Gribov Copies in the Maximally Abelian Gauge and Confinement.

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Abstract

We fix SU(2) lattice gauge fields to the Maximally Abelian gauge in both three and four dimensions. We extract the corresponding U(1) fields and monopole current densities and calculate separately the confining string tensions arising from these U(1) fields and monopole ‘condensates’. We generate multiple Gribov copies and study how the U(1) fields and monopole distributions vary between these different copies. As expected, we find substantial variations in the number of monopoles, their locations and in the values of the U(1) field strengths. The string tensions extracted from ‘extreme’ Gribov copies also differ but this difference appears to be no more than about 20%. We also directly compare the fields of different Gribov copies. We find that on the distance scales relevant to confinement the U(1) and monopole fluxes that disorder Wilson loops are highly correlated between these different Gribov copies. All this suggests that while there is indeed a Gribov copy problem the resulting ambiguity is, in this gauge and for the study of confinement, of limited importance.

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1 Introduction.

The idea of 't Hooft [1] that confinement in non-Abelian gauge theories might be associated with monopoles in suitable Abelian projections of the fields, has been the subject of extensive numerical investigation in recent years. This interest was provoked by the observation [2] that in a particular gauge, the Maximally Abelian (MA) gauge [3], the string tension one obtains from the corresponding Abelian Wilson loops appears to equal the full non-Abelian string tension [4]. Subsequently it was shown that the same is true for the string tension that one obtains from the monopole currents (e.g. [5]).

The MA gauge is, however, plagued by Gribov copies; each non-Abelian field configuration has multiple gauge copies along its gauge orbit which satisfy the MA gauge condition. The $U(1)$ fields and monopole currents that one extracts from these different non-Abelian gauge copies are gauge-variant and differ amongst the various Gribov copies [6, 7]. Thus one expects that the $U(1)$ and monopole string tensions will differ according to which Gribov copies one chooses to use in the calculation. This has cast doubt on the significance of the comparison with the non-Abelian string tension, given that those calculations relied on choosing the Gribov copies at random.

What is the proper way to treat these Gribov copies is not known. We focus in this paper on a useful preliminary question: what is the magnitude of the variations in those properties of the $U(1)$ fields and monopole currents which determine the string tension? If the variation is large then no further progress is possible without addressing the Gribov copy problem. If, on the other hand, the variation is small then calculations which ignore the Gribov copy problem should be reasonably reliable. We find that the latter appears to be the case in both 3 and 4 dimensions.

We carry out calculations simultaneously in 3 and 4 space-time dimensions. $SU(N)$ gauge theories appear to possess linear confinement in both cases and in both cases this is a non-perturbative phenomenon. That confinement should be driven by monopoles is an argument that can be made equally well in either case.

The contents of this paper are as follows. The procedure for fixing to the Maximally Abelian gauge is reviewed in section 2 and we indicate there why Gribov copies will naturally arise. In section 3 we show that Gribov copies are a non-perturbative phenomenon: if the volume is so small that the gauge field coupling is $\leq O(1)$ on all length scales then, effectively, no extra Gribov copies are generated. On larger volumes such copies appear and we show how they differ in relevant quantities such as the monopole density. In section 4 we outline a speculative picture of monopole confinement, in order to provide a specific framework within which to discuss the possible differences between Gribov copies. In section 5 we present our calculations of the string tensions for different subsets of Gribov copies. Although our calculations are not unambiguous, they indicate a small but systematic variation of the string tension with the choice of Gribov copy ensemble. Then in section 6 we compare directly the $U(1)$ fields and monopole distributions of different Gribov copies, with a focus on the long range structures that lead to confinement. We find that any such differences are remarkably small. We finish with some conclusions in section 7.

The work in this paper is an outgrowth of a wider-ranging study of confinement in various Abelian
gauges. We refer the reader to that work [8] for various technical details, and for some of the calculations that are alluded to below.

2 Fixing to the Maximally Abelian Gauge.

An \( SU(2) \) lattice field configuration consists of a set of \( SU(2) \) matrices \( \{U_\mu(n)\} \) assigned to the links of a lattice of \( V \) sites. To put this into the MA gauge we find a local gauge transformation \( \{g(n)\} \) which when applied to this field configuration

\[
U_\mu(n) \rightarrow U_\mu^g(n) = g(n)U_\mu(n)g^\dagger(n + \vec{\mu}).
\]

maximises the gauge–dependent functional, \( R \), of the links:

\[
R = \frac{1}{V} \sum_n \text{Tr} (X(n)\sigma_3)
\]

where

\[
X(n) = \sum_{\mu > 0} U_\mu(n)\sigma_3 U^\dagger_\mu(n)
\]

Maximising \( R \) is equivalent to making \( X(n) \) diagonal (i.e. proportional to \( \sigma_3 \)) for all \( n \). It is also equivalent to maximising the sum over all links, \( l \), of the difference between the 11 and 12 components of the link matrices, i.e. \( \sum_l |u_{11}(l)|^2 - |u_{12}(l)|^2 \). That is to say, it is the gauge in which the link matrices are made to align, as closely as possible, along the \( \sigma_3 \) direction. Since matrices proportional to a given generator commute, this is called the Maximally Abelian gauge.

Once the field has been placed into this gauge, we write the link matrices as the product of matter fields \( c \) and Abelian fields, represented as link angles, \( \{\theta_\mu(n)\} \), via

\[
U_\mu^g(n) = \begin{pmatrix}
  c_{11}(n,\mu) & c_{12}(n,\mu) \\
  -c^*_{12}(n,\mu) & c_{11}(n,\mu)
\end{pmatrix} \times \begin{pmatrix}
  \exp i\theta_\mu(n) & 0 \\
  0 & \exp -i\theta_\mu(n)
\end{pmatrix}
\]

where \( c_{11} \) is real. The MA gauge fixing is incomplete and it is easy to see that the remaining gauge transformations correspond to local Abelian gauge transformations on the above Abelian fields. Abelian fields generically contain topological singularities which correspond to magnetic monopoles, and these can be located using the usual method of DeGrand and Toussaint [9].

We can measure a \( U(1) \) string tension by calculating large Wilson loops with the above \( U(1) \) fields. We can also calculate the string tension produced by the monopole currents, by iteratively solving [8] a set of dual Maxwell equations [10] to obtain a scalar potential in \( D = 3 \), or a vector potential in \( D = 4 \).

Each \( X(n) \) depends on the gauge transformations not only at the site \( n \) but also at neighbouring sites, so we cannot obtain an immediate solution as we would if we had chosen to diagonalise an operator such as the plaquette operator, \( U_{\mu\nu}(n) \) (for some values of \( \mu, \nu \)). Instead we proceed iteratively using standard techniques [3]. If \( x_{12}(n) \) is the 12 component of the matrix \( X(n) \) then ideally we should iterate until \( x_{12}(n) = 0 \) for all \( n \). In practice we iterate until, typically, \( |x_{12}(n)| \leq 10^{-7}, \forall n \). We have checked that with this criterion any gauge fixing systematic errors are far below our statistical errors.
The functional $R$ defined over the gauge orbit corresponding to some particular field configuration has, in general, many maxima. This is illustrated schematically in Figure 1. The gauge copy at each maximum satisfies the MA gauge condition, and is a Gribov copy. (In this paper we shall refer to all these fields as Gribov copies, rather than singling out one as the ‘original’ and the remainder as ‘copies’. In addition we disregard the obvious degeneracies that arise through the remnant $U(1)$ gauge invariance etc.) Although gauge invariant quantities are the same for all these copies, the $U(1)$ fields defined above, together with their associated magnetic current distributions, are only gauge invariant under the remnant $U(1)$ sub-group, and so will in general differ. So, for example, if we were to calculate the $U(1)$ string tension using always the field configuration corresponding to the global maximum of $R$, then we might expect to get a different value than the one we would get if we chose a Gribov copy at random [7]. Since the correct selection is not known, there is an obvious problem of interpretation.

