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On the absolutely continuous subspace
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Abstract

The paper continues investigations of the absolutely continuous and singular subspaces which were started in [1]. The definitions of [1] of these subspaces are applied to closed non-selfadjoint operators of the form $L = A_0 + V_0 + iW$, where A_0 , V_0 and W are self-adjoint operators. If the boundary values $\Theta_L^\pm(k) = s - \lim_{\epsilon \rightarrow +0} \Theta_L(k \pm i\epsilon)$ of the characteristic function $\Theta_L(z) = I + i\text{sign}(W)\sqrt{2|W|}(L^* - z)^{-1}\sqrt{2|W|}$, $z \in \varrho(L^*) \cap \mathbf{C}_\pm$, of L exist for a.e. $k \in \mathbf{R}^1$, then the singular subspace $\mathcal{H}^s(L^*)$ of L^* , which is defined as the orthogonal complement to the absolutely continuous subspace $\mathcal{H}^{ac}(L)$, admits the characterization

$$\mathcal{H}^s(L^*) = \left\{ h \in \mathcal{H} : \begin{array}{l} \lim_{\epsilon \rightarrow +0} ((L^* - k - i\epsilon)^{-1}h, g) = \lim_{\epsilon \rightarrow +0} ((L^* - k + i\epsilon)^{-1}h, g) \\ \text{for a.e. } k \in \mathbf{R}^1 \text{ and } \forall g \in \mathcal{H} \end{array} \right\}.$$

It is shown that for non-selfadjoint Schrödinger operators $L = -\Delta + q$, $q : \mathbf{R}^n \rightarrow \mathbf{C}$, with potentials decaying like $q(x) = O(|x|^{-\beta})$, $x \in \mathbf{R}^n$, $1 < \beta < \infty$, at infinity, the boundary values $\Theta_L^\pm(k)$ exist for a.e. $k \in \mathbf{R}^1$. The same holds for arbitrary nuclear perturbations $V = V_0 + iW$. In both cases the absolutely continuous subspace $\mathcal{H}^{ac}(L)$ cannot disappear provided for the self-adjoint operator $A = A_0 + V_0$ the absolutely continuous subspace is not empty. For the Schrödinger operator this is well-known.

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1 Introduction

The paper is devoted to the important question how to define the absolutely continuous subspace for a non-selfadjoint operator. We recall that for self-adjoint operators this is done using the spectral measure which corresponds to each self-adjoint operator in a unique manner. However, for non-selfadjoint operators there is no such correspondence. Hence we have to look for a new approach. This new approach was developed in [1] and is essentially due to Tikhonov [27], [28]. For this purpose let us introduce the Hardy classes $H_{\pm}^2(\mathbf{R}^1)$ which consist of those functions which are analytic in $\mathbf{C}_{\pm} = \{z \in \mathbf{C} : \pm \Im(z) > 0\}$ and obey

$$\sup_{\epsilon > 0} \int_{-\infty}^{+\infty} dk |f(k \pm i\epsilon)|^2 < +\infty. \quad (1.1)$$

Note that the Hardy classes can be identified with subspaces of the Hilbert space $L^2(\mathbf{R}^1)$ of square Lebesgue integrable functions. One has

$$L^2(\mathbf{R}^1) = H_+^2(\mathbf{R}^1) \oplus H_-^2(\mathbf{R}^1). \quad (1.2)$$

Let L be a densely defined closed operator on the separable Hilbert space $\mathcal{H}$ such that $\varrho(L) \cap \mathbf{C}_{\pm} \neq \emptyset$ and let

$$\Omega_{g,h}(z) \stackrel{\text{def}}{=} ((L - z)^{-1}g, h), \quad z \in \varrho(L), \quad g, h \in \mathcal{H}. \quad (1.3)$$

We set

$$M(L) = \left\{ g \in \mathcal{H} : \begin{array}{l} \Omega_{g,h}(z) \text{ admits an analytic continuation to } \mathbf{C}_+ \text{ and} \\ \mathbf{C}_- \text{ such that } \Omega_{g,h}(z) \in H_{\pm}^2(\mathbf{R}^1) \text{ for } \forall h \in \mathcal{H} \end{array} \right\}. \quad (1.4)$$

The absolutely continuous subspace $\mathcal{H}^{ac}(L)$ of L is defined as the the closure of $M(L)$, i.e.

$$\mathcal{H}^{ac}(L) \stackrel{\text{def}}{=} \overline{M(L)}. \quad (1.5)$$

The subspace $\mathcal{H}^s(L^*) \stackrel{\text{def}}{=} \mathcal{H} \ominus \mathcal{H}^{ac}(L)$ is called the singular subspace of L^* .

Note that for self-adjoint operators these definitions lead to the usual absolutely continuous and singular subspaces. If $\mathcal{H}$ is a completely non-selfadjoint maximal dissipative operator, then $\mathcal{H}^{ac}(H) = \mathcal{H}_1(H)$ and $\mathcal{H}^s(H^*) = \mathcal{H}_0(H)$. See [1].

