Killing spinors, the adS black hole and \( I(ISO(2,1)) \) gravity

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ABSTRACT

We construct a supersymmetric extension of the \( I(ISO(2,1)) \) Chern-Simons gravity and show that certain particle-like solutions and the adS black-hole solution of this theory are supersymmetric.
1. Introduction

Pure gravity in 2+1 dimensions has no propagating degrees of freedom and can be formulated as a Chern-Simons (CS) gauge theory with gauge group \(ISO(2,1)\) [1]. As such, it is integrable and can be consistently quantised [2]. For a generic coupling of 2+1 gravity to matter, though, the above novel integrability properties of the theory are not retained. However there are couplings of 2+1 gravity to matter such that the combined gravity-plus-matter theory again admits a Chern-Simons formulation. Such theories are also integrable and can be consistently quantised. There are several 2+1 gravity-matter systems that admit a CS formulation: for example all CS theories with gauge group \(G\) that have \(ISO(2,1)\) as subgroup. For the purposes of this paper we will restrict ourselves to the model suggested in [3]; this theory is a Chern-Simons theory with gauge group \(I(ISO(2,1))\). The novel property of this theory is [3] that one of its classical solutions is the adS black hole solution of [4]; this is the case because it can be arranged such that topological matter can play the role of the cosmological constant of the adS gravity.

The non-vanishing commutators of the Lie algebra of \(I(ISO(2,1))\) are

\[
[M_a, M_b] = -\epsilon^c_{ab} M_c , \quad [M_a, P_b] = -\epsilon^c_{ab} P_c \\
[M_a, J_b] = -\epsilon^c_{ab} J_c , \quad [M_a, K_b] = -\epsilon^c_{ab} K_c \\
[J_a, K_b] = -\epsilon^c_{ab} P_c ,
\]

where \(\{M_a; a=1,2,3\}\) are the Lorentz generators and \(\{P_a; a=1,2,3\}\) are the momentum generators associated with the gravitational sector, and \(\{J_a; a=1,2,3\}\) and \(\{K_a; a=1,2,3\}\) are two sets of additional generators that are associated with the matter sector. An invariant non-degenerate inner product is

\[
\langle M_a, P_b \rangle = \mu \eta_{ab} , \quad \langle J_a, K_b \rangle = \mu \eta_{ab} ,
\]

where \(\mu\) is a non-zero real number. Given the algebra (1.1) and the invariant non-degenerate inner product (1.2), it is straightforward to construct the associated
Chern-Simons theory. The fields of the theory are the connections $A$ with gauge group $I(ISO(2, 1))$ and the action is the Chern-Simons functional of these connections. This action describes the coupling of Poincaré gravity to topological matter, is invariant under the gauge transformations with gauge group $I(ISO(2, 1))$ up to surface terms, and the field equations are the zero curvature conditions $\mathcal{F}(A) = 0$ of the connection $A$.

Recently there has been some progress in finding the necessary and sufficient conditions for the existence of Killing spinors in 2+1-dimensional spacetimes for both Poincaré and adS gravities [5,6, 7]. In the Poincaré case, the existence of Killing spinors necessitates the generalisation of the standard Poincaré supergravities [6]. The new Poincaré supergravities are Chern-Simons theories whose Lie algebras have generators that are either (i) those of the super-Poincaré algebra with central charges or (ii) some of the outer automorphisms of the super-Poincaré algebra with central charges that rotate its supersymmetry and central charges. The novelty here is that the gravitino supersymmetry transformation of these new Poincaré supergravities is modified by terms that include the gauge field associated with the automorphism generators. Such modification of the gravitino supersymmetry transformations is necessary for the existence of the Killing spinors in these theories [6].

