tensor would diverge as one approached the chronology horizon. We remark on the one hand that this evidence was of a heuristic nature and it is now known not to be true in general that the stress-energy tensor really does diverge as one approaches the chronology horizon for “all” initial states. In fact, recently Krasnikov [4], in the context of a scalar field on two-dimensional Misner space, and Sushkov in the context of automorphic fields on four-dimensional Misner space have given examples of (Hadamard) states for which the stress-energy vanishes on the initial globally hyperbolic region. (Additionally, Boulware [6], in the case of Gott space, and Tanaka and Hiscock [7], in the case of Grant space, have shown that for a sufficiently massive field, the stress-energy is bounded on the initial globally hyperbolic region. On the other hand, a slightly different statement (Theorem 2 of [3]) has recently been rigorously proven by Kay, Radzikowski and Wald: namely, that, for the model consisting of the real covariant Klein-Gordon equation on a spacetime with compactly generated chronology horizon, for any Hadamard state on the initial globally hyperbolic region, the expectation value of the renormalized stress-energy tensor (and also of other similarly defined quantities such as the renormalized expectation value of $\phi^2$) is necessarily singular on the chronology horizon. (See before the statement of our theorem below for a more precise statement/discussion.)

We shall focus on Sushkov’s model which concerns a generalization [5], which we shall call here the Sushkov state, of the Hiscock-Konkowski state [9] for a massless automorphic field on the initially globally hyperbolic region of (four-dimensional) Misner space. The Sushkov state $\langle \alpha | \cdot | \alpha \rangle$ is labelled by a phase, $\alpha$, called the automorphic parameter and Sushkov pointed out that there must exist a value of this automorphic parameter for which the expectation value $\langle \alpha | T_{ab} | \alpha \rangle$ in the Sushkov state of the renormalized stress-energy tensor vanishes (and another value for which the renormalized expectation value of $\phi^2$ vanishes).

The Kay-Radzikowski-Wald theorem was not explicitly proved for the case of automorphic fields. However, the extension to this case appears to be straightforward. In fact, one expects there to be no difficulty [10] in extending the theorems of [3] to

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3The theorem also does not strictly apply because the chronology horizon of four-dimensional Misner space is not compactly generated. However, one easily sees that the theorems of [3] still hold in this case since the conclusions of Proposition 2 of [3] still hold in the case of any spacetime which arises as the product with a Riemannian manifold of dimension $4 - d$ of a spacetime with compactly generated chronology horizon with dimension $d < 4$. 

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the case of a complex scalar field in an external electromagnetic field, and the case of an automorphic field may of course be regarded as a special case of this. Applying this result to the situation discussed by Sushkov, we thus face the rather surprising occurrence of an energy-momentum tensor which, while bounded (in fact zero!) up to a (four-dimensional) chronology horizon is nevertheless singular on the chronology horizon with the conclusion that even in this case, the Chronology Protection Conjecture does not, after all, appear to be contradicted. (We shall also explain at the end of this letter that the states proposed by Boulware and Tanaka-Hiscock for Gott space and Grant space [8] will also have a stress-energy tensor which is singular on the chronology horizon in a slightly different sense, and for slightly different reasons.)

Rather than elaborating on the details of the general extension of the general theorem of Kay-Radzikowski-Wald, in the present letter we shall provide an elementary proof that the conclusions of the Kay-Radzikowski-Wald theorem apply to the specific situation discussed by Sushkov, i.e. we shall show directly that for any value of the automorphic parameter $\alpha$, $\langle \alpha | T_{ab} | \alpha \rangle$ and $\langle \alpha | \phi^2 | \alpha \rangle$ must necessarily be singular on the chronology horizon of Misner space. To accomplish this we express the Sushkov two-point function for an automorphic field by an image-sum formula which generalizes that of Hiscock and Konkowski [9].

We remark that the proof of the theorem in [3] appeals to powerful general theorems of Duistermaat and Hörmander on the “Propagation of Singularities” [11] whereas, in the specific situation discussed here, one may bypass any appeal to these general theorems since one sees explicitly, by direct inspection of the image sum formula, how this propagation of singularities occurs. In this way, we hope the discussion below provides a useful example of, and could serve as an entrée to, the general theorems of [3].