If H is a self-adjoint operator, then the scalar-valued function $\Xi_{h,g}(z)$,

$$\Xi_{h,g}(z) \stackrel{\text{def}}{=} \overline{\Omega_{g,h}(\bar{z})} = ((H - z)^{-1}h, g), \quad z \in \mathbf{C}_{\pm} \quad h, g \in \mathcal{H}, \quad (1.6)$$

admits boundary values $\Xi_{h,g}^{\pm}(k)$,

$$\Xi_{h,g}^{\pm}(k) \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow +0} \Xi_{h,g}(k \pm i\epsilon), \quad (1.7)$$

and, moreover, the singular subspace $\mathcal{H}^s(H)$ can be characterized by

$$\mathcal{H}^s(H) = \{h \in \mathcal{H} : \Xi_{h,g}^+(k) = \Xi_{h,g}^-(k) \text{ for a.e. } k \in \mathbf{R}^1 \text{ and } \forall g \in \mathcal{H}\}. \quad (1.8)$$

Naturally the question arises whether this characterization carries over to non-selfadjoint operators. In more detail, let us introduce the scalar-valued function $\Xi_{h,g}(z)$,

$$\Xi_{h,g}(z) \stackrel{\text{def}}{=} \overline{\Omega_{g,h}(\bar{z})} = ((L^* - z)^{-1}h, g), \quad z \in \varrho(L^*), \quad h, g \in \mathcal{H}. \quad (1.9)$$

Hence one has to answer the question whether this function admits a meromorphic continuation to $\mathbf{C}_\pm$ for each $h, g \in \mathcal{H}$ such that the boundary values $\Xi_{h,g}^\pm(k)$ exist for a.e. $k \in \mathbf{R}^1$ and

$$\mathcal{H}^s(L^*) = \{h \in \mathcal{H} : \Xi_{h,g}^+(k) = \Xi_{h,g}^-(k) \text{ for a.e. } k \in \mathbf{R}^1 \text{ and } \forall g \in \mathcal{H}\}. \quad (1.10)$$

In contrast to [23] and [24], where this relation was used as a definition of singular subspace of a non-selfadjoint operator, we are going to verify the relation (1.10) for elements of the singular subspace like in [10], where firstly this relation was proved for bounded non-selfadjoint operators.

In [1] it was shown that this true for maximal dissipative operators which have a discrete spectrum in $\mathbf{C}_+$ and its characteristic function $\Theta_H(z)$, $z \in \varrho(H^*)$, admits boundary values $\Theta_H^-(k) \stackrel{\text{def}}{=} s - \lim_{\epsilon \rightarrow +0} \Theta_H(k - i\epsilon)$ for a.e. $k \in \mathbf{R}^1$.

The aim of this paper is to extend this result to non-selfadjoint operators L of the form

$$L_\kappa = A_0 + V_0 + \frac{i}{2}\alpha\kappa\alpha, \quad (1.11)$$

where A_0 , V_0 , α and κ are self-adjoint operators and $\alpha \geq 0$. For details see Section 2. With respect to scattering theory for non-selfadjoint operators this form of operators was considered by Naboko [11] - [18]. It turns out that under conditions which guarantee the existence of wave operators $W_\pm(A_0 + V_0, L_\kappa)$ [17] the relation (1.10) holds. Moreover, one gets that the absolutely continuous subspace cannot disappear provided $\mathcal{H}^{ac}(A_0 + V_0) \neq \{0\}$.

In outline the proof follows [10]. However, in [10] model consideration were widely used. Since they are difficult to understand for people who are not familiar with them we have avoid models.

However, in contrast to [10] and in accordance with [23] and [23] it turns out that (1.10) holds if the characteristic function $\Theta_L(z)$ admits boundary values of both sides $\Theta_L(k)$ for a.e. $k \in \mathbf{R}^1$.

Furthermore, in [10] an essential assumption was the existence of scalar multiples of certain bounded operator-valued analytic functions. In general those assumptions are difficult to verify. So we have replaced these strong assumptions by weaker ones which need only a certain local behaviour of certain analytic functions near the real axis $\mathbf{R}^1$.

Finally, we apply the results to non-selfadjoint Schrödinger operators. It is shown that if the complex potential $q(\cdot) : \mathbf{R}^n \longrightarrow \mathbf{C}$, $n = 1, 2, \dots$, behaves like $q(x) = O(|x|^{-1-\epsilon})$, $\epsilon > 0$, at infinity, then the singular subspace of the non-selfadjoint Schrödinger operator can be characterized by (1.10). Furthermore, one gets that the absolutely continuous subspace is not empty in this case. In particular, this yields the existence of elements $g \in L^2(\mathbf{R}^n)$, $n = 1, 2, \dots$, such that $((L - z)^{-1}g, h) \in H_{\pm}^2(\mathbf{R}^1)$ for each $g \in \mathcal{H} = L^2(\mathbf{R}^n)$.

2 Perturbed non-selfadjoint operator

In the following we consider non-selfadjoint operators L of the form $L = A_0 + V$ where V is a non-selfadjoint operator which has the structure

$$V = V_0 + iW, \quad (2.1)$$

where V_0 is a selfadjoint operator and W is a symmetric one. Let us assume that W has the representation

$$W = \frac{1}{2}\alpha\kappa\alpha, \quad (2.2)$$

where α is a non-negative self-adjoint operator and κ is a self-adjoint contraction acting from $\mathcal{E}$ into $\mathcal{E}$, $\mathcal{E} = \overline{\text{ran}(\alpha)}$. This means that

$$\text{dom}(W) = \{f \in \text{dom}(\alpha) : \kappa\alpha f \in \text{dom}(\alpha)\} \quad (2.3)$$

and

$$Wf = \frac{1}{2}\alpha\kappa\alpha f, \quad f \in \text{dom}(W). \quad (2.4)$$

If the set $\mathcal{D}_{\kappa}$,

$$\mathcal{D}_{\kappa} = \text{dom}(A) \cap \text{dom}(V_0) \cap \text{dom}(\alpha\kappa\alpha), \quad (2.5)$$

is dense in $\mathcal{H}$, then the operator

$$\tilde{L}_{\kappa}f = A_0f + V_0f + \frac{i}{2}\alpha\kappa\alpha f, \quad f \in \text{dom}(\tilde{L}_{\kappa}) = \mathcal{D}_{\kappa}, \quad (2.6)$$

is densely defined. Since $\mathcal{D}_{\kappa} \subseteq \text{dom}(\tilde{L}_{\kappa}^*)$ the operator $\tilde{L}_{\kappa}$ is closable.

