The Gross–Neveu model on a sphere with a magnetic monopole

E. Elizalde\textsuperscript{a,b,1}, S. Naftulin\textsuperscript{c}, and S.D. Odintsov\textsuperscript{b,2}

\textsuperscript{a}Center for Advanced Studies CEAB, CSIC, Camí de Santa Bàrbara, 17300 Blanes, Spain
\textsuperscript{b}Department ECM and IFAE, Faculty of Physics, University of Barcelona, Diagonal 647, 08028 Barcelona, Spain
\textsuperscript{c}Institute for Single Crystals, 310141 Kharkov, Ukraine

Abstract

We study, for the first time, the phase structure of the Gross–Neveu model with a combination of a (constant) gravitational and a magnetic field. This has been made possible by our finding of an exact solution to the problem, namely the effective potential for the composite fermions. Then, from the corresponding implicit equation the phase diagram for the dynamical fermion mass is calculated numerically for some values of the magnetic field. For a small magnetic field the phase diagram hints to the possibility of a second order phase transition at some critical curvature. With growing magnetic field only the phase with broken chiral symmetry survives, because the magnetic field prevents the decay of the chiral condensate. This result is bound to have important consequences in early universe cosmology.

PACS: 04.62.+v, 04.60.-m, 02.30.+g

\textsuperscript{1}E-mail: eli@zeta.ecm.ub.es
\textsuperscript{2}On leave of absence from Tomsk Pedagogical University, 634041 Tomsk, Russia. E-mail: sergei@ecm.ub.es
Because of its remarkable properties of being a renormalizable and asymptotically free theory, the Gross–Neveu (GN) model [1] has been often considered as a reliable scenario for studying basic phenomena in particle physics, as chiral symmetry breaking and the formation of composite bound states. The study of the model under external conditions—as non-zero temperature, non-zero fermionic number density, or an electromagnetic field—has been carried out in Refs. [2]-[9] (see also the references therein). In particular, an investigation of space-dependent configurations at non-zero temperature has been done in Refs. [7]. The phase structure of the GN model in an external gravitational field has been studied in Refs. [10, 11], and a discussion of the chiral condensate in two-dimensional curved space can be found in Ref. [12]. For models of the early universe, the relevance of the study that we are going to present here originates in the fact that it is very probable that primordial magnetic fields have existed in combination with a strong curvature. Recently, there have been intense discussions on the role of such magnetic fields in relation with inflationary universe models [13]. Thence the interest in considering the combined effect of both gravitational and magnetic fields, which can be extended to quite different situations. In the case of the early universe this combined effect can produce a significant increase in the number of created particles [14].

As a concrete example, we investigate in this letter the phase structure of the GN model on the two-dimensional Euclidean sphere $S^2$ with a magnetic monopole inside (in the three-dimensional language), that is, we consider the combined effect in the GN model of a magnetic and a constant curvature gravitational field. As we will see, the magnetic monopole manages to prevent the decay of the chiral condensate. In fact, when the intensity of the magnetic field grows, the critical curvature tends rapidly to zero, and chiral symmetry is always broken. Our starting point is the action of the GN model with the auxiliary field $\sigma = \lambda \bar{\psi} \psi / N$ on the sphere $S^2$ of radius $r$ (curvature $R = 2/r^2$)

$$S = \int d^2x \sqrt{g} \left( \bar{\psi} \gamma^\mu \nabla_\mu \psi - \lambda \sigma \bar{\psi} \psi \frac{N}{2} \sigma^2 \right), \quad (1)$$

where $\psi$ is an $N$-component spinor. After performing the $1/N$-expansion and calculating as usually the effective potential for the non-zero condensate $\langle \sigma \rangle = \text{const} \neq 0$, we can interpret the appearance of the condensate as a process of dynamical mass generation, so that $M = \lambda \langle \sigma \rangle$. To first order in the $1/N$-expansion, we have

$$V(M) = \frac{M^2}{2\lambda^2} - \frac{1}{\lambda} \text{Tr} \ln (\gamma^\mu \nabla_\mu - M), \quad (2)$$
where $V$ is the volume and $\text{Tr}$ includes a summation over the spinor indices and the trace of the covariant operator modes too. For the calculation of (2) we will use zeta function regularization. It is convenient to work with the derivative of the potential

$$\frac{\partial V}{\partial M} = \frac{M}{\lambda^2} + \frac{1}{V} \text{Tr} \left( (\gamma^\mu \nabla_\mu - M)^{-1} \right) = \frac{M}{\lambda^2} - \frac{M}{V} \text{Tr} \left( M^2 + \frac{R}{4} - \nabla^2 \right)^{-1}. \tag{3}$$