A convenient mathematical description of (four dimensional) Misner space may be obtained by taking the region $\mathcal{R}$

$$U \in (-\infty, 0)$$

(1)

$$V \in (-\infty, \infty)$$

(2)

of Minkowski space with the metric

$$ds^2 = -dUdV + dY^2 + dZ^2$$

(3)

(where $U$ and $V$ are the double null coordinates $U = T - X$, $V = T + X$) and identifying points related by a fixed Lorentz boost $L$ (with rapidity say $\alpha$) in the


In terms of the representation of Misner space as the region $\mathcal{R}$ of Minkowski space with the identifications (4) we may think of a classical automorphic field as a complex solution $\phi$ of the Klein-Gordon equation on $\mathcal{R}$ satisfying the condition

$$\phi(Lx) = e^{2\pi i \alpha} \phi(x)$$

(5)

Fig1.
for all $x \in \mathcal{R}$ where $\alpha$ is a parameter lying in the interval $[0, 1)$. (The case $\alpha = 0$ corresponds to an ordinary complex field, the case $\alpha = \frac{1}{2}$ to a complex twisted field.)

For each $\alpha$, the Sushkov state $\langle \alpha \cdot | \alpha \cdot \rangle$ for a quantum complex automorphic field on the initial globally hyperbolic region $\text{IGH}$ is defined in [5] by a mode-sum method. It is possible to show [10] that the two-point function $G^\alpha$ in this state arises as the image sum

$$G^\alpha(x, x') := \langle \alpha | \phi(x) \phi^\dagger(x') | \alpha \rangle = \sum_{n=-\infty}^{\infty} e^{2\pi i n \alpha} G_M(x, L^n x')$$

where $G_M(x, x') = \langle M | \phi(x) \phi^\dagger(x') | M \rangle$ is the two point function in the Minkowski vacuum which, for non-null separated pairs of points is equal to $(2\pi)^{-2}/\sigma(x, x')$ where $\sigma$ is the square of the geodesic distance between $x$ and $x'$. In particular, in the case $\alpha = 0$, the Sushkov two-point function coincides with the Hiscock-Konkowski two-point function. (We remark in passing that this identity of the Hiscock-Konkowski image sum with Sushkov’s mode sum settles in the affirmative the – as far as we are aware, hitherto open – question as to whether the Hiscock-Konkowski two-point function satisfies the positivity conditions required for it to arise from a quantum state.)

Generalizing the point-splitting procedure of [9], the renormalized expectation value of the (conformally improved) stress-energy tensor in the Sushkov state is given by

$$\langle T_{ab}(y) \rangle = \lim_{(x,x') \to (y,y)} \left( \frac{2}{3} \partial_a \partial_b y - \frac{1}{6} \eta_{ab} \eta^{cd} \partial_c \partial_d y - \frac{1}{3} \partial_a \partial_b \right) G^\alpha_{\text{ren}}(x, x').$$

where

$$G^\alpha_{\text{ren}}(x, x') = \frac{1}{(2\pi)^2} \sum_{n=-\infty}^{\infty} \frac{e^{2\pi i n \alpha}}{\sigma(x, L^n x')}$$

is the renormalized two-point function.

Using eq. (7) and (8) we find that the renormalized stress-energy is given by

$$\langle T_{b}^a \rangle = \frac{K}{12\pi^2} t^{-4} \text{diag}(1, -3, 1, 1)$$

where

$$K = \sum_{n=1}^{\infty} \cos(2\pi n a) \frac{2 + \cosh(n a)}{(\cosh(n a) - 1)^2}.$$
We emphasize that in performing this calculation it is necessary to justify inter-
changing the derivative operator in (7) with the sum in (8). This may be justified
by observing that any point \( y \) in \( IGH \) has a neighbourhood such that, for all pairs
\((x, x')\) of points in this neighbourhood, each term in the sum in (7) is differentiable,
and moreover, the sums which result when one acts on each term in (7) with one or
two unprimed or primed derivatives are all uniformly convergent.

Sushkov [5] pointed out that, since (see [9]) the expectation value of the stress-
energy tensor for a twisted field is equal to that for an untwisted field multiplied
by a negative constant (and since a [complex] twisted field may be regarded as an
automorphic field for parameter value \( \alpha = 1/2 \)) one expects there to be some value
of \( \alpha \) for which the expectation value of the stress-energy tensor vanishes. (One
can similarly argue that there is another value of \( \alpha \) for which the renormalized
expectation value of \( \phi^2 \) vanishes.)