Proposition 2.1 *Let A_0 , V_0 and α , $\alpha \geq 0$, be self-adjoint operators such that*

- (i) $\mathcal{D}_{\kappa} = \text{dom}(A_0) \cap \text{dom}(V_0) \cap \text{dom}(\alpha\kappa\alpha)$ is dense in $\mathcal{H}$,
- (ii) there are non-negative constants a_0 , a , b_0 and b such that

$$0 \leq a_0 + \frac{a}{2} < 1 \quad (2.7)$$

and

$$\left\| \sqrt{|V_0|} \right\|^2 \leq a_0 \|f\|^2 + b_0 \|f\|^2, \quad f \in \mathcal{D}_\kappa, \quad (2.8)$$

$$\|\alpha f\|^2 \leq a \left\| \sqrt{|A_0|} f \right\|^2 + b \|f\|^2, \quad f \in \mathcal{D}_\kappa, \quad (2.9)$$

(iii) $\mathcal{D}_\kappa$ is a core of $\sqrt{|A_0|}$ and α .

Then there is a unique closed extension L_κ of $\tilde{L}_\kappa$ such that

$$\text{dom}(L_\kappa) \subseteq \text{dom} \left(\sqrt{|A_0|} \right) \quad \text{and} \quad \text{dom}(L_\kappa^*) \subseteq \text{dom} \left(\sqrt{|A_0|} \right) \quad (2.10)$$

and $\pm i\eta \in \varrho(L_\kappa)$ for $\eta > 0$ sufficiently large.

Proof: The proof repeats in essentially the considerations of Proposition 5.1 of [1]. As in [1] we introduce the operators $T_0 = \sqrt{|V_0|}(A_0^2 + \eta^2)^{-1/4}$ and $\beta = \alpha(A_0^2 + \eta^2)^{-1/4}$ and we set

$$\begin{aligned} \text{dom}(L_\kappa) = & \quad (2.11) \\ = & \left\{ f \in \text{dom} \left(\sqrt{|A_0|} \right) : \{J_\eta + T_0^* \text{sign}(V_0) T_0 + \frac{i}{2} \beta^* \kappa \beta\} (A_0^2 + \eta^2)^{1/4} f \in \text{dom} \left(\sqrt{|A_0|} \right) \right\}, \end{aligned}$$

where $J_\eta = \text{sign}(A_0 + i\eta) = (A_0 + i\eta)(A_0^2 + \eta^2)^{-1/2}$, and

$$L_\kappa f = -i\eta f + (A_0^2 + \eta^2)^{1/4} \{J_\eta + T_0^* \text{sign}(V_0) T_0 + \frac{i}{2} \beta^* \kappa \beta\} (A_0^2 + \eta^2)^{1/4} f, \quad f \in \text{dom}(L_\kappa). \quad (2.12)$$

From Proposition 5.1 of [1] we know that the operator $\{J_\eta + T_0^* \text{sign}(V_0) T_0 + \frac{i}{2} \beta^* \kappa \beta\}^{-1}$ exists and is bounded. We set

$$Z_\eta = (A_0^2 + \eta^2)^{-1/4} \{J_\eta + T_0^* \text{sign}(V_0) T_0 + \frac{i}{2} \beta^* \kappa \beta\}^{-1} (A_0^2 + \eta^2)^{-1/4}. \quad (2.13)$$

A straightforward computation shows that $\text{ran}(Z_\eta) \subseteq \text{dom}(L_\kappa)$ and

$$(L_\kappa + i\eta) Z_\eta f = f, \quad f \in \mathcal{H}, \quad (2.14)$$

as well as

$$Z_\eta (L_\kappa + i\eta) f = f, \quad f \in \text{dom}(L_\kappa). \quad (2.15)$$

Similar one has $\text{ran}(Z_{-\eta}) \subseteq \text{dom}(L_\kappa)$ and

$$(L_\kappa - i\eta) Z_{-\eta} f = f, \quad f \in \mathcal{H}, \quad (2.16)$$

and

$$Z_{-\eta} (L_\kappa - i\eta) f = f, \quad f \in \text{dom}(L_\kappa). \quad (2.17)$$

Hence $\pm i\eta \in \varrho(L_\kappa)$ for sufficiently large $\eta > 0$.

Since $(L_\kappa + i\eta)^{-1} = Z_\eta$ we obtain $\text{dom}(L_\kappa) \subseteq \text{dom}(\sqrt{|A_0|})$. Taking the adjoint we get $(L_\kappa^* - i\eta)^{-1} = Z_\eta^*$ which yields $\text{dom}(L_\kappa^*) \subseteq \text{dom}(\sqrt{|A_0|})$. The uniqueness can be proved as in Proposition 5.1 of [1]. •

Convention 2.2 *Using the symbol L_κ in the following we mean the closed operator which is determined by Proposition 2.1. In particular this means that the conditions (i) - (iii) of Proposition 2.1 are assumed.*

Corollary 2.3 *The operator L_κ obeys*

$$s - \lim_{\eta \rightarrow +\infty} (\pm i\eta)(L_\kappa \pm i\eta)^{-1} = I. \quad (2.18)$$