The main purpose of this paper is to construct a supersymmetric extension of 2+1 gravity-matter system that is described by the $I(ISO(2, 1))$ Chern-Simons theory above and to investigate the existence of Killing spinors for some classical solutions of this theory. For this a supersymmetric extension of the algebra (1.1) will be constructed. The generators of this supersymmetry algebra will be the bosonic generators $\{P_a, M_a, J_a, K_a\}$ of (1.1), two supersymmetry charges $\{Q^i; i = 1, 2\}$ and two additional bosonic charges $\{Z, T\}$ where $Z$ is a central charge and $T$ is the generator of the outer automorphisms. The gravitino supersymmetry transformation will determine the Killing spinor equation and then the methods of [6] will be applied to solve it for various solutions of the supergravity theory. We will find that the adS black hole solution of this theory admits a Killing spinor.
This paper is organised as follows: In section two we will construct a supersymmetric extension of the $I(ISO(2, 1))$ Chern-Simons theory and give the gravitino supersymmetry transformations. In section three we present several classical solutions of this theory and then we will investigate the conditions under which such spacetimes admit Killing spinors. Finally, in appendix one, we use the covariant canonical approach to define the non-abelian charges of a CS theory and compute their Poisson bracket algebra; this Poisson bracket algebra is a Kac-Moody algebra with a central extension.

2. Supergravity

To construct a supersymmetric extension of the the bosonic gravity-matter system described by the $I(ISO(2, 1))$ Chern Simons theory [3], we must find a supersymmetric extension of the Lie algebra of $I(ISO(2, 1))$ together with an invariant non-degenerate inner product. The non-vanishing commutators of such a superalgebra are as follows:

\[
\begin{align*}
[M_a, M_b] &= -\epsilon_{ab}^c M_c, \quad [M_a, P_b] = -\epsilon_{ab}^c P_c \\
[M_a, J_b] &= -\epsilon_{ab}^c J_c, \quad [M_a, K_b] = -\epsilon_{ab}^c K_c \\
[J_a, K_b] &= -\epsilon_{ab}^c P_c, \quad a, b, c = 1, 2, 3, \\
\{Q^i_\alpha, Q^j_\beta\} &= -\frac{1}{2}\delta^{ij}(\gamma^a)_{\alpha\beta}P_a + i\epsilon_{\alpha\beta}^\epsilon\epsilon^{ij}Z \\
[M_a, Q^i_\alpha] &= i\frac{1}{2}(Q^i\gamma_a)_\alpha, \quad [T, Q^i_\alpha] = -\epsilon^{ij}Q^j_\alpha,
\end{align*}
\]

where $\{Q^i; i = 1, 2\}$ are the supersymmetry charges, $Z$ is a central charge, $T$ is an automorphism generator that rotates the two supersymmetry charges, and $\alpha, \beta = 1, 2$ are spinor indices. The rest of the generators are as those of the $I(ISO(2, 1))$ algebra of (1.1). We use the ‘mostly-minus’ metric convention and hence gamma matrices that are pure imaginary. Note that

\[
\begin{align*}
\gamma^a \gamma^b &= \eta^{ab} + i\epsilon^{abc}\gamma_c \\
(\bar{\psi})_\alpha &= \psi^\beta \epsilon_{\beta\alpha}.
\end{align*}
\]

We also introduce a formal conjugation with respect to which all the even genera-
tors of the superalgebra are antihermitian whereas the odd ones are hermitian, and we adopt the standard convention that complex conjugation of a fermion bi-linear introduces an additional minus sign.

Next we introduce the following inner product

\[ \langle M_a, P_b \rangle = \mu \eta_{ab} \quad \langle Q^i_a, Q^j_b \rangle = i \mu \epsilon_{\alpha \beta} \delta^{ij}, \tag{2.3} \]

\[ \langle T, Z \rangle = -\mu, \quad < J_a, K_b > = \mu \eta_{ab}, \]

where \( \mu \) is a real non-zero constant with dimensions of mass. This inner product is invariant, non-degenerate and hermitian with respect to the formal conjugation introduced above. The (anti-hermitian) gauge field \( \mathcal{A} \) is

\[ \mathcal{A} = e^a P_a + \omega^a M_a + C Z + A T + B^a J_a + H^a K_a + \psi^i Q_i, \tag{2.4} \]

where \( e^a \) is interpreted as the frame, \( \omega^a \) is the spin-connection of the gravitational sector, \( \{ \psi^i; i = 1, 2 \} \) are the gravitinos, and the rest \( \{ C, A, B^a, H^a \} \) are matter gauge fields associated with the remaining generators of the superalgebra (2.1).