Calculation of the trace yields

$$\text{Tr} \left( M^2 + \frac{R}{4} - \nabla^2 \right)^{-1} = \frac{V}{2\pi} \sum_{l=0}^{\infty} \frac{2(l+1)}{(l+1)^2 + r^2 M^2}, \tag{4}$$

where the well-known expressions for the spinor spectrum on $S^2$ have been used (eigenvalues $\lambda_l = (l+1)^2$ with degeneracies $d_l = 2(l+1)$). With the notation $M_0 \equiv M|_{R=0}$ and making use of the zeta-function regularization procedure (see Ref. [15] for recent review of the method), we obtain

$$\frac{\partial V}{\partial M} \bigg|_{\text{reg}} = \frac{M}{\lambda^2} + \frac{M}{\pi} \text{Re} \psi(1 + irM), \tag{5}$$

where $\psi(z)$ is here the digamma function. Here standard zeta-function regularization has been employed, in the sense of converting the divergent $l$-sum into a derivative of an Epstein zeta function of the kind $F(z; a, b) \equiv \sum_{n=0}^{\infty} [(n+a)^2 + b]^{-z}$ with respect to $a$ at $a = 1/2$, being $b = r^2 M^2 + 1/4$. After that one just makes the analytic continuation of this zeta function to $z = 1$, taking due care of the poles (see [15]). Note that, written in this form, the result coincides with the one obtained in Refs. [11], where a full explanation of the regularization procedure, as applied to the present situation, is given.

Having in mind the subsequent extension of the procedure to include a magnetic field (which is our goal here) we will introduce —already in the present case— the following renormalization condition:

$$\frac{\partial V}{\partial M} \bigg|_{M=M_0, r \to \infty} = 0.$$  

Doing so, for the renormalized potential we get

$$\frac{\partial V}{\partial M} \bigg|_{\text{ren}} = \frac{M}{\pi} \left[ \text{Re} \psi(1 + irM) - \ln(r M_0) \right]. \tag{6}$$

A graphic representation of the dependence of $M$ on $r$, which is implicitly contained in this expression —equated to zero— is given in Fig. 1 (case $k = 0$, i.e., the curve below). As has been discussed in [11] in detail, this curve shows the behavior that corresponds to a second order phase transition. The critical curvature is defined by the condition $\ln(r_c M_0) = \text{Re} \psi\left(\frac{1+i}{2}\right)$, and has the value $r_c = 0.42$. One should observe that we are not talking here about an absolutely real phase transition (since the only difference affects the discrete symmetry $\psi \to \gamma_5 \psi$, $\sigma \to -\sigma$). A delicate point of the approach is that of the possible influence of
space-dependent configurations, which could modify the whole picture substantially, leading to the single phase where $M \neq 0$ necessarily.