Proof: From the proof of Proposition 5.1 of [1] it follows the existence of a constant $\tau > 0$ such that

$$\|T_0\|^2 + \frac{1}{2}\|\beta\|^2 \leq 1 - \tau < 1 \quad (2.19)$$

for $\eta > 0$ sufficiently large. Since $Z_{\pm\eta} = (L_\kappa \pm i\eta)^{-1}$ one gets the estimate

$$\|(L_\kappa \pm i\eta)^{-1}\| \leq \frac{1}{\eta} \left(1 - \|T_0\|^2 - \frac{1}{2}\|\beta\|^2\right)^{-1} \leq \frac{1}{\eta\tau} \quad (2.20)$$

or

$$\|(\pm i\eta)(L_\kappa \pm i\eta)^{-1}\| \leq \frac{1}{\tau} \quad (2.21)$$

for sufficiently large $\eta > 0$. Since

$$s - \lim_{\eta \rightarrow +\infty} (L_\kappa \pm i\eta)^{-1} = 0 \quad (2.22)$$

and

$$\lim_{\eta \rightarrow +\infty} (\pm i\eta)(L_\kappa \pm i\eta)^{-1}f = \lim_{\eta \rightarrow +\infty} \{f - (L_\kappa \pm i\eta)^{-1}L_\kappa f\} = f \quad (2.23)$$

for $f \in \text{dom}(L_\kappa)$ we obtain (2.18) taking into account (2.21). •

Let us introduce the symmetric operator $\tilde{A}$,

$$\tilde{A}f = A_0f + V_0f, \quad f \in \text{dom}(\tilde{A}) = \mathcal{D}_\kappa. \quad (2.24)$$

By Proposition 2.1 there is a unique self-adjoint extension A of $\tilde{A}$ such that

$$\text{dom}(A) \subseteq \text{dom}(\sqrt{|A_0|}). \quad (2.25)$$

The self-adjoint extension A is called the associated pseudo-Friedrichs extension.

By (2.10) the operator-valued functions

$$G(z) = \alpha(L_\kappa - z)^{-1}, \quad z \in \varrho(L_\kappa), \quad (2.26)$$

$$G_*(z) = \alpha(L_\kappa^* - z)^{-1}, \quad z \in \varrho(L_\kappa^*), \quad (2.27)$$

$$B(z) = \alpha(A - z)^{-1}, \quad z \in \varrho(A), \quad (2.28)$$

are well-defined and are analytic families of bounded operators. With respect to (2.26) - (2.28) one gets the representations

$$\begin{aligned} (L_\kappa - z)^{-1} &= \\ &= (A - z)^{-1} - \frac{i}{2} G_*(\bar{z})^* B(z) = (A - z)^{-1} - \frac{i}{2} B(\bar{z})^* G(z), \quad z \in \varrho(A) \cap \varrho(L_\kappa), \end{aligned} \quad (2.29)$$

and

$$\begin{aligned} (L_\kappa^* - z)^{-1} &= \\ &= (A - z)^{-1} + \frac{i}{2} G(\bar{z})^* B(z) = (A - z)^{-1} + \frac{i}{2} B(\bar{z})^* G_*(z), \quad z \in \varrho(A) \cap \varrho(L_\kappa^*). \end{aligned} \quad (2.30)$$

A closed operator is called completely non-selfadjoint if there is no subspace which reduces the operator to a self-adjoint one. Let S_κ be the Cayley transform of L_κ , i.e.

$$S_\kappa = (i\eta - L_\kappa)(i\eta + L_\kappa)^{-1}. \quad (2.31)$$

In terms of the Cayley transform the closed operator L_κ is completely non-selfadjoint if and only if S_κ is completely non-unitary, i.e. if and only if there is no subspace which reduces S_κ to a unitary operator. By [4] the operator S_κ is completely non-unitary if and only if

$$f \perp \bigvee_{n=0}^{\infty} S_\kappa^n \text{ran}(I - S_\kappa S_\kappa^*) \quad \text{and} \quad f \perp \bigvee_{n=0}^{\infty} S_\kappa^{*n} \text{ran}(I - S_\kappa^* S_\kappa) \quad (2.32)$$

yields $f = 0$.

Corollary 2.4 *The operator L_κ is completely non-selfadjoint if and only if*

$$\mathcal{H} = \bigvee_{z \in \varrho(A)} B(\bar{z})^* \text{ran}(\kappa). \quad (2.33)$$

If $\ker(\kappa) = \{0\}$, then this is the case if and only if

$$\mathcal{H} = \bigvee_{z \in \varrho(A)} (A - z)^{-1} \text{ran}(\alpha). \quad (2.34)$$

Proof: A straightforward computation shows that

$$I - S_\kappa S_\kappa^* = 2\eta G_*(i\eta)^* \kappa G_*(i\eta) \quad (2.35)$$

and

$$I - S_\kappa^* S_\kappa = 2\eta G(-i\eta)^* \kappa G(-i\eta). \quad (2.36)$$

Let us calculate $\overline{\text{ran}(G_*(i\eta))}$. Since $\mathcal{D}_\kappa \subseteq \text{dom}(L_\kappa) \subseteq \text{dom}(\alpha)$ and $\mathcal{D}_\kappa$ is a core of α one gets $\overline{\text{ran}(G_*(i\eta))} = \overline{\text{ran}(\alpha)} = \mathcal{E}$. Hence $\overline{\text{ran}(\kappa G_*(i\eta))} = \overline{\text{ran}(\kappa)}$ which yields

$$G_*(i\eta)^* \text{ran}(\kappa) \subseteq \overline{\text{ran}(I - S_\kappa S_\kappa^*)} = \overline{G_*(i\eta)^* \text{ran}(\kappa)}. \quad (2.37)$$