In a gauge where the operator $X(n)$ transforms purely in the adjoint representation, such as the plaquette gauge mentioned above or the Polyakov loop gauge [11], the gauge fixing transformation can be calculated exactly without iteration and there is just a single Gribov copy.

### 3 Gribov Copies - a First Comparison.

The numerical procedure for fixing to the MA gauge smoothly deforms a field along its gauge orbit to a field at which $R$ is a local maximum. Since the local deformations increase $R$, this procedure provides a natural partition of the gauge orbit into subsets of gauge copies, each of which subsets is associated with a specific Gribov copy. In this picture, illustrated schematically in Figure 1, the portion of the gauge orbit between neighbouring minima of $R$ forms a ‘basin of attraction’ for the intervening maximum. (Of course the exact details of the gauge fixing algorithm must have some effect on the boundaries of these subsets. We shall return to this question at the end of this Section.)

For a given $SU(2)$ lattice field configuration, this partition can be mapped out by applying an ensemble of random gauge transformations to it. Upon fixing to the MA gauge, each of these transformed fields will be deformed into one of the Gribov copies. The fraction of these transformed fields that is associated with a particular Gribov copy provides a natural measure of the fractional volume of the gauge orbit associated with that Gribov copy.

If the field configuration is a typical field corresponding to the coupling being weak at all length scales (as one would obtain in a sufficiently small space-time volume) then one would expect to find only one Gribov copy, just as in perturbation theory. That is to say, as the coupling vanishes, the fraction of the gauge orbit volume corresponding to one particular copy will go to unity. (As remarked previously we ignore the trivial degeneracies that arise because of the remnant $U(1)$ symmetry.) One would expect that this copy should be the one corresponding to the absolute maximum of $R$.

We test this numerically in $SU(2)$ (using the standard Wilson plaquette action) in both $D = 3$ and
$D = 4$. We shall denote the couplings in the two theories by $\beta_3 \equiv \frac{4}{ag^2}$ and $\beta_4 \equiv \frac{4}{g^2}$ respectively. All calculated quantities will be in lattice units. We begin with the $D = 3$ case. We have taken an $8^3$ lattice and have generated 20 independent $SU(2)$ field configurations at values of $\beta_3 \equiv 4/ag^2$ ranging from 4 to 12. Over this range of $\beta_3$ the string tension, $\sigma$, varies from $\sqrt{\sigma} = 0.41$ to 0.12 in lattice units [17]. In units of the physical length scale, $\xi \equiv 1/\sqrt{\sigma}$, the lattice volume is small at $\beta_3 = 12$ and reasonably large at $\beta_3 = 4$. That is to say, at $\beta_3 = 4$ the typical field configuration contains long-distance non-perturbative physics, while at $\beta_3 = 12$ the typical field configuration will correspond to relatively weak coupling at all accessible length scales. From each of the $SU(2)$ field configurations, we generate $N_{GT} = 50$ random gauge copies. This is intended to provide an approximation to the full gauge orbit. Each of these gauge copies is then fixed to the MA gauge.

We focus on three quantities, listed in Table 1. The first is the fraction, $f$, of the gauge orbit that belongs to the Gribov copy corresponding to the largest (observed) value of $R$. The second is the probability, $p$, that the Gribov copy with the largest associated fraction of the gauge orbit is in fact the one for which the value of $R$ is a maximum. This is intended as an estimate of the probability that the Gribov copy corresponding to the absolute maximum of $R$ has the largest volume of the gauge orbit. The third is the average number of different Gribov copies, $n_G$, obtained when fixing the $N_{GT} = 50$ gauge copies to the MA gauge. The quoted errors should obviously be taken as being no more than indicative.

We see that at small $\beta_3$, where the lattice volume is large enough to accommodate non-perturbative physics, we have many Gribov copies and the Gribov copy with the largest value of $R$ plays a much diminished rôle. (Although it is interesting to note that this rôle is still much greater than that of other individual Gribov copies.) As $\beta_3$ grows the number of Gribov copies rapidly decreases and the Gribov copy with the largest value of $R$ becomes the only important one. The transition between these two regimes occurs for $\beta_3 \sim 7$ where the lattice size in physical units is $8/\xi \sim 1.7$. All this is consistent with the general expectations we outlined earlier in this section. (We cannot, of course, prove that the Gribov copy we find with the largest value of $R$ actually corresponds to the absolute maximum; however given the pattern of our results it seems that this must be the case except possibly at sufficiently small values of $\beta_3$.)

We have performed similar calculations in the $D = 4$ theory, using $N_{GT} = 100$ random gauge copies of each of 20 independent $SU(2)$ configurations on an $8^4$ lattice. We do this over the range $2.4 \leq \beta_4 \leq 2.7$ where the volume changes from being reasonably large in physical units to being very small. The corresponding values of $f$, $p$, $n_G$ are listed in Table 2, where the behaviour is clearly very similar to that in the $D = 3$ case. The transition between the regime with many copies to the one with few occurs at $\beta_4 \sim 2.5$ which corresponds to a lattice size in physical units of $8/\xi \sim 1.5$ [12]; again similar to $D = 3$.

To see how the various Gribov copies differ with respect to the $U(1)$ fields that we extract from them, we use a simple quantity that is sensitive to the local fluctuations of the $U(1)$ fields, the $U(1)$ plaquette action,

\[ S = \frac{1}{V} \sum_p (1 - \cos \theta_p) \]
where $\theta_p$ is the sum of the $U(1)$ link angles around the plaquette $p$. Note that this ‘action’ has nothing to do with whatever is the effective $U(1)$ action that describes the projected Abelian fields; that is expected to be highly non–local [4]. In Figure 2 we display a scatter plot of $S$ versus $R$ for 500 MA gauge fixings obtained from a ‘typical’ $8^4$ configuration at each of the values of $\beta_4$ indicated. Although there is a large scatter, there is an evident correlation between larger values of $R$ and smaller values of $S$; that is, Gribov copies corresponding to larger values of $R$ lead to $U(1)$ fields with weaker fluctuations [6]. This is also the case in $D = 3$.

If we look at the monopole content of various Gribov copies, we find that there are on average fewer monopoles in Gribov copies with larger $R$, in both $D = 3$ and $D = 4$. If confinement is driven by monopoles then, all other things being equal, the strength of the confining force will be proportional to the strength of the monopole condensate. Our above observations would then imply that the string tension is smaller if one uses Gribov copies of larger $R$. This is a question we shall address more directly below.

Given that Gribov copies do differ it is natural to ask if there is any convincing criterion for selecting which Gribov copies should be used for extracting the $U(1)$ fields and monopole distributions.