With the present approach, this argument can be made precise as follows: If
we let the automorphic parameter \( \alpha \) equal 1/2 then \( K \) in (10) is negative since the
first term in the sum (10) is negative and the terms alternate in sign and decrease
monotonically in absolute value. If we instead let \( \alpha \) equal zero then trivially
\( K \) is positive. Next, using the property that \( K \) is a continuous function of \( \alpha \), since every
term in the sum is continuous as a function of \( \alpha \) and since the sum which defines
\( K \) is uniformly convergent in \( \alpha \), it immediately follows from the intermediate value
theorem that there has to exist a value of the automorphic parameter such that
\( K \) and hence the expectation value of the stress-energy vanishes.

One way of understanding why the formula (7) leads to a finite result for the
expectation value of \( \langle T_{ab} \rangle \) in the initial globally hyperbolic region is to observe
that \( G_{\alpha \text{ ren}} \) arises as the difference of two two-point functions, \( G^\alpha \) and \( G_M \) each of
which have the same local singularity structure (“Hadamard form”) so that \( G_{\alpha \text{ ren}} \)
itslf is smooth, and, in consequence, the limit in (7) is well-defined (and finite).
(\( G_M \) plays here the role, which in the general procedure (see e.g. [12]) for defining
\( \langle T_{ab} \rangle \) is played by the subtraction of a “locally Hadamard parametrix”.) In the
setting of a real linear scalar field on a general spacetime with compactly generated
Cauchy horizon, Kay Radzikowski and Wald [3] showed however (see especially
“Theorem 2 (alternative statement)” in Section 5 of [3]) that there are necessarily
certain points (“base points”) on the Cauchy horizon such that in the intersection
of any neighbourhood, \( N \), of any base point with the initial globally hyperbolic
region, the difference between any bisolution which takes the Hadamard form on
the initial globally hyperbolic region and any “Hadamard parametrix” defined on \( N \)
is necessarily singular in the sense that it not only fails to be smooth, but fails even
to be bounded. In consequence, $\langle T_{ab} \rangle$ will necessarily be singular (or ill-defined) at these base points as we mentioned in the opening paragraph. (For exactly similar reasons, other renormalized quantities such as $\langle \phi^2 \rangle$ will also be singular.)

We now present our direct argument that in the case of complex automorphic fields on Misner space (where every point on the chronology horizon is a base point in the sense of [3]) and the Sushkov state, the statement of the theorem continues to hold. In other words:

**Theorem.** Let $b$ be any point on the chronology horizon of Misner space, and let $N$ be any neighbourhood of $b$. Then, for any value of the automorphic parameter $\alpha$, there exists at least one pair of non-null related points $(x, x') \in (N \cap IGH) \times (N \cap IGH)$, where $IGH$ is the initial globally hyperbolic region, such that the renormalized two-point function $G^\alpha_{\text{ren}}(x, x')$ is singular (in the sense that $G^\alpha_{\text{ren}}(x, y)$ diverges as $y$ approaches $x'$).

**Proof.** Regarding Misner space as the region $R$ of Minkowski space with the identifications (4), the renormalized two-point function $G^\alpha_{\text{ren}}$ is given by the formula (8) where the square of the geodesic distance is given, in double null coordinates by

$$\sigma(x, L^n x') = -UV - U'V' + e^{-ma}UV + e^{ma}V'U + (Y - Y')^2 + (Z - Z')^2.$$ 

Clearly $N \cap IGH$ will contain a neighbourhood which arises, when we coordinatize it in this way, as the product of a rectangle $R = \{(U, V) : U_1 < U < U_2, V_0 < V < 0\}$ with a rectangle $S = \{(Y, Z) : Y_1 < Y < Y_2, Z_1 < Z < Z_2\}$. So it will suffice to exhibit a pair of points contained in such a product having the properties stated in the theorem. In fact, we shall exhibit such a pair for which the $Y$ and $Z$ coordinates are the same – say $(Y_0, Z_0) \in S$. Choose a point $x = (U, V, Y_0, Z_0), (U, V) \in R$. Then it is easy to see that there exists a point $\hat{x} = (\hat{U}, \hat{V}, Y_0, Z_0), (\hat{U}, \hat{V}) \in R$, spacelike separated from $x$, such that every point $x_\delta = (\hat{U}, \delta, Y_0, Z_0), V < \delta \leq 0$ along the line connecting $\hat{x}$ to the point $(U, 0, Y_0, Z_0)$ on the chronology horizon (see figure 2) is also spacelike separated from $x$. (Alternatively, one can choose $\hat{x}$ to be timelike separated from $x$ and such that all the points $x_\delta$ are also timelike separated etc.) We now show that one can choose $\delta$ so that $L^m x_\delta$ is null related to $x$ for some integer $m$ so that $\sigma(x, L^m x_\delta) = 0$. To see this, we note that for any $\delta$ which takes the form