By

$$S_\kappa = -I + 2i\eta(L_\kappa + i\eta)^{-1} \quad (2.38)$$

we obtain

$$\bigvee_{n=0}^{\infty} S_\kappa^n \mathcal{H}_0 = \bigvee_{n=0}^{\infty} (L_\kappa + i\eta)^{-n} \mathcal{H}_0, \quad (2.39)$$

where $\mathcal{H}_0$ is a subspace of $\mathcal{H}$. However this yields

$$\bigvee_{n=0}^{\infty} (L_\kappa + i\eta)^{-n} \mathcal{H}_0 = \bigvee_{z \in \varrho(L_\kappa) \cap \mathcal{U}_{-i\eta}} (L_\kappa - z)^{-1} \mathcal{H}_0, \quad (2.40)$$

where $\mathcal{U}_{-i\eta}$ is that component of the resolvent set of L_κ which contains $-i\eta$. Note that $\mathcal{U}_{-i\eta}$ contains $-i\infty$. Taking into account (2.37) and (2.40) this shows

$$\bigvee_{n=0}^{\infty} S_\kappa^n \overline{\text{ran}(I - S_\kappa S_\kappa^*)} = \quad (2.41)$$

$$= \bigvee_{z \in \varrho(L_\kappa) \cap \mathcal{U}_{-i\eta}} (L_\kappa - z)^{-1} \overline{G_*(i\eta)^* \text{ran}(\kappa)} = \bigvee_{z \in \varrho(L_\kappa) \cap \mathcal{U}_{-i\eta}} (L_\kappa - z)^{-1} G_*(i\eta)^* \text{ran}(\kappa). \quad (2.42)$$

By the identity

$$G_*(\bar{z})^* - G_*(i\eta)^* = (i\eta + z)(L_\kappa - z)^{-1} G_*(i\eta)^*, \quad z \in \varrho(L_\kappa), \quad (2.43)$$

we find

$$\bigvee_{n=0}^{\infty} S_\kappa^n \text{ran}(I - S_\kappa S_\kappa^*) = \bigvee_{z \in \varrho(L_\kappa) \cap \mathcal{U}_{-i\eta}} G_*(\bar{z})^* \text{ran}(\kappa). \quad (2.44)$$

Similarly one proves

$$\bigvee_{n=0}^{\infty} S_\kappa^{*n} \text{ran}(I - S_\kappa^* S_\kappa) = \bigvee_{z \in \varrho(L_\kappa^*) \cap \mathcal{U}_{i\eta}} G(\bar{z})^* \text{ran}(\kappa), \quad (2.45)$$

where $\mathcal{U}_{i\eta}$ is that component of the resolvent set of L_κ^* which contains $i\eta$. Hence the closed operator L_κ is completely non-selfadjoint if and only if

$$f \perp \bigvee_{z \in \varrho(L_\kappa) \cap \mathcal{U}_{-i\eta}} G_*(\bar{z})^* \text{ran}(\kappa) \quad \text{and} \quad f \perp \bigvee_{z \in \varrho(L_\kappa^*) \cap \mathcal{U}_{i\eta}} G(\bar{z})^* \text{ran}(\kappa) \quad (2.46)$$

yields $f = 0$. However using (2.29) and (2.30) one gets

$$\bigvee_{z \in \varrho(L_\kappa) \cap \mathcal{U}_{-i\eta}} G_*(\bar{z})^* \text{ran}(\kappa) = \bigvee_{z \in \mathbf{C}_-} B(\bar{z})^* \text{ran}(\kappa) \quad (2.47)$$

and

$$\bigvee_{z \in \varrho(L_\kappa^*) \cap \mathcal{U}_{i\eta}} G(\bar{z})^* \text{ran}(\kappa) = \bigvee_{z \in \mathbf{C}_+} B(\bar{z})^* \text{ran}(\kappa). \quad (2.48)$$

For details see [1]. However, this shows that L_κ is completely non-selfadjoint if and only if (2.33) is satisfied.

If $\ker(\kappa) = \{0\}$, then $\overline{\text{ran}(\kappa)} = \mathcal{E}$. Therefore

$$\mathcal{H} = \bigvee_{z \in \varrho(A)} B(\bar{z})^* \text{ran}(\kappa) = \bigvee_{z \in \varrho(A)} (A - z)^{-1} \text{ran}(\alpha). \quad (2.49)$$

Hence (2.33) is equivalent to (2.34). •

3 Associated maximal dissipative operator H

Let us assume that the set $\mathcal{D}$,

$$\mathcal{D} = \text{dom}(A_0) \cap \text{dom}(V_0) \cap \text{dom}(\alpha^2), \quad (3.1)$$

is dense in $\mathcal{H}$ and that the assumptions of Proposition 2.1 are satisfied where $\mathcal{D}_\kappa$ is replaced by $\mathcal{D}$. Then the results of Section 5 of [1] can be used. In particular, there is a unique maximal dissipative extension H of the dissipative operator $\tilde{H}$,

$$\tilde{H}f = Af + V_0f + \frac{i}{2}\alpha^2f, \quad f \in \text{dom}(\tilde{H}) = \mathcal{D}, \quad (3.2)$$

such that

$$\text{dom}(H) \subseteq \text{dom}(\alpha) \quad \text{and} \quad \text{dom}(H^*) \subseteq \text{dom}(\alpha) \quad (3.3)$$

and $i\eta \in \varrho(H)$ provided $\eta > 0$ is sufficiently large.