One increasingly frequent suggestion is that the Gribov copy corresponding to the absolute maximum of $R$ should be used. This is partly motivated by an analysis of the corresponding problem in the Landau gauge (see [13] and references therein). A concrete example is provided by Landau gauge calculations of the photon propagator in the Coulomb phase of $U(1)$ lattice gauge theory. One finds [14] that one gets an incorrect propagator unless one makes a selection on the Gribov copies. If one selects the Gribov copy corresponding to the absolute maximum (of the functional appropriate to the Landau gauge) one indeed obtains the correct perturbative propagator [15]. This provides an argument for using the Gribov copy corresponding to the absolute maximum of $R$ in the case where one wants to obtain quantities that are essentially perturbative. (One should note however that even here this choice is not unique [15].) It is however not at all clear that the same criterion should apply when considering quantities that are non-perturbative - or indeed what lessons the Landau gauge has for the Maximally Abelian gauge.

Part of the difficulty in motivating the choice of one particular Gribov copy is that if we simply look at the various Gribov copies without any theoretical prejudice then we find that the values of $R$ corresponding to the various Gribov copies, show very little variation. For example, on our $8^3$ lattice at $\beta_3 = 4$ the average value of $R$ for all Gribov copies is 0.820 while the average value for those copies with the maximum value of $R$ is 0.824. On $12^4$ at $\beta_4 = 2.4$ the corresponding values are 0.7308 and 0.7320. These are very small differences and it is hard to see what it is that might pick out one maximum as the correct one to use.

Finally we briefly return to the question raised at the beginning of this section. We described there how the gauge orbit may be naturally partitioned into subsets, each of which is associated with a single Gribov copy. We then remarked that our gauge-fixing algorithm would deform a particular gauge copy into the associated Gribov copy, and so by gauge-fixing an ensemble of randomly generated gauge copies we would generate the Gribov copies with a probability that was proportional to the volume of the associated subset of the gauge orbit. This is clearly an idealisation.
and a measure of our deviation from this idealised picture can be provided by comparing the results of different ways of gauge fixing. In particular we can see how the incorporation of over-relaxation alters the probability distribution of the resulting Gribov copies. An added motivation for this comparison is that intuitively the idealised picture holds best if we make small gauge-fixing steps. However in order to be reasonably efficient we are forced to incorporate the ‘larger’ over-relaxation steps. One would obviously like to know how much this biases our resulting ensemble of Gribov copies. (Clearly, by using steps that were large enough, one could imagine an arbitrarily large bias of the distribution.)

We have taken an $8^3 SU(2)$ lattice field generated at $\beta_3 = 5$ and have generated 250 random gauge copies from it. We have then gauge fixed in three different ways. The first contained no over-relaxation steps. The second contained 2 over-relaxation steps every 3 iterations and corresponded to the value we typically used in our $D = 3$ calculations. The last had 4 over-relaxation steps every 5 iterations. The corresponding average values of $R$ turned out to be 0.8548, 0.8547, and 0.8551 respectively. The standard deviations of the corresponding distributions were 0.0025, 0.0025, and 0.0020 respectively. Thus the overall properties of the distributions of the Gribov copies are very similar. Comparing the distributions in detail we find that the differences are located in the long tail of Gribov copies with values of $R$ much below the average. These Gribov copies have small weights, individually, and so it is not clear whether we are seeing a statistical or systematic effect. 

The bulk of the Gribov copy distribution is almost identical and our results appear consistent with the differences being statistical. In particular there appears to be no particular enhancement of the number of Gribov copies with the larger values of $R$ when we alter the gauge-fixing procedure.

The example described in the previous paragraph appears to be typical. It suggests that our idealised picture does indeed make sense, except perhaps for the long tail of Gribov copies with very small values of $R$.

4 A Simple Model.

To assess the significance of any differences that we observe between different Gribov copies, it would be useful to have some picture of the dynamics within which we can use our physical intuition. Since this dynamics is not known, we shall sketch a picture which has some plausibility even if it cannot be justified in detail (and indeed is not completely consistent as it stands and may well be incorrect).

In this picture one supposes that at some length scale the gauge fields produce a composite adjoint scalar field and that there is a dynamical symmetry breaking on a longer distance scale analogous to the explicit breaking in the Georgi-Glashow model. Such a theory should possess ‘t Hooft-Polyakov monopoles. Outside the cores of any monopoles the fields will be effectively $U(1)$, while within the cores they become fully $SU(2)$. If these monopoles condense then they will generate a linearly rising potential between static fundamental charges [16].

If the volume of space outside the monopole cores is much larger than the volume within the cores,
so that the fields are effectively $U(1)$ throughout most of space, then we would clearly expect the MA
gauge to pick out this $U(1)$ field. That is to say if we go to the gauge where the composite scalar
is proportional to $\sigma_3$, then the gauge fields will be essentially Abelian over most of space-time and
we would expect that this is what we would obtain by going to a gauge that maximises the Abelian
character of the $SU(2)$ fields. Now if we consider a closed surface that encloses a monopole and if
we locate this surface outside the monopole core, i.e. in the region where the fields are Abelian,
then there will be a net magnetic flux out of that surface. Therefore if we interpolate the $U(1)$ field
within the region of space-time occupied by the monopole core, this will necessarily produce a Dirac
magnetic monopole somewhere in that region. Since within the core there is no physical $U(1)$ field,
there is no reason why the interpolated $U(1)$ field should not contain several (anti)monopoles in the
region of space occupied by the core. The only constraint is that the net magnetic charge within
the core should equal the charge of the 't Hooft-Polyakov monopole. So any such extra monopoles
will come in monopole-antimonopole pairs whose separation is less than the core size. Similarly
even if there is a single monopole in the core it need not be at the centre of the core. However one
may think of a shift in the position of a monopole as equivalent to the addition of an appropriately
positioned monopole-antimonopole pair. Thus we would expect that by going to the MA gauge
not only would we obtain the $U(1)$ field that is produced by the symmetry breaking, but that we
would also obtain a gas of Dirac monopoles whose positions would be located within the cores of
the corresponding 't Hooft-Polyakov monopoles. The Dirac monopoles are of course unphysical,
but they serve to trace out the locations of the physical 't Hooft-Polyakov monopoles. This gas
of monopoles will in general include a gas of monopole-antimonopole pairs whose separation is no
larger than the size of the core. That is to say, a gas of magnetic dipoles.

If the monopoles condense then we will get the same string tension whether we calculate it from
the 't Hooft-Polyakov monopoles or from the $U(1)$ monopoles. The reason is that the extra Dirac
monopoles are located in dipoles of a limited spatial extent, and these can only contribute to the
shorter distance pieces of the potential.

In this picture we see that the density of $U(1)$ monopoles does not of itself effect the string tension.
Thus if Gribov copies differ in that some contain extra dipoles in the 't Hooft-Polyakov monopole
cores, then this will be harmless from the point of view of confinement. Clearly we should compare
the monopole gases in different Gribov copies and try to see whether this is the case or not. We
shall do this in a later section.

In practice, of course, our non-Abelian theory has one overall scale and so there is no reason why
the fraction of the volume inside the monopole cores should not be comparable to that outside.
Once a significant fraction of the space-time volume (i.e. that within the monopole cores) does not
contain a real $U(1)$ field, it is harder to see why the MA gauge fixing should pick out the correct
$U(1)$ field, or, if it does, why the Gribov copy for which the $SU(2)$ field is made as Abelian as
possible everywhere should be the one that best corresponds to the dynamically generated $U(1)$
field. Indeed given the small observed differences in $R$ between the various copies, it is quite
possible that it is one of the copies with $R$ less than the absolute maximum that most faithfully
maps out the effective $U(1)$ fields and 't Hooft-Polyakov monopoles of this picture.
5 Gribov Copies and the String Tension.