We compute the curvature two-form \( \mathcal{F} = d\mathcal{A} + \mathcal{A}^2 \) to be

\[ \mathcal{F} = T^a P_a + F^a(\omega) M_a + F(C) Z + F(A) T + F^a(B) J_a + F^a(H) K_a + D\psi^i Q_i, \tag{2.5} \]

where

\[ T^a = de^a - \epsilon^a_{\ b c} \omega^b e^c - \frac{1}{4} \bar{\psi}^i \gamma^a \psi^i - \epsilon^a_{\ b c} B^b H^c \]

\[ F^a(\omega) = d\omega^a - \frac{1}{2} \epsilon^a_{\ b c} \omega^b \omega^c \]

\[ F(C) = dC \]

\[ F(A) = dA \]

\[ F^a(B) = dB^a - \epsilon^a_{\ b c} \omega^b B^c \]

\[ F^a(H) = dH^a - \epsilon^a_{\ b c} \omega^b H^c \]

\[ D\psi^i = d\psi^i + \frac{i}{2} \omega^c \gamma_c \psi^i + A \epsilon^{ij} \psi^j. \tag{2.6} \]

Note that \( \omega^a \) has torsion proportional not only to gravitinos, as is expected in a

* Note that in our conventions the field \( \psi \) anticommutes with \( Q \).
supergravity theory, but also proportional to other matter fields like \(B\) and \(H\).

The action of the theory is

\[
S = \mu \int d^3x \left[ eR(\omega) - i\varepsilon^{mnp}\bar{\psi}_m^i \partial_n \psi^i_p + 2\varepsilon^{mnp}C_m \partial_n A_p \right. \\
+ \left. 2\varepsilon^{mnp}H_m^a \partial_n B_p^a - 2\epsilon_{abc}\varepsilon^{mnp}\omega_m^a B_n^b H_p^c \right],
\]

where the ‘Ricci’ scalar \(R(\omega)\) is the trace of the frame \(e\) with Hodge dual tensor of the curvature \(F(\omega)\) and the spin-connection \(\omega_m^a\) is an independent field, i.e. this is the first-order form of the supergravity action. The action (2.7) is invariant (up to a surface term) under gauge transformations of the connection \(A\). The supersymmetry transformations are the gauge transformations along the fermionic generators \(Q^i\) of the supersymmetry algebra. The non-zero supersymmetry transformations of the fields are

\[
\begin{align*}
\delta e^a &= \frac{1}{2} \bar{\zeta}^i \gamma^a \psi^i \\
\delta \psi^i &= \mathcal{D} \zeta^i \\
\delta C &= i\epsilon_{ij} \bar{\zeta}^i \psi^j 
\end{align*}
\]

where \(\zeta^i\) are anticommuting spinor parameters and

\[
\mathcal{D}\zeta^i = d\zeta^i + \frac{i}{2} \omega^c \gamma_c \zeta^i + A \epsilon^{ij} \zeta^j .
\]

Note that the non-vanishing supersymmetry transformations of the fields are precisely those of the Poincaré supergravity theories of [6].
3. Killing Spinors

3.1 Particle-like solutions

A class of stationary classical solutions of the 2+1 gravity-matter system of the previous section is given by generalising the multi-point particle solutions of the 2+1 Poincaré gravity of [8, 9]. For this, we use the ansatz

$$ds^2 = \left( dt + \sum_{L=1}^{N} J_L(r) \frac{\vec{r} - \vec{r}_L}{|\vec{r} - \vec{r}_L|^2} \cdot d\vec{r} \right)^2 - h^2(r) d\vec{r} \cdot d\vec{r}, \quad \omega^i = 0, \quad H^0 = 0,$$

$$B^i = \epsilon^i_j H^j, \quad H^i = \phi dx^i,$$  \hspace{1cm} (3.1)

where $\phi, h$ are functions of the Euclidean 2-space, $\mathbb{E}^2$, whose coordinates are $\vec{r} = (x, y)$, the underline denotes frame indices, and $\epsilon$ is the $\epsilon$-tensor on Euclidean 2-space $\mathbb{E}^2$, i.e. $\epsilon^{ij}_k = \delta^{ik}\epsilon_{kj}$.