Let us assume that

$$\check{\mathcal{D}}_\kappa = \mathcal{D}_\kappa \cap \mathcal{D} = \text{dom}(A) \cap \text{dom}(V_0) \cap \text{dom}(\alpha\kappa\alpha) \cap \text{dom}(\alpha^2) \quad (3.4)$$

is dense in $\mathcal{H}$ and $\check{\mathcal{D}}_\kappa$ is a core of $\sqrt{|A_0|}$ and α . In general, the closed extension L_κ depends on the chosen domain. Starting with the domain $\mathcal{D}_\kappa$ or with $\check{\mathcal{D}}_\kappa$ we obtain two extensions L_κ and $\check{L}_\kappa$, respectively. However, the relation $\check{\mathcal{D}}_\kappa \subseteq \mathcal{D}_\kappa$ and the uniqueness of the extensions L_κ and $\check{L}_\kappa$ immediately implies that $L_\kappa = \check{L}_\kappa$. Note that the same holds for the pseudo-Friedrichs extension A which is the same for the sets $\mathcal{D}_\kappa$, $\mathcal{D}$ and $\check{\mathcal{D}}_\kappa$. Similarly the operator H is the same for $\mathcal{D}$ and $\check{\mathcal{D}}_\kappa$.

Convention 3.1 *In addition to convention 2.2 we always assume that the set $\check{\mathcal{D}}_\kappa$ is dense in $\mathcal{H}$ and is core of $\sqrt{|A_0|}$ and α . The operator H is called the associated maximal dissipative operator.*

Furthermore, we remark that by Corollary 2.4 and Corollary 5.2 of [1] the closed extension L_κ and the associated maximal dissipative operator H are simultaneously completely non-selfadjoint provided $\ker(\kappa) = \{0\}$.

Assume that the maximal dissipative operator H is completely non-selfadjoint, i.e. condition (2.34) is satisfied. Then the operator-valued function $\Theta_H(\cdot) : \mathbf{C}_+ \longrightarrow \mathcal{B}(\mathcal{E})$,

$$\Theta_H(z) \stackrel{\text{def}}{=} I + i\alpha(H^* - z)^{-1}\alpha, \quad z \in \mathbf{C}_+, \quad (3.5)$$

is called the characteristic function of the maximal dissipative operator H . The function $\Theta_{H^*}(z) \stackrel{\text{def}}{=} (\Theta_H(\bar{z}))^*$, $z \in \mathbf{C}_-$, is called the characteristic function of H^* . This function is related to the characteristic function of the maximal dissipative operator $-H^*$ by

$$\Theta_{\{-H^*\}}(z) = \Theta_{H^*}(-z), \quad z \in \mathbf{C}_+. \quad (3.6)$$

Each maximal dissipative operator admits a minimal self-adjoint dilation [5]. For the maximal dissipative operator H this minimal self-adjoint dilation can be written down explicitly [29]. To this purpose let us introduce the larger Hilbert space $\mathcal{K}$,

$$\mathcal{K} = \begin{array}{c} L^2(\mathbf{R}_-^1, \mathcal{E}) \\ \oplus \\ \mathcal{H} \\ \oplus \\ L^2(\mathbf{R}_+^1, \mathcal{E}) \end{array} \quad (3.7)$$

and the symmetric operator $\tilde{K}$,

$$\text{dom}(\tilde{K}) = \left\{ \begin{pmatrix} f_- \\ g \\ f_+ \end{pmatrix} : \begin{array}{ll} f_\pm \in W_2^1(\mathbf{R}_\pm^1, \mathcal{E}), & g \in \text{dom}(H), \\ f_-(0) \in \text{dom}(\alpha), & f_+(0) - f_-(0) = i\alpha g \end{array} \right\} \quad (3.8)$$

and

$$\tilde{K} \begin{pmatrix} f_- \\ g \\ f_+ \end{pmatrix} = \begin{pmatrix} i \frac{d}{dx} f_-(x) \\ Hg + \alpha f_-(0) \\ i \frac{d}{dx} f_+(x) \end{pmatrix}. \quad (3.9)$$

The symmetric operator $\tilde{K}$ is essentially self-adjoint and its closure K is the desired minimal self-adjoint dilation of H .

For the resolvent $(K - z)^{-1}$ one has the following representations:

$$(K - z)^{-1} \begin{pmatrix} f_- \\ g \\ f_+ \end{pmatrix} = \begin{pmatrix} -i \int_{-\infty}^x dy e^{-iz(x-y)} f_-(y) \\ (H - z)^{-1} g + i(H - z)^{-1} \alpha \int_{-\infty}^0 dy e^{izy} f_-(y) \\ -i \int_0^x dy e^{-iz(x-y)} f_+(y) - i\Theta_{H^*}(z) \int_{-\infty}^0 dy e^{-iz(x-y)} f_-(y) + ie^{-izx} \alpha (H - z)^{-1} g \end{pmatrix} \quad (3.10)$$

for $z \in \mathbb{C}_-$. Taking the adjoint we obtain the representation

$$(K - z)^{-1} \begin{pmatrix} f_- \\ g \\ f_+ \end{pmatrix} = \begin{pmatrix} i \int_x^0 dy e^{iz(y-x)} f_-(y) + i\Theta_H(z) \int_0^\infty dy e^{iz(y-x)} f_+(y) - ie^{-izx} \alpha (H^* - z)^{-1} g \\ (H^* - z)^{-1} g - i(H^* - z)^{-1} \alpha \int_0^\infty dy e^{izy} f_+(y) \\ i \int_x^\infty dy e^{iz(y-x)} f_+(y) \end{pmatrix} \quad (3.11)$$

for $z \in \mathbb{C}_+$.