We have seen in the previous section that just because Gribov copies differ in the apparent strengths of their magnetic condensates, this does not imply that they differ in their confining properties. To address the latter issue, we shall compare the confining properties of those Gribov copies that possess larger values of $R$ with those that possess smaller values, and we shall compare both subsets with Gribov copies that are simply chosen at random (as is usually done in calculations of the Abelian string tension). There are many ways to perform such a selection and we have chosen the following procedure.

On each of the Monte Carlo generated $SU(2)$ fields we perform $N_{GT}$ random gauge transformations. Each of these gauge copies is fixed to the MA gauge. This provides us with a number of Gribov copies that is $\leq N_{GT}$. (Since we work with volumes that are reasonably large, the number of different copies will be close to $N_{GT}$.) From these Gribov copies we select the one with the smallest value of $R$, the one with the largest value of $R$ and one at random. In this way we obtain from our ensemble of $SU(2)$ fields three ensembles of $U(1)$ fields: one ensemble comes from copies with values of $R$ that are smaller than average, one from copies with $R$ larger than average and one from copies where $R$ is average (on the average). We denote these ensembles as $\{R_{\text{min}}\}$, $\{R_{\text{max}}\}$ and $\{R_{\text{av}}\}$ respectively. Within these three ensembles we calculate various quantities of interest to determine in what ways Gribov copies that have been so selected differ from each other. Clearly the larger $N_{GT}$, the greater the potential difference between the ensembles. (Note that if we construct an ensemble from all the gauge copies with equal weighting then this is equivalent to an ensemble where we choose one at random, but should produce results that are more accurate.)

The quantities that we shall consider here are the $U(1)$ and monopole string tensions and, for comparison, the monopole density and the average $U(1)$ plaquette. The last two quantities are straightforward but the calculation of the string tension requires some explanation. The usual procedure would be as follows. On each of the configurations we calculate the values of the Wilson loops, $W(r,t)$, both from the Abelian fields and from the magnetic monopoles. This provides us with averages on each of our three ensembles. Working within each ensemble separately, the quark potentials (in lattice units) are extracted using the usual expression:

$$V(r) = -\lim_{t \to \infty} \left[ \ln \left( \frac{\langle W(r,t+1) \rangle}{\langle W(r,t) \rangle} \right) \right]$$

The potential is then fitted with a sum of terms that are linear, Coulomb and constant in the distance $r$. The string tension, $\sigma$, is the coefficient of the linear term.

This basic method is not very efficient and in other contexts it is usually improved upon in two ways. Firstly the fluctuations in the time-like links can be reduced by a self-averaging procedure. This makes most difference at smaller values of $\beta$. For larger values of $\beta$ it can be improved upon enormously by appropriately smearing the fields and by applying a variational criterion to the extraction of $V(r)$. This second technique also allows $\sigma$ to be calculated more simply, and at least as accurately, from correlations of Polyakov loops.

The first method of improvement cannot be applied here because its details depend on the action
and the effective action for the $U(1)$ fields obtained in the MA gauge is not known (and in any case is certain to be so non-local as to render this method inapplicable).

The second improvement is only guaranteed to work within a proper field theory; that is to say it depends on the existence of a well-defined transfer matrix. Our $U(1)$ fields are derived from the $SU(2)$ fields by a very non-local procedure and so there is no guarantee that the corresponding ensembles have the desired properties. What we find, from calculations of correlation functions of smeared Polyakov loops in both 3 and 4 dimensions, is that if such a transfer matrix does exist then it certainly does not have the positivity properties of the $SU(2)$ theory. In principle this need not be a problem as long as it does not affect the large eigenvalues of the transfer matrix. (After all, most ‘improved’ non-Abelian actions break positivity in this sense.) Unfortunately the loss of positivity has the practical defect of making it difficult to apply the smearing+variational techniques that have proved so powerful in extracting string tensions for non-Abelian gauge theories.

So if we want to calculate $\sigma$ we must rely on the inefficient basic method outlined above. In the 4 dimensional case it turned out that we were able to obtain accurate calculations of the monopole string tension by this means. However we were not able to do so for the $U(1)$ fields themselves. This is both because they are relatively more noisy and also because they contain larger sub-leading terms. (An alternative, and useful method is to calculate the Wilson loops from ‘cooled’ $U(1)$ fields [8].) In 3 dimensions there is the added complication that the Coulomb potential grows logarithmically. This is a particular problem in trying to obtain $V(r)$ and $\sigma$ from monopoles because here the Coulomb potential is not screened. This additional long-range component makes the extraction of the potential and string tension quite involved. For that reason we shall not present any results, in this paper, for the $D = 3$ monopole string tension.

To obtain an estimate of the confining properties of the $U(1)$ fields we use the oldest and simplest method of all, that of Creutz ratios. We define Creutz ratios

$$C(r, t) = \frac{\langle W(r, t) \rangle \times \langle W(r - 1, t - 1) \rangle}{\langle W(r, t - 1) \rangle \times \langle W(r - 1, t) \rangle}$$

and an effective string tension $\sigma_{\text{eff}}(r) = -\ln C(r, r)$. The string tension (in lattice units) is then obtained from

$$\sigma = \lim_{r \to \infty} \sigma_{\text{eff}}(r).$$

This definition can clearly be extended to make use of Creutz ratios with $r \neq t$.

We start with the case of 3 dimensions. Taking advantage of the relative speed of 3 dimensional computations, we perform calculations with a large number of gauge copies, $N_{GT} = 30$, so ensuring that our ensembles $\{R_{\text{min}}\}$ and $\{R_{\text{max}}\}$ are really quite extreme. We do this on an ensemble of 400 independent $SU(2)$ gauge fields. In Table 3 we present the average $U(1)$ plaquette and the average number of monopoles obtained on the three Gribov copy ensembles, $\{R_{\text{max}}\}$, $\{R_{\text{min}}\}$ and $\{R_{\text{av}}\}$, that we defined above. The variation in the plaquettes is significant but small. On the other hand, the variation in the monopole densities is large. This is of concern since, all other things being equal, the string tension will be proportional to the monopole density. Of course, as discussed in the previous section, it may well be that all other things are not equal. For example,
if the difference in densities is due to magnetic dipoles then there will be no effect on $\sigma$. Whether there is an effect on $\sigma$ or not is the question we shall now address.

In Table 4 we present the values of the effective string tension for the three ensembles of Gribov copies. We see that $\sigma_{\text{eff}}$ appears to decrease towards its asymptotic value, just as one would find if one carried out this calculation directly with the $SU(2)$ fields. Moreover at any given value of $r$ we see that $\sigma_{\text{eff}}(r)$ decreases as $R$ increases. What can we say about $\sigma$ itself? One would normally estimate $\sigma$ from $\sigma_{\text{eff}}(r)$ at values of $r$ where the latter had become independent of $r$ within errors. Of course, this criterion will only work if the errors are sufficiently small for the $r$-independence to be statistically compelling. Clearly our data is rather marginal in this respect. Nonetheless, using this criterion Table 4 suggests that there is indeed a variation of $\sigma$ with the ensemble used. If we take Gribov copies at random we find $\sigma \sim 0.89(3)$. Compared to this, $\sigma$ is reduced by about 10% for the Gribov copies with the largest values of $R$ and increased by about 15% for the Gribov copies with the smallest values of $R$. These differences are not negligible, but we find them remarkably small given that we are effectively comparing the top $\frac{1}{30}$th (in $R$) of the gauge orbit against the bottom $\frac{1}{30}$th. Certainly they are not so large as to render meaningless a comparison, at least semi-quantitatively, with the full $SU(2)$ string tension, which happens to be 0.0983(16) at $\beta_3 = 5$ [17].