Choosing the triad so that

$$e^0 = dt + \sum_{L=1}^{N} J_L(r) \frac{\vec{r} - \vec{r}_L}{|\vec{r} - \vec{r}_L|^2} \times d\vec{r},$$

$$e^i = h \ dx^i,$$  \hspace{1cm} (3.2)

it is straightforward to show that $de^0 = 0$ provided the functions $J_L$ are all constants. Incorporating this into the above ansatz, the field equations $F(B) = 0$ and $F(H) = 0$ yield

$$dB^0 = 0, \quad \omega^0_i = \epsilon^i_j \partial_j \log \phi,$$  \hspace{1cm} (3.3)

so

$$B^0 = df.$$  \hspace{1cm} (3.4)

From the field equations $F(\omega) = 0$, we find that $\log \phi$ is a harmonic function on
\[ \phi^2 = \prod_{L=1}^N |\vec{r} - \vec{r}_L|^{-\frac{\rho_L}{\pi}}, \quad (3.5) \]

where \( \{\rho_L; L = 1, \ldots, N\} \) are real constants and \( \{\vec{r}_L; L = 1, \ldots, n\} \) are the centres of the harmonic functions. Next, the field equation \( T = 0 \) implies that

\[ f = \frac{h}{\phi}, \quad (3.6) \]

and \( \log f \) is a harmonic function on \( \mathbb{E}^2 \). Finally the field equations for \( A \) and \( C \) decouple and any flat connection solves their field equations.

It is clear from the above that since \( \log \phi \) and \( \log h \) are harmonic functions on \( \mathbb{E}^2 \), \( \log h \) is a harmonic function on \( \mathbb{E}^2 \) as well, so we may choose

\[ h^2 = \prod_{L=1}^N |\vec{r} - \vec{r}_L|^{-\frac{m_L}{\pi}}, \quad (3.7) \]

in which case the metric (3.1) is that of a conical \( N \)-particle spacetime located at the positions \( \{\vec{r}_L; L = 1, \ldots, N\} \) where \( \{m_L; L = 1, \ldots, N\} \) are the masses of the particles, and \( \{J_L; L = 1, \ldots, N\} \) are their spins. To explain this identification of the mass and spin of the \( L \)th-particle with the parameters \( m_L \) and \( J_L \) of the metric (3.1), respectively, we remark that the mass and the spin of the \( L \)th-particle are determined in terms of the geometry of the configuration as follows:

\[ m_L \equiv \oint_{\Gamma_L} \Omega^0, \quad J_L \equiv \frac{1}{2\pi} \oint_{\Gamma_L} e^0, \quad (3.8) \]

where \( \Omega \) is the Levi-Civita connection of the metric (3.1) and \( \Gamma_L \) is a non-trivial fundamental homology cycle of the space \(^*\). The total mass of the configuration is

\(^*\) The fundamental homology cycles are the closed paths in space that their interior contain the position of only one particle.
the deficit angle of the conical spacetime and therefore it should be less than $2\pi$.

Finally, the function $f$ is easily computed using (3.5), (3.6) and (3.7) as follows:

$$f^2 = \prod_{L=1}^{N} |\vec{r} - \vec{r}_L|^2 \left(\frac{m_L - \rho_L}{\pi}\right). \quad (3.9)$$

The gravitino supersymmetry transformation is expressed in terms of the connections $\omega$ and $A$. For the solutions above both $\omega$ and $A$ connections are flat, so a Killing spinor exists (after an appropriate projection) for the subclass of the above solutions that the holonomies of the connections $\omega$ and $A$ obey the holonomy matching condition of [6]. This condition can be expressed in terms of the charges (holonomies) of the connections $\omega$ and $A$ evaluated at the non-trivial fundamental homology cycles $\{\Gamma_L; L = 1, \ldots, N\}$ of spacetime. The holonomy of the connection $\omega$ evaluated at the fundamental cycles is given by the set of numbers, $\{\rho_L; L = 1, \ldots, N\}$, i.e.