4 Lax-Phillips scattering theory

Let $\mathcal{K}_0 = L^2(\mathbf{R}^1, \mathcal{E})$. By $\mathcal{D}_-$ and $\mathcal{D}_+$ we denote the subspaces $L^2(\mathbf{R}_-^1, \mathcal{E})$ and $L^2(\mathbf{R}_+^1, \mathcal{E})$, respectively. Note that $\mathcal{K}_0 = \mathcal{D}_- \oplus \mathcal{D}_+$. Further, let us introduce the identification operators $J_\pm : \mathcal{K}_0 \longrightarrow \mathcal{K}$,

$$J_- f = \begin{pmatrix} \chi_{\mathbf{R}_-^1}(x) f(x) \\ 0 \\ 0 \end{pmatrix} \quad (4.1)$$

and

$$J_+ f = \begin{pmatrix} 0 \\ 0 \\ \chi_{\mathbf{R}_+^1} f(x) \end{pmatrix}. \quad (4.2)$$

as well as the differentiation operator K_0 defined by

$$\text{dom}(K_0) = W_2^1(\mathbf{R}^1, \mathcal{E}) \subseteq L^2(\mathbf{R}^1, \mathcal{E}) \quad (4.3)$$

and

$$(K_0 f)(x) = i \frac{d}{dx} f(x), \quad f \in \text{dom}(K_0). \quad (4.4)$$

The operator K_0 is self-adjoint and generates the translation group, i.e.

$$(e^{itK_0}f)(x) = f(x-t), \quad f \in L^2(\mathbf{R}^1, \mathcal{E}). \quad (4.5)$$

We note that

$$e^{itK}J_{\pm}f_{\pm} = J_{\pm}e^{itK_0}f_{\pm}, \quad f_{\pm} \in \mathcal{D}_{\pm}, \quad \pm t \geq 0. \quad (4.6)$$

Using that we obtain the existence of the Lax-Phillips wave operators $W_{\pm}$ [9], [2].

$$W_{\pm} = s - \lim_{t \rightarrow \pm\infty} e^{-itK}J_{\pm}e^{itK_0}. \quad (4.7)$$

The adjoint wave operators admit the representations

$$W_{\pm}^* = s - \lim_{t \rightarrow \pm\infty} e^{-itK_0}J_{\pm}^*e^{itK}. \quad (4.8)$$

Since the strong convergence implies the existence of the Abelian limit one has the representations

$$W_{\pm}^* = s - \lim_{\epsilon \rightarrow +0} \epsilon \int_0^{+\infty} dt e^{-\epsilon t} e^{\mp itK_0} J_{\pm}^* e^{\pm itK}. \quad (4.9)$$

Let $E_0(\cdot)$ be the spectral measure of K_0 . Using spectral integrals [2] we obtain the representations

$$W_{\pm}^* = s - \lim_{\epsilon \rightarrow +0} \pm i\epsilon \int_{-\infty}^{+\infty} dE_0(\lambda) J_{\pm}^* (K - \lambda \pm i\epsilon)^{-1}. \quad (4.10)$$

Using the resolvent formula (3.10) we obtain

$$\begin{aligned} \left(J_+^* (K - \lambda + i\epsilon)^{-1} \begin{pmatrix} f_- \\ g \\ f_+ \end{pmatrix} \right) (x) &= -i \int_0^x dy e^{-i(\lambda - i\epsilon)(x-y)} f_+(y) + \\ &- i\Theta_{H^*}(\lambda - i\epsilon) \int_{-\infty}^0 dy e^{-i(\lambda - i\epsilon)(x-y)} f_-(y) + i e^{-i(\lambda - i\epsilon)x} \alpha(H - \lambda + i\epsilon)^{-1} g. \end{aligned} \quad (4.11)$$

Let $F : L^2(\mathbf{R}^1, \mathcal{E}) \longrightarrow L^2(\mathbf{R}^1, \mathcal{E})$ be the Fourier transformation

$$(Ff)(k) = \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx e^{ikx} f(x), \quad f \in L^2(\mathbf{R}^1, \mathcal{E}). \quad (4.12)$$

A straightforward computation proves that

$$\begin{aligned} \left(FJ_+^* (K - \lambda + i\epsilon)^{-1} \begin{pmatrix} f_- \\ g \\ f_+ \end{pmatrix} \right) (k) &= \\ &= \frac{1}{k - \lambda + i\epsilon} \hat{f}_+(k) + \frac{1}{k - \lambda + i\epsilon} \Theta_{H^*}(\lambda - i\epsilon) \hat{f}_-(\lambda - i\epsilon) - \frac{1}{k - \lambda + i\epsilon} \frac{1}{\sqrt{2\pi}} \alpha(H - \lambda + i\epsilon)^{-1} g. \end{aligned} \quad (4.13)$$

We note that

$$(FE_0(\lambda)F^{-1}h)(k) = \chi_{(-\infty, \lambda)}(k) \hat{h}(k), \quad h \in L^2(\mathbf{R}^1, \mathcal{E}). \quad (4.14)$$

Let $h(\cdot, \cdot) : \mathbf{R}^1 \times \mathbf{R}^1 \longrightarrow \mathcal{E}$ be measurable. If for each $\lambda \in \mathbf{R}^1$ one has $h(\lambda) = h(\lambda, \cdot) \in L^2(\mathbf{R}^1, \mathcal{E})$ and

$$\hat{h}(\lambda, k) = (Fh(\lambda))(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx e^{ikx} h(\lambda, x) \quad (4.15)$$

is continuous and, moreover, the spectral integral

$$h = \int_{-\infty}^{+\infty} E_0(d\lambda) h(\lambda) \in L^2(\mathbf{R}^1, \mathcal{E}) \quad (4.16)$$

exists, then we have

$$\left(F \int_{-\infty}^{+\infty} E_0(d\lambda) h(\lambda) \right) (k) = \hat{h}(k, k). \quad (4.17)$$