In the 4 dimensional case we have performed calculations on a $12^4$ lattice at $\beta_4 = 2.3$ and $\beta_4 = 2.4$. In the former case we have produced ensembles of Gribov copies with $N_{GT} = 10$ and in the latter, where the statistical errors are smaller, with $N_{GT} = 5$. Because the values of $N_{GT}$ are much smaller we expect the different $R$ ensembles to be much less extreme than in the $D = 3$ case. In Table 5 we show how the average $U(1)$ plaquette and the summed dual monopole current links vary across these different ensembles for $\beta_4 = 2.4$. The pattern of these results is similar to that obtained in $D = 3$.

As stated earlier, we have been able to extract the $D = 4$ monopole string tensions and these are listed in Table 7. We see a significant variation between the string tensions at $\beta_4 = 2.4$: the string tension in the ensemble $R_{\text{min}}$ is $17 \pm 5\%$ greater than that in $R_{\text{max}}$, while that in $R_{\text{av}}$ is about $10 \pm 5\%$ greater. The $\beta_4 = 2.3$ values of $\sigma$ show a similar variation with $R$. For the $U(1)$ fields we follow the same procedure as in $D = 3$ and extract effective string tensions from Creutz ratios. These are listed Table 6. If we compare the values of $\sigma_{\text{eff}}(r)$ at, say, $r = 3$ we observe a similar trend to that observed in the $D = 3$ case.

In all the above cases the trend is the same: the string tension decreases as $R$ increases. Moreover it would appear from these calculations that if we used the Gribov copy with the largest value of $R$ rather than choosing a copy at random (as has usually been done in previous calculations) this would reduce $\sigma$ by $O(10\%)$.

We conclude from the above that while we obtain very different monopole densities on ensembles of Gribov copies selected according to whether $R$ is large or small, the differences in the string tensions are much more modest.
6  Gribov copies - a Direct Comparison.

So far we have compared the average properties of different subsets of Gribov copies. The long-distance fluctuations can, however, be completely different even if the average properties are the same. For example if we compare the $U(1)$ fields obtained from two independent sets of $SU(2)$ gauge fields, the average properties will be the same even though the long-distance fluctuations are completely uncorrelated. Clearly to obtain a complete picture of how Gribov copies differ we need to determine how correlated are the long-distance fluctuations of Gribov copies from the same $SU(2)$ gauge field.

The ‘long-distance fluctuations’ that are relevant to this study are those that have to do with confinement. This has to do with the area decay of large Wilson loops. In a given $U(1)$ field configuration the value of a space-like Wilson loop along a contour $C$ is just a phase and can be written as

$$W(C) = e^{i B(C)}$$

where $B(C)$ is the magnetic flux through a surface spanning that contour. (In Euclidean space-time, and at zero temperature, we can confine our discussion to space-like loops with no loss of generality. Note also that in 3 dimensions we follow convention and refer to the fluxes as ‘magnetic’, as though we were looking at static fields within a time-slice of the 4 dimensional theory.) The expectation value of $W(C)$ will depend on the distribution of magnetic fluxes in the vacuum. In particular, the fact that the average value of $W(C)$ decreases exponentially with the minimal area spanned by the contour $C$, tells us that the flux $B$ is effectively the sum of a number of elementary fluxes which are mutually independent and sufficiently localised so that the number of such fluxes passing through the contour $C$ is proportional to the minimal area spanned by $C$.

One possible source of such fluxes is a gas of magnetic monopoles. Such a gas has to have particular characteristics; for example, if the (anti)monopoles pair off into dipoles then this will not disorder large Wilson loops sufficiently to make them decay exponentially with the area of the loop. A screened plasma would, on the other hand, confine. However the locations of the monopoles are not really the relevant degrees of freedom. For example if one has a confining monopole gas and shifts the locations of the (anti)monopoles by a fixed distance in random directions then this new gas is equivalent to the old gas plus a gas of randomly oriented dipoles. The latter have no effect on the confining properties so the two gases are identical for confinement at sufficiently large distances. So if we are to compare two Gribov copies with respect to those properties that are responsible for confinement, we should really compare the pattern of fluxes in the two configurations rather than just the numbers and locations of the monopoles.

The most direct method to do this is to subtract the fluxes from each other and calculate the Wilson loop with respect to this difference of magnetic fluxes:

$$W^{Diff}(C) = e^{i (B_1(C) - B_2(C))}$$

where $B_1$ and $B_2$ are the magnetic fluxes, through the same contour $C$, in the two different field configurations. So if we want to know whether different Gribov copies of the same $SU(2)$ gauge
field have the same confining fluctuations, we can take two copies from each $SU(2)$ field, calculate the Wilson loop from the difference of the fluxes, and average this over an ensemble of $SU(2)$ fields. If the resulting expectation value decreases less rapidly than exponentially with the area, this tells us that the fluctuations in the differences of the fluxes correspond to a theory with zero string tension and that Gribov copies have identical confining fluctuations. A non-zero area term provides us with a non-zero ‘difference’ string tension, $\sigma_{\text{Diff}}$. By comparing this with $2\sigma$, which is what we would find if the fluctuations in the two Gribov copies were completely uncorrelated, we can say whether the effect is ‘small’ or ‘large’.

In this section we will describe calculations in which we compare the confining fluctuations of Gribov copies in the above way. We can do so separately for the $U(1)$ fluxes and for the monopole fluxes.

If the $U(1)$ link angles of the two Gribov copies are $\{\theta_1^l\}$ and $\{\theta_2^l\}$ then we form the difference links $\theta_{\text{Diff}}^l \equiv \theta_1^l - \theta_2^l$ and we use these to calculate Wilson loops. Averaging over an ensemble of $SU(2)$ fields we can obtain the corresponding potential, $V_{\text{Diff}}(r)$ (or Creutz ratios), and the string tension, $\sigma_{\text{Diff}}$.

For monopoles we can form a difference gas of monopoles defined by $m_{\text{Diff}}(n) = m_1(n) - m_2(n)$ where $n$ labels the cube and $m$ is the magnetic charge in that cube. (This is for $D=3$; in $D=4$ one subtracts the currents on the dual lattice.) An alternative procedure is to extract the monopoles directly from the difference links, $\theta_{\text{Diff}}^l$, in the usual way. We call the latter monopole gas $m_{\text{oDiff}}$. These two difference gases should certainly have the same long-distance properties, although the sub-leading contributions might be quite different (as indeed will turn out to be the case). We calculate the dual potential for the difference gas and hence the fluxes through Wilson loops. From this we can again extract potentials (or Creutz ratios) and string tensions. In the monopole picture of confinement one would, of course, expect to find that the $U(1)$ and monopole string tensions were equal.

We note that this technique of directly comparing fields has a practical advantage over the calculations described in the previous section, in that the correlations between the fluctuations in the different copies are properly taken care of so that the statistical errors on comparative quantities will be both more accurate and (probably) smaller.

We begin with our 3 dimensional calculations and, as in the previous section, we shall only compare the $U(1)$ fields. We have performed calculations with $N_{\text{GT}} = 2$ on a $16^3$ lattice at $\beta_3 = 5$ and on a $24^3$ lattice at $\beta_3 = 9$. In addition we also have the calculations described previously, which were on a $12^3$ lattice at $\beta_3 = 5$ with $N_{\text{GT}} = 30$. In physical units the $24^3$ lattice at $\beta_3 = 9$ is about the same size as the $12^3$ lattice at $\beta_3 = 5$. Thus we have some check on both the scaling and volume dependence of our results.