$$\delta = \frac{e^{-ma}UV - UV}{\hat{U} - e^{ma}U},$$

$L^m x_\delta$ for $x_\delta = (\hat{U}, \delta, Y_0, Z_0)$ will be null separated from $x$, and, by taking $m$ to be sufficiently large and positive, we can arrange for $\delta$ to lie in the interval $(\hat{V}, 0)$. Taking $x' = x_\delta$, we thus have exhibited a pair $(x, x')$ of points in $(N \cap IGH) \times (N \cap IGH)$
such that the \( m \)th term in the sum (8) is singular. On the other hand, it is easy to see that the remaining terms in this sum are uniformly convergent in some neighbourhood of \((x, x')\). We conclude that \( G^{\alpha}_{\text{ren}} \) is singular at \((x, x')\). ☐

We note that the above theorem can be given a geometrical interpretation: On the region \( \mathcal{R} \), \( x_\delta \) has the property that, in the neighbourhood \( N \cap IGH \) it is spacelike separated from \( x \), but one of its images \( L^m x_\delta \) is null separated from \( x \). Reinterpreting \( \mathcal{R} \) as Misner space by making the identifications (4), we would rather say that \( x \) and \( x_\delta \) are spacelike separated in the neighbourhood \( N \cap IGH \) but globally null separated. Thus the phenomenon of “propagation of singularities” (see the introductory discussion) is, in the case of the bisolution \( G^\alpha \), inherent in the image sum formula.

Next, we remark that one might have thought that one could argue that, since there is a value of \( \alpha \) for which \( \langle T_{ab} \rangle \) vanishes in the initial globally hyperbolic region, then, by continuity, for this value of \( \alpha \), \( \langle T_{ab} \rangle \) would also vanish on the chronology horizon. However, the uniform convergence properties mentioned earlier do not hold for neighbourhoods of points on the chronology horizon and thus the required continuity property may (and, in view of the above theorem does!) fail. What we have exhibited in fact is a situation where \( \langle T_{ab} \rangle \) has a finite limit as one approaches the chronology horizon, but nevertheless is singular on the chronology horizon. Further-
more, and in consequence of this singularity, we would not regard it as legitimate to view Misner space together with an automorphic field in the Sushkov state for this value of $\alpha$ as a solution to semiclassical gravity. Thus we would not regard this model as evidence against the Chronology Protection Conjecture.

Finally, we note that one can prove a similar theorem for the analogues of the Sushkov state on the spacetime (with compactly generated Cauchy horizon) obtained by taking the product of two-dimensional Misner space with a two torus. (In this case one simply adds further terms to the image sum (8), and these terms are non-singular and have a convergent sum when evaluated at the pair $(x, x')$). The cases of Gott space and Grant space are only slightly different. These have chronology horizons which are not compactly generated and which contain no base points ([3]). Nevertheless the statement of “Theorem 2” (but not now of “Theorem 2 (alternative statement)” of [3]) will still hold (see Section 6 of [3]) since there are null geodesics (on the so-called polarized hypersurfaces – see [8, 1]) which self-intersect arbitrarily close to the chronology horizon. This result can also be reproduced by elementary methods similar to those of this letter [13] in the case of the states discussed by Boulware [6] (on Gott space) and Tanaka-Hiscock [7] (on Grant space) (and also, e.g. the analogue of the Sushkov state for a massless automorphic field on Gott space or Grant space). In fact, all these states are singular in the neighbourhood of any point on the chronology horizon in the sense that a statement similar to the above theorem still holds, where, however, the pair $(x, x')$ must now be chosen so that at least one of $x, x'$ lies in the “CTC” side of the chronology horizon (where one extends the two-point functions of the states to this side of the horizon in an obvious way).

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References