On the other hand, one could also think in modifying the method through the introduction of some explicit symmetry breaking phenomenon (as an external magnetic field, for instance), which effectively supports the phase with $M \neq 0$, even in the absence of a gravitational field. We shall now investigate this possibility in detail, taking as illustrative example the case of a monopole placed in the center of the spherical manifold $S^2$ (in the three-dimensional point of view). This issue of combining a gravitational and a magnetic field over a compact manifold had never been discussed before. We will see in the following that a precise model for such situation exists, and that sensible and interesting results can arise from this case. Since it is obviously impossible to introduce a magnetic field in a two-dimensional curved space of constant curvature, we came to the alternative idea that consists in describing the situation from a three-dimensional viewpoint, e.g., embedding $S^2$ in $R^3$. From the point of view of this embedding only the normal component of the three-dimensional magnetic field $B$ will influence the dynamics of the problem. However, one cannot impose that $|B| = \text{const.}$ on the sphere (a compact manifold) since the total flow of $B$ through the sphere is equal to zero (this was the original problem). Nevertheless, we can obtain an homogeneous flow of the magnetic field through the sphere by introducing in it a magnetic monopole (since then $\text{div} B \neq 0$). If the monopole is located in the center of the sphere, its magnetic field has the form: $B = k \frac{r}{r^2}$, $k = \text{const.} > 0$, and it can be considered as the natural analogue of a constant magnetic field $B$ [16] distributed on the sphere. Actually, this trick of introducing a monopole as a mean to study effects of a homogeneous magnetic field has been used extensively to simulate the quantum Hall effect (see, for instance, Ref. [17]). In our situation, the role of $B$ is played by the total flow $\phi = 4\pi k$ (or by the quantity $k$ itself). We will also assume that Dirac’s quantization condition is fulfilled, and hence $k = 1/2, 1, 3/2, 2, \ldots$ For simplicity, we shall take $e = 1$.

The change in Eq. (3) due to the presence of the magnetic monopole comes from an additional contribution in the connection $[\nabla_\mu, \nabla_\nu]$, which yields a term of the form $\epsilon_{\mu\nu}k/r^2$ and, as a result, $\gamma^\rho \gamma^\nu \nabla_\mu \nabla_\nu = \nabla^2 - \frac{R}{4} - \frac{5}{2} \gamma_5 R$. Now the operator $\nabla^2$ contains the potential $A_\mu$ and coincides with the Hamiltonian for a spinorial charged particle in the field of a Dirac monopole [18]. The monopole does not break the spherical symmetry of the problem. The eigenvalues will actually change, but in a simple way. The effective potential (more exactly, its derivative) for the GN model on a sphere $S^2$ with a magnetic monopole in its center (as
described above), is given by
\[
\frac{\partial V}{\partial M}\bigg|_{reg} = \frac{M}{\lambda^2} - \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \frac{\gamma^{\mu} \nabla_{\mu} + M}{M^2 - \nabla^2 + \frac{R}{4} + \frac{4k^2}{M^2} \gamma_5} \\
= \frac{M}{\lambda^2} - \frac{M}{2\pi} \sum_{s=\pm 1} \sum_{l=k}^{\infty} \frac{2(l+1)}{(l+1)^2 + r^2M^2 - k^2 + ks},
\]
(7)
where \( s = \pm 1 \) corresponds to the two different chiralities and \( k \) describes the magnetic field effect. Performing again a regularization via the zeta function [15], and using the renormalization condition \( \frac{\partial V}{\partial M}\bigg|_{M=M_0, r \to \infty, k \text{ fixed}} = 0 \), we obtain the renormalized potential. The condition \( \frac{\partial V}{\partial M} = 0 \) leads to an implicit dependence of the dynamically generated mass \( M \) in terms of \( r \) and \( k \), \( M = M(r, k) \) (in place of (6):
\[
\sum_{s,s'=\pm 1} \psi(k + 1 + is\sqrt{r^2M^2 + ks' - k^2}) = 4\ln(rM_0).
\]
(8)
In Fig. 1 the dependence of \( y \equiv M/M_0 \) on \( x \equiv rM_0 \) is represented for the first three values of \( k \), namely \( k = 0, 1/2, 1 \) (lower to upper curve, respectively). The corresponding curve for \( k = 5 \) is depicted in Fig. 2. Here, the loss of monotonicity through the appearance of a pick for higher values of \( k \) (starting at \( k = 1 \)) is clearly confirmed. Fig. 3 shows the value of the critical curvature \( x_c \) as a function of \( k \). As we see, with growing \( k \) the critical curvature decreases quickly, what leaves the phase with \( M \neq 0 \) only. In other words, chiral symmetry is always broken then. We thus observe that the magnetic field has a stronger effect on the phase transition pattern than the gravitational field. For \( R \to 0 \) one can easily find the asymptotic form of \( M = M(R, k) \) from (8) analytically (for fixed \( k \))
\[
\frac{M^2}{M_0^2} = 1 - \frac{R}{12M_0^2} + \frac{1 + 30k^2}{360M_0^2} R^2 + \mathcal{O}(R^3).
\]
(9)
From this last expression we see explicitly how the magnetic field acts against the decay of the chiral condensate. However, when \( k \) is not too big there exists a finite critical value \( x_c \) for which \( M(x_c) = 0 \). The phase diagram shows a phase transition of second order.