For each $SU(2)$ field we take the pair of Gribov copies that we have generated and calculate the difference angles, $\{\theta_{\text{Diff}}^l\}$, as defined above. From these we calculate Wilson loops and from the averages of the Wilson loops we obtain Creutz ratios. From the latter we extract effective (difference) string tensions, $\sigma_{\text{Diff}}(r)$, as in the previous section. These are listed in Table 8 for
the $16^3$ and $24^3$ lattices. For comparison the string tensions one obtains from randomly chosen
gauge copies (the ensemble $\{R_{\text{av}}\}$ of the previous section) are $\sigma = 0.087(3)$ and $\sigma = 0.0240(5)$
respectively. We see from Table 8 that $\sigma_{\text{eff}}^{\text{Diff}}(r)$ decreases as we go to larger $r$. At some point the
signal gets lost in the noise so it not possible for us to say whether it goes to zero or not. What
we can safely do is to put an upper bound

$$\sigma_{\text{Diff}} \leq \frac{1}{5} \sigma$$

on the string tension of the ‘difference’ fields.

In Table 9 we return to our $12^3$ lattice at $\beta_3 = 5$. We show in the first column the values of
$\sigma_{\text{eff}}^{\text{Diff}}(r)$ obtained from randomly chosen pairs of Gribov copies. In the second column we show
the corresponding values when the difference field is formed by subtracting the $U(1)$ field with
the largest value of $R$ amongst our 30 gauge copies, from the field with the smallest value of $R$. We see a qualitatively similar pattern to that in Table 8, albeit with larger statistical errors.

(Calculations with 30 gauge fixings per $SU(2)$ field are computationally very expensive and this
limits our statistics.) As one would expect, at each value of $r$ the difference between extreme Gribov
copies is larger than that between randomly chosen ones. The difference, however, decreases with $r$ and is consistent with going to zero; in any case we can certainly conclude that $\sigma_{\text{Diff}} \leq \frac{1}{3} \sigma$.

For purposes of comparison, it is interesting to see what $\sigma_{\text{eff}}^{\text{Diff}}(r)$ looks like if we construct our
$U(1)$ difference angles from two fields that do not come from the same $SU(2)$ field, but instead
come from two independent $SU(2)$ fields. In practice our ‘independent’ fields are ones which are
separated by 50 Monte Carlo sweeps. We perform this calculation on $24^3$ fields at $\beta_3 = 9$, where
our calculations are the most accurate. In Figure 3 we display the values of $\sigma_{\text{eff}}^{\text{Diff}}(r)$ as calculated
in this way. We also show the string tension extracted from the difference of two Gribov copies
that come from the same $SU(2)$ field (i.e. the values listed in the appropriate column of Table 8)
and, in addition, the effective string tension extracted simply from a single randomly chosen gauge
copy of each $SU(2)$ field.

The first thing we note is that in this last case the effective string tension rapidly becomes independent of $r$, making the extraction of the desired $r \to \infty$ limit straightforward. The second thing
we note is that this effective string tension is exactly half the difference string tension calculated
from pairs of independent $SU(2)$ fields. This is what one expects for independent fields, where the
fluxes are independent and $\langle W_{\text{Diff}} \rangle$ factorises. Finally we observe that the values of the effective
string tension as calculated from the difference of two Gribov copies from the same field, are very
much smaller and are still decreasing at values of $r$ where the other two effective string tensions
have already become independent of $r$. This is our best $D = 3$ calculation, with respect to both
statistical and systematic errors. It is consistent with the pattern of the long-distance fluctuations
which produce confinement, being exactly the same on different Gribov copies.

In 4 dimensions we are able to obtain useful calculations of Wilson loops not only from the differences of $U(1)$ fields, but also from differences of monopole current distributions, as described
above. In neither case are we able to extract potentials and string tensions unambiguously. For
the $U(1)$ fields the reason is the same as before. For the monopoles the reason is that the Wilson
loops calculated with difference gases possess much larger sub-leading contributions than with the individual monopole gases. We shall therefore use Creutz ratios to extract values of $\sigma_{\text{diff}}^\text{eff}$ in all cases.

Our calculations have been performed on $12^4$ lattices at $\beta_4 = 2.3$ and 2.4. In Fig 4 we plot the values of $\sigma_{\text{diff}}^\text{eff}$ that we have obtained from the $U(1)$ difference fields that have been extracted from pairs of randomly chosen Gribov copies from each $SU(2)$ field configuration. They are shown as a function of the area of the (largest) Wilson loop used in the evaluation of the Creutz ratio. We have included not only square (largest) loops, as in the previous section, but also some rectangular loops. (Although we have limited ourselves to using ones which are nearly square.) We see from Fig 4 that $\sigma_{\text{diff}}^\text{eff}$ drops smoothly towards zero with increasing area. Whether it actually asymptotes to zero or not is something that we cannot say, because of the increasing errors at larger areas. Nonetheless what we can safely infer is that the asymptotic $U(1)$ difference string tension is bounded by $\sigma_{\text{diff}}^\text{eff} \leq \frac{2}{3} \sigma$.

We now extract the monopole gases from the $U(1)$ difference fields of the previous paragraph, and calculate the corresponding Creutz ratios. The resulting values of $\sigma_{\text{diff}}^\text{eff}$ are plotted in Fig 5. We also show on these plots the $\sigma_{\text{diff}}^\text{eff}$ one obtains from the original monopole gases at the corresponding values of $\beta_4$. We observe that $\sigma_{\text{diff}}^\text{eff}$ drops monotonically towards zero. This is especially striking when compared to the behaviour of $\sigma_{\text{diff}}$. From these figures we obtain a typical bound $\sigma_{\text{diff}}^\text{eff} \leq \frac{2}{3} \sigma$.

As mentioned earlier, an alternative to the above is to subtract the the two monopole gases so as to produce a difference monopole gas. From this we can calculate Creutz ratios and these are displayed in Fig 6. Comparing to Fig 5 we see that these effective string tensions decrease much more slowly as a function of loop area. It appears that this method of comparing monopole properties greatly enhances the sub-leading shorter distance fluctuations (for reasons that we have not completely understood). The bound one obtains from Fig 6 on $\sigma_{\text{diff}}^\text{eff}$ is clearly weaker than our previous bounds and this method is clearly much less efficient.

We have found that in both 3 and 4 dimensions the long-distance fluctuations that drive linear confinement are remarkably similar in pairs of randomly chosen Gribov copies.

This result naturally provokes the question: perhaps the difference between the monopole gases on different Gribov copies is manifestly trivial? For example it might be due to monopole-antimonopole pairs one or two lattice spacings apart.

One way of probing this question is to calculate the lengths of the monopole loops and to examine the number of loops of each length (the loop ‘spectrum’). If the monopole difference gas were trivial then we would expect a spectrum greatly more peaked at small loop lengths than that of either of the configurations contributing to the difference gas. In fact this is not what we find. Instead we obtain a spectrum which extends to large loops. Indeed, apart from the overall normalisation, we find that this spectrum is indistinguishable from that which we obtain in the usual monopole gas. An analogous conclusion is reached in 3 dimensions. Thus the difference is certainly not trivial. A natural possibility is that it might consist of dipoles whose extent is limited by some physical length scale, such as the extent of the ’t Hooft-Polyakov cores in the simple ‘model’ we introduced.
earlier. With a separation of several lattice spacings, any constraint on the loop lengths would be much less severe. This is a much harder possibility to test and we shall not attempt to do so here.