$$\rho_L = \oint_{\Gamma_L} \omega^0. \quad (3.10)$$

Using large gauge transformations, we can restrict these number as follows: $0 \leq \rho_L < 2\pi$. Similarly the holonomy of the connection $A$ is

$$Q_L = \oint_{\Gamma_L} A. \quad (3.11)$$

Then the holonomy matching conditions are given as follows:

$$\rho_L = 2|Q_L| \quad (3.12)$$

for periodic boundary conditions for Killing spinors on the holonomy fundamental cycles (even spin structure), and

$$\rho_L = 2|Q_L \pm \pi| \quad (3.13)$$

for antiperiodic boundary conditions for Killing spinors on the holonomy fundamental cycles (odd spin structure).
3.2 The adS black hole solution

Another solution of interest of this supergravity plus matter theory is the adS black hole [4]. This can be easily seen by taking the gravitino equal to zero and then observing that the field equations for gauge fields $A$ and $C$ decouple from the field equations of the remainning fields. This black hole solution is given as follows

\[ ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (W dt + d\phi)^2 \]
\[ B^0 = r_+ d\phi + \frac{r_-}{\ell} dt, \quad B^1 = -\ell d(\nu + \sqrt{\nu^2 - 1}) , \quad B^2 = \frac{r_+}{\ell} dt - r_- d\phi \]  
\[ H^0 = -\frac{1}{\ell} B^0, \quad H^1 = d(\nu - \sqrt{\nu^2 - 1}) , \quad H^2 = \frac{1}{\ell} B^2 , \]

where

\[ N^2(r) = -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2} , \quad W = -\frac{J}{2r^2} \]

\[ \nu^2 = \frac{r_+^2 - r_-^2}{r_+^2 - r_-^2} , \]

\(-\infty < t < \infty, 0 \phi < 2\pi \) and \( 0 < r < +\infty \).

The parameters $\ell$, $M$ and $J$ are constants of integration in (3.14). For the usual adS black hole [4], $\ell$ is a coupling constant inversely proportional to the square root of the cosmological constant, and $M$ and $J$ are the quasilocal mass and angular momentum of the black hole [10]. However for the gravity-plus-matter solution (3.14), (3.15) given above, it has been shown that $M\ell$ is the angular momentum parameter and $J/\ell$ is the mass parameter [3].

To determine whether or not this solution admits Killing spinors observe that the connection $\omega$ of the theory is identically zero. So it is flat and has zero holonomy for every closed path in space-time. So the black hole space-time admits Killing spinors provided that we take $A$ to be gauge equivalent to a trivial $U(1)$ connection.
4. Conclusion

We have investigated the existence of killing spinors in the CS theory with gauge group $I(ISO(2,1))$ suggested in [3]. This theory describes Poincaré gravity coupled to topological matter and one of its classical solutions is the adS black hole. For this we have constructed a supersymmetric extension of this theory using a super-extension of $I(ISO(2,1))$ and CS methods. We have then shown that some of the classical solutions of this theory are supersymmetric, i.e. they admit Killing spinors. Amongst the supersymmetric solutions are spinning particle-like configurations and the adS black hole solution of this theory.

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**APPENDIX**

**Non-abelian charges**

In this appendix, we will summarize the properties of the currents of a CS theory using the covariant canonical approach. This will complement the relevant part of section three. Let $\mathcal{L}(\phi)$ be a Lagrangian at most quadratic in the time derivatives of the fields. Moreover let us suppose that $\mathcal{L}(\phi)$ is invariant under the transformations

$$\delta_u \phi = f(u, \phi), \quad [\delta_u, \delta_v] \phi = \delta_{[u,v]} \phi \equiv f([u,v], \phi) \quad (A.1)$$

with parameters $u, v$ up to a surface term, i.e.

$$\delta_u \mathcal{L}(\phi) = \partial_\mu X^\mu (u, \phi), \quad (A.2)$$

where $\mu$ is a spacetime index. The field equations of the theory are

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \pi^{\mu} = 0 \quad (A.3)$$
where $\pi^\mu = \frac{\partial L}{\partial \partial_\mu \phi}$. One can define a symplectic current in the theory

$$S^\mu = d\phi \wedge d\pi^\mu$$  \hspace{1cm} (A.4)$$

which is conserved subject to field equations and closed as a form on the space of fields of the theory. Assuming that the spacetime is globally hyperbolic, one can define a closed two-form

$$\Omega = \int_\Sigma S^\mu d\Sigma_\mu$$  \hspace{1cm} (A.5)$$

where $\Sigma$ is a Cauchy surface. Due to the conservation of the symplectic current $\Omega$ is independent from the choice of $\Sigma$. If the Lagrangian is not invariant under local (gauge) transformations then the form $\Omega$ is non-degenerate and therefore symplectic (at least weakly).