Up to now \( k \) has been kept fixed. However, in the situation when \( R \to 0 \), in order to have a magnetic field, \( B \), we must scale \( k \) as \( k = Br^2 \), \( B = \text{const}, r \to \infty \). In this limit \( B \) is to be identified with the usual homogeneous magnetic field on a plane. Taking this limit \( (B/M_0^2 = \mathcal{O}(1)) \) one sees that \( M(r, k) \) tends to a finite value \( \tilde{M}(B) \), defined implicitly by
\[
\psi\left(\frac{\tilde{M}}{2B}\right) + \psi\left(1 + \frac{\tilde{M}}{2B}\right) = 2\ln \frac{M_0^2}{2B}.
\]
(10)
This expression confirms a well-known result for the dynamical fermion mass in the GN model with an homogeneous magnetic field. The first curvature corrections to the dynamical fermion mass are as follows (here $\tilde{M}(B) \geq M_0$, $k \simeq Br^2$, $R \ll B$, $\tilde{M}_0^2$, and $B$ is finite):

$$M^2 \simeq \tilde{M}^2 + R \left\{ -\frac{1}{4} - \frac{\tilde{M}^4}{8B^2} + \frac{2B^2 + \tilde{M}^4}{2B\tilde{M}^4} \left[ \psi' \left( 1 + \frac{\tilde{M}^2}{2B} \right) + \psi' \left( \frac{\tilde{M}^2}{2B} \right) \right] \right\}. \quad (11)$$

Note that when $B \to 0$, we have from (10) that $\tilde{M}^2 \simeq M_0^2 + \frac{B^2}{3M_0^2}$.

Summing up, after being able to introduce a constant magnetic field on the surface of a two-dimensional sphere (serving as a generic example of a compact, curved manifold), we have studied the phase structure of the GN model under the combined influence of a gravitational and a magnetic field. The critical curvature has been shown to be a quickly decreasing function of $k$. With growing intensity of the magnetic field, chiral symmetry is always broken and hence there is no massless phase. We conclude, as a consequence, that the role of the magnetic field in this problem is more significant than the role of the gravitational field. It would be of interest to consider the generalization of the above problem to four dimensions, e.g., to study $D = 4$ four-fermion models influenced by a combination of a gravitational field, of Friedman-Robertson-Walker type, and a magnetic field, as the one created by a primordial monopole. Under the perspective of such more realistic situation, the neat (albeit preliminary) results that we have obtained here acquire a deep cosmological importance.

**Acknowledgments.** We would like to thank Yu.I. Shil’nov for pointing us a mistake in the original version of this paper. The work has been supported by DGICYT (Spain), project PB93-0035 and grant SAB93-0024, by CIRIT (Generalitat de Catalunya), grant GRQ94-8001, and by RFFR (Russia), project 94-02-03234.
References


**Figure captions**

**Fig. 1.** Curves showing the dependence of $y \equiv M/M_0$ on $x \equiv rM_0$ represented for the first three values of $k$, namely $k = 0, 1/2, 1$ (lower to upper curve, respectively).

**Fig. 2.** The corresponding curve, as in Fig. 1, for $k = 5$. Loss of monotonicity caused by the appearance of a pick for higher values of $k$ (starting at $k = 1$) is clearly observed.

**Fig. 3.** The critical curvature $x_c$ represented as a function of $k$ for the values compatible with Dirac’s quantization condition. With growing $k$ the critical curvature decreases very quickly. This leaves the phase with $M \neq 0$ only. In other words, chiral symmetry is always broken for $k$ large.