7 Conclusions.

The Maximally Abelian gauge has provided a promising framework within which to explore the monopole approach to confinement. There are, however, Gribov copies in this gauge and quantities such as the monopole density vary quite strongly between the different copies of the same non-Abelian gauge field. This naturally raises the question of what significance should be attached to the observation that, if one performs calculations on the $U(1)$ fields that are extracted from randomly chosen Gribov copies, then the Abelian and non-Abelian string tensions are approximately equal.

In this paper we have confirmed that Gribov copies differ in various respects, such as the monopole density. We have however argued that this need not be an important difference, and we illustrated this within a dynamical ‘picture’ where confinement arises from the condensation of ‘t Hooft-Polyakov monopoles following the formation of composite scalars and a dynamical symmetry breaking in the non-Abelian fields.

We then attempted to establish in a direct fashion whether or not the confining properties of different Gribov copies are the same. We first calculated the $U(1)$ and monopole (effective) string tensions on Gribov copies that had been selected into separate ensembles on the basis of the value of the quantity $R$ which is maximised when one fixes to the MA gauge. What we found was that the (effective) string tension does indeed vary, and that it decreases as we increase the value of $R$. This variation is, however, small compared to the typical value of the string tension. For example, we find that the string tension calculated on Gribov copies with the largest values of $R$ is about 10% less than that obtained with randomly chosen Gribov copies.

Comparing string tensions on such selected ensembles of Gribov copies provides a rather crude probe of the differences between such copies. What one really wants to do is to compare, between different Gribov copies of the same non-Abelian field, those $U(1)$ field fluctuations that drive confinement. We suggested that this could be achieved most simply by subtracting the Abelian gauge potentials of two copies and calculating Wilson loops on this new field which is the difference of the old ones. If the pattern of long-distance confining fluctuations is identical on different Gribov copies, then the resulting ‘difference’ string tension, $\sigma_{\text{Diff}}$, should be zero. If the fluctuations are completely uncorrelated then we should find $\sigma_{\text{Diff}} = 2\sigma$, where $\sigma$ is the string tension one obtains from Wilson loop calculations on randomly chosen Gribov copies. What we actually found, in both 3 and 4 dimensions, is that $\sigma_{\text{Diff}} \leq \frac{1}{6}2\sigma$ and, indeed, is compatible with zero. That is to say, as far as the confining fluctuations are concerned, different Gribov copies of the same non-Abelian field are very strongly correlated.

The fact that there does appear to be some variation of $\sigma$ with $R$, has further implications. Consider the calculation of the string tension in the ensemble of randomly chosen Gribov copies. We have
assumed that there is a particular value of $\sigma$ associated to this ensemble. However, by definition, this ensemble contains Gribov copies with all possible values of $R$. Thus one would na"ïvely expect that Wilson loops of large enough area $A$ will be given by

$$\langle W(A) \rangle \sim \int_{\sigma_-}^{\sigma_+} d\sigma \rho(\sigma) e^{-\sigma A}$$

where $\rho(\sigma)$ is some suitable density function which takes into account the fact that there is a variation of $\sigma$ with $R$ between limits $\sigma_-$ and $\sigma_+$. Since we observe that the value of $\sigma$ appears to decrease monotonically with increasing $R$, we may assume that $\sigma_-$ is the value of $\sigma$ for the Gribov copies with the largest values of $R$. Now we observe that

$$\lim_{A \to \infty} \langle W(A) \rangle \sim e^{-\sigma_- A}$$

However if the variation of $\sigma$ with $R$ is as small as we find it to be, one can easily estimate that this ‘true’ asymptotic behaviour would only become visible for Wilson loops that are far larger than those we have been able to consider. Thus it is no surprise that we have not observed this effect. Nonetheless the implication is that the ‘true’ asymptotic value of $\sigma$, as obtained with randomly chosen Gribov copies, is in fact the value of $\sigma$ that one would obtain if one were to use the Gribov copies with the largest value of $R$. Clearly the most efficient way to calculate this string tension is to choose such extreme Gribov copies from the start. This provides a concrete argument for focusing on these extreme Gribov copies.

Of course one must be careful with the ‘na"ïve’ argument of the previous paragraph. The basic assumption was that there is a different $\sigma$ for different $R$, so that the string tension we obtain with randomly chosen Gribov copies is a sub-leading phenomenon that will be transformed, for sufficiently large areas, into $\sigma_-$. It might be that the situation is the reverse, i.e. there is a definite string tension, $\bar{\sigma}$, associated with randomly chosen Gribov copies and the $R$-dependent values of $\sigma$ that we observe are in fact a sub-leading phenomenon and will be transformed for sufficiently large areas into $\bar{\sigma}$. Even if our first assumption is correct, there are some obvious questions that need to be answered such as what is the large volume dependence of $\rho(\sigma)$, how does the onset of the area law vary with $R$ etc., before we can take the conclusion of the previous paragraph too seriously.

We conclude that even though we see a modest variation of the string tension with the particular ensemble of Gribov copies used, the confining Abelian fluctuations of different Gribov copies are in fact remarkably correlated. This is so in the zero temperature confining phase for both 3 and 4 dimensions. It suggests that while Gribov copies do indeed pose a real problem in principle - for the study of confinement in the Maximally Abelian gauge - the practical ambiguities are really quite limited and do not seriously undermine the evidence that has made this gauge so interesting.

**Note Added**

When this work was completed we received a paper [18] in which techniques are developed to
calculate string tensions on the Gribov copies corresponding to the absolute maximum of \( R \). These authors also observe that \( \sigma \) decreases as \( R \) increases.

**Acknowledgments**

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References

[15] V. Mitrjushkin — private communication
Figure 1: A schematic diagram of the gauge orbit of a single $SU(2)$ configuration. The dotted lines denote the effect of iteratively fixing to the MA gauge. This diagram is not designed to be accurate, but to illustrate why two gauge transformations, 1 and 2, of a single configuration may fix to different Gribov copies.

<table>
<thead>
<tr>
<th>$\beta_3$</th>
<th>$f$</th>
<th>$p$</th>
<th>$n_G$</th>
</tr>
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<tbody>
<tr>
<td>4.0</td>
<td>0.128 (25)</td>
<td>0.35 (11)</td>
<td>28.0 (17)</td>
</tr>
<tr>
<td>5.0</td>
<td>0.363 (37)</td>
<td>0.80 (10)</td>
<td>14.8 (15)</td>
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<tr>
<td>6.0</td>
<td>0.684 (59)</td>
<td>0.95 (5)</td>
<td>5.7 (9)</td>
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<td>7.0</td>
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<td>1.00 (5)</td>
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<td>8.0</td>
<td>0.938 (31)</td>
<td>1.00 (5)</td>
<td>1.55 (20)</td>
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<td>9.0</td>
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<td>1.00 (5)</td>
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<td>10.0</td>
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<td>11.0</td>
<td>0.989 (5)</td>
<td>1.00 (5)</td>
<td>1.25 (10)</td>
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<tr>
<td>12.0</td>
<td>0.986 (5)</td>
<td>1.00 (5)</td>
<td>1.35 (11)</td>
</tr>
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Table 1: Some properties of Gribov copies on an $8^3$ lattice in $D = 3$. 
Table 2: Some properties of Gribov copies on an $8^4$ lattice in $D = 4$.