The conserved currents, subject to field equations, of the theory associated with the symmetries (A.1) are

$$J^\mu(u, \phi) = f(u, \phi) \frac{\partial L}{\partial \partial_\mu \phi} - X^\mu(u, \phi) .$$  \hspace{1cm} (A.6)$$

The conserved charges are

$$Q[u, \phi] = \int_\Sigma J^\mu(u, \phi) d\Sigma_\mu .$$  \hspace{1cm} (A.7)$$

The Poisson bracket of two of the above charges in the context of the covariant canonical approach is defined as follows:

$$\{Q[u, \phi], Q[v, \phi]\} \equiv \delta_u Q[v, \phi] - \delta_v Q[u, \phi]$$  \hspace{1cm} (A.8)$$

Due to the surface terms in (A.6) that are necessary for the definition of conserved charges, the Poisson algebra of the charges need not be isomorphic to the Lie algebra of the transformations on the fields.
Next we apply this general formalism to the case of Chern-Simons theory in 2+1 dimensions; see also [11]. To define a Chern-Simon theory in 2+1 dimensions, we introduce a Lie algebra \( \mathcal{L}(H) \) of a group \( H \), an invariant non-degenerate inner product \( \langle \cdot, \cdot \rangle \) on \( \mathcal{L}(H) \) and gauge fields \( A \) with gauge group \( H \). The action is

\[
S = \mu \int_\mathcal{M} \left( \langle A, dA \rangle + \frac{1}{3} \langle A, [A, A] \rangle \right), \tag{A.9}
\]

where \( \mu \) is a constant. The symplectic current is

\[
S = \mu \ast \langle dA, dA \rangle \tag{A.10}
\]

and the closed two-form \( \Omega \) is

\[
\Omega = \mu \int_\Sigma \left( \ast \langle dA, dA \rangle \right) \mu d\Sigma^\mu, \tag{A.11}
\]

where star is the Hodge star. The form \( \Omega \) vanishes, subject to field equations, along the directions tangent to the gauge orbits and therefore is degenerate*. The theory is invariant under, up to a surface term, under the gauge transformations

\[
\delta A = Du \equiv du + [A, u] \tag{A.12}
\]

of the connection \( A \) with parameter \( u \). The conserved currents are

\[
J = \mu \ast d \langle u, A \rangle. \tag{A.13}
\]

* An alternative way to use (A.11) is to view it as a symplectic form on the space of connections of a two-dimensional Riemann surface. Evaluating (A.11) along the vectors tangent to orbits of the gauge group, one gets as a momentum map the curvature of the connection \( A \). Using Hamiltonian reduction, one can define a symplectic structure on the space of gauge equivalence classes of flat connections.
The associated charges are

\[ Q[u, A] = \int_{\Sigma} J^\mu(a, \phi) d\Sigma_\mu = \mu \int_{\partial \Sigma} <a, A>, \quad (A.14) \]

where \( \partial \Sigma \) is the boundary of \( \Sigma \). The associated Poisson brackets are

\[ \{Q[u, A], Q[v, A]\} = -2Q[[u, v], A] - 2\mu \int_{\partial \Sigma} <u, dv>. \quad (A.15) \]

In the case that \( \Sigma \) is a disc, then \( \partial \Sigma = S^1 \) and the Poisson bracket algebra of charges in this case is isomorphic to a Kac-Moody algebra. In addition, using the field equations \( F(A) = 0 \), one can write \( A = h^{-1} dh \) in which case (A.15) becomes the Poisson bracket algebra of the left invariant currents of a Wess-Zumino-Witten model whose target space is the group \( H \) in agreement with the results of [12, 13].

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