<table>
<thead>
<tr>
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<th>$p$</th>
<th>$n_G$</th>
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<td>0.01 (2)</td>
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<td>2.4</td>
<td>0.07 (17)</td>
<td>0.35 (5)</td>
<td>77.8 (252)</td>
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<td>2.5</td>
<td>0.43 (33)</td>
<td>0.70 (5)</td>
<td>22.1 (266)</td>
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<tr>
<td>2.6</td>
<td>0.68 (38)</td>
<td>0.75 (5)</td>
<td>8.6 (139)</td>
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<tr>
<td>2.7</td>
<td>0.91 (16)</td>
<td>1.00 (5)</td>
<td>2.1 (19)</td>
</tr>
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</table>

Figure 2: Scatter plots of $S$ versus $R$ in $D = 4$ at $\beta_4 = 2.4, 2.5, 2.6, 2.7$ (reading horizontally). $N_{GT} = 500$ for $\beta_4 = 2.4, 2.5$, $N_{GT} = 100$ for $\beta_4 = 2.6, 2.7$. 
Table 3: Average $R$, $U(1)$ plaquette and monopole number in $D = 3$ on a $12^3$ lattice at $\beta_3 = 5$, using $N_{GT} = 30$.

<table>
<thead>
<tr>
<th></th>
<th>$R_{\text{min}}$</th>
<th>$R_{\text{av}}$</th>
<th>$R_{\text{max}}$</th>
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<tbody>
<tr>
<td>$\langle R \rangle$</td>
<td>0.8477 (3)</td>
<td>0.8533 (4)</td>
<td>0.8566 (3)</td>
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<td>$\langle \cos \theta_p \rangle$</td>
<td>0.8856 (5)</td>
<td>0.8884 (4)</td>
<td>0.8901 (5)</td>
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<tr>
<td>$\langle n_M \rangle$</td>
<td>5.875 (83)</td>
<td>4.752 (44)</td>
<td>4.018 (72)</td>
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</table>

Table 4: Effective $U(1)$ string tensions in $D = 3$ on a $12^3$ lattice at $\beta_3 = 5$, using $N_{GT} = 30$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\sigma_{\text{eff}}(r)$</th>
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<tr>
<td></td>
<td>$R_{\text{min}}$</td>
</tr>
<tr>
<td>2</td>
<td>0.1078 (15)</td>
</tr>
<tr>
<td>3</td>
<td>0.1028 (30)</td>
</tr>
<tr>
<td>4</td>
<td>0.107 (7)</td>
</tr>
<tr>
<td>5</td>
<td>0.092 (16)</td>
</tr>
</tbody>
</table>

Table 5: Average $R$, $U(1)$ plaquette and summed lengths of monopole loops (normalised by the number of lattice links) in $D = 4$ on a $12^4$ lattice at $\beta_4 = 2.4$, using $N_{GT} = 5$.

<table>
<thead>
<tr>
<th></th>
<th>$R_{\text{min}}$</th>
<th>$R_{\text{av}}$</th>
<th>$R_{\text{max}}$</th>
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<tr>
<td>$\langle R \rangle$</td>
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<td>0.7308 (1)</td>
<td>0.7320 (1)</td>
</tr>
<tr>
<td>$\langle \cos \theta_p \rangle$</td>
<td>0.7066 (5)</td>
<td>0.7094 (7)</td>
<td>0.7132 (5)</td>
</tr>
<tr>
<td>$\langle P_M \rangle$</td>
<td>0.0541 (4)</td>
<td>0.0514 (6)</td>
<td>0.0489 (4)</td>
</tr>
</tbody>
</table>
Table 6: Effective $U(1)$ string tensions in $D = 4$ on a $12^4$ lattice, using $N_{GT} = 10$ at $\beta_4 = 2.3$ and $N_{GT} = 5$ at $\beta_4 = 2.4$ respectively.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\sigma_{\text{eff}}(r)$</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$R_{\text{min}}$</td>
<td>$R_{\text{av}}$</td>
<td>$R_{\text{max}}$</td>
</tr>
<tr>
<td>2</td>
<td>0.230 (3)</td>
<td>0.224 (3)</td>
<td>0.223 (3)</td>
</tr>
<tr>
<td>3</td>
<td>0.176 (9)</td>
<td>0.190 (10)</td>
<td>0.162 (10)</td>
</tr>
<tr>
<td>4</td>
<td>0.128 (39)</td>
<td>0.113 (47)</td>
<td>0.161 (41)</td>
</tr>
</tbody>
</table>

Table 7: Monopole string tensions in $D = 4$ on a $12^4$ lattice, using $N_{GT} = 10$ at $\beta_4 = 2.3$ and $N_{GT} = 5$ at $\beta_4 = 2.4$.

<table>
<thead>
<tr>
<th>$\beta_4$</th>
<th>$\sigma$</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$R_{\text{min}}$</td>
<td>$R_{\text{av}}$</td>
<td>$R_{\text{max}}$</td>
</tr>
<tr>
<td>2.3</td>
<td>0.136 (2)</td>
<td>0.127 (2)</td>
<td>0.121 (2)</td>
</tr>
<tr>
<td>2.4</td>
<td>0.068 (2)</td>
<td>0.064 (2)</td>
<td>0.058 (2)</td>
</tr>
</tbody>
</table>

Table 8: Effective $U(1)$ string tensions of difference gases between random Gribov copies in $D = 3$ on lattices of comparable physical sizes.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\sigma_{\text{eff}}^{\text{diff}}(r)$</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$16^3; \beta_3 = 5$</td>
<td>$24^3; \beta_3 = 9$</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.1106 (12)</td>
<td>0.0549 (4)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.0387 (13)</td>
<td>0.0320 (5)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.0158 (23)</td>
<td>0.0195 (6)</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.0127 (51)</td>
<td>0.0103 (10)</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>0.0122 (95)</td>
<td>0.0085 (14)</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>-0.0150 (110)</td>
<td>0.0055 (20)</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>0.0018 (27)</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>-0.0013 (31)</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>0.0060 (62)</td>
<td></td>
</tr>
</tbody>
</table>
Table 9: Effective $U(1)$ string tensions of difference gases between Gribov copies that are extreme in $R$ using $N_{GT}$ gauge transformations in $D = 3$ on a $12^3$ lattice at $\beta_3 = 5$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$\sigma_{\text{eff}}^\text{Diff}(r)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$N_{GT} = 2$</td>
</tr>
<tr>
<td>2</td>
<td>0.1116 (33)</td>
</tr>
<tr>
<td>3</td>
<td>0.0351 (40)</td>
</tr>
<tr>
<td>4</td>
<td>0.0111 (90)</td>
</tr>
<tr>
<td>5</td>
<td>-0.013 (15)</td>
</tr>
</tbody>
</table>

Figure 3: Comparison of $U(1)$ effective string tensions in $D = 3$ on a $24^3$ lattice at $\beta_3 = 9$. Key: + full $U(1)$ fields ($R_{av}$), $\times$ Difference fields from independent configurations, $*$ Difference fields from random Gribov copies.
Figure 4: Effective string tensions for the $U(1)$ difference fields from extremal Gribov copies at $eta_4 = 2.3, 2.4$ respectively.
Figure 5: Effective string tensions for; + monopoles, × monopoles identified from the U(1) difference fields from extremal Gribov copies, at β₄ = 2.3, 2.4 respectively
Figure 6: Effective string tensions for the monopole difference gases from extremal Gribov copies at $\beta_4 = 2.3, 2.4$ respectively.