Hence we find

$$\begin{aligned} & \left(F \left(i\epsilon \int_{-\infty}^{+\infty} E_0(d\lambda) J_+^*(K - \lambda + i\epsilon)^{-1} \begin{pmatrix} f_- \\ g \\ f_+ \end{pmatrix} \right) \right) (k) = \\ & = \hat{f}_+(k) + \Theta_{H^*}(k - i\epsilon) \hat{f}_-(k - i\epsilon) - \frac{1}{\sqrt{2\pi}} \alpha(H - k + i\epsilon)^{-1} g. \end{aligned} \quad (4.18)$$

Since $\Theta_{H^*}(k - i\epsilon) \hat{f}_-(k - i\epsilon)$ and $\alpha(H - k + i\epsilon)^{-1} g$ converges in the L^2 -sense to $\Theta_{H^*}(k) \hat{f}_-(k)$ and $\alpha(H - k + i0)^{-1} g$ we obtain the representation

$$\left(FW_+^* \begin{pmatrix} f_- \\ g \\ f_+ \end{pmatrix} \right) (k) = \Theta_{H^*}(k) \hat{f}_-(k) - \frac{1}{\sqrt{2\pi}} \alpha(H - k + i0)^{-1} g + \hat{f}_+(k). \quad (4.19)$$

Similarly, we get

$$\left(FW_-^* \begin{pmatrix} f_- \\ g \\ f_+ \end{pmatrix} \right) (k) = \hat{f}_-(k) - \frac{1}{\sqrt{2\pi}} \alpha(H^* - k - i0)^{-1} g + \Theta_H(k) \hat{f}_+(k). \quad (4.20)$$

The operator $S = W_+^* W_-$ is called the Lax-Phillips scattering operator [2]. The scattering operator commutes with the free evolution, i.e.

$$e^{itK_0} S = S e^{itK_0}, \quad t \in \mathbf{R}^1. \quad (4.21)$$

Hence the operator FSF^{-1} commutes with the multiplication operator generated by the independent variable k . Therefore there is a measurable operator-valued function $S(\cdot) : \mathbf{R}^1 \longrightarrow \mathcal{B}(\mathcal{E})$ such that

$$(FSF^{-1}f)(k) = S(k)f(k), \quad f \in L^2(\mathbf{R}^1, \mathcal{E}). \quad (4.22)$$

Here and in the following by $\mathcal{B}(\cdot)$ we denote the set of bounded operators on some Hilbert space. The measurable function $S(\cdot)$ is called the scattering matrix. Since $W_- f_- = J_- f_-$, $f_- \in \mathcal{D}_-$, one gets

$$S f_- = W_+^* J_- f_-, \quad f_- \in \mathcal{D}_-, \quad (4.23)$$

which yields

$$FSf = FW_+^* J_- f_-, \quad f_- \in \mathcal{D}_-. \quad (4.24)$$

Using (4.19) we obtain

$$(FSf_-)(k) = S(k)(Ff_-)(k) = \Theta_{H^*}(k)\hat{f}_-(k), \quad f_- \in \mathcal{D}, \quad (4.25)$$

for a.e. $k \in \mathbf{R}^1$ which proves

$$S(k) = \Theta_{H^*}(k) \quad (4.26)$$

for a.e. $k \in \mathbf{R}^1$.

The wave operators $W_\pm$ are called complete if $\mathcal{R}_\pm \stackrel{\text{def}}{=} \text{ran}(W_\pm) = \mathcal{K}$. If this is not the case, then we have to do with a Lax-Phillips scattering theory with losses. The subspaces $\mathcal{K}_- \stackrel{\text{def}}{=} \mathcal{K} \ominus \mathcal{R}_-$ and $\mathcal{K}_+ \stackrel{\text{def}}{=} \mathcal{K} \ominus \mathcal{K}_+$ are called the emission and absorption subspaces, respectively.

If the wave operators $W_\pm$ are complete, then the scattering operator is a unitary operator, which yields that the scattering matrix $S(k)$ consists of unitary operators for a.e. $k \in \mathbf{R}^1$. In particular this means the the characteristic function $\Theta_H(z)$ is an inner function of both sides and that H belongs to C_{00} (for notation see [5]).

5 Auxiliary subspace

By $H_\pm^2(\mathbf{R}^1, \mathcal{E})$ we denote the so-called Hardy classes which consist of those elements of $L^2(\mathbf{R}^1, \mathcal{E})$ which admit an analytic continuation to the upper or, respectively, lower half plane such that

$$\sup_{\epsilon > 0} \int_{-\infty}^{+\infty} dk \|f(k \pm i\epsilon)\|^2 < \infty. \quad (5.1)$$

Obviously we have

$$L^2(\mathbf{R}^1, \mathcal{E}) = H_-^2(\mathbf{R}^1, \mathcal{E}) \oplus H_+^2(\mathbf{R}^1, \mathcal{E}) \quad (5.2)$$

and

$$FL^2(\mathbf{R}_\pm^1, \mathcal{E}) = H_\pm^2(\mathbf{R}^1, \mathcal{E}) \quad (5.3)$$

Further let us introduce an important subspace of the dilation space $\mathcal{K}$. We set

$$\mathcal{N}(L_\kappa) \stackrel{\text{def}}{=} \{f \in \mathcal{K} : (L_\kappa - z)^{-1} P_\mathcal{H}^\mathcal{K} f = P_\mathcal{H}^\mathcal{K} (K - z)^{-1} f, \quad \forall z \in \varrho(L_\kappa) \cap \mathbf{C}_\pm\} \quad (5.4)$$

Since

$$\mathcal{N}(L_\kappa) = \bigcap_{z \in \varrho(L_\kappa) \cap \mathbf{C}_\pm} \ker \left\{ (L_\kappa - z)^{-1} P_\mathcal{H}^\mathcal{K} - P_\mathcal{H}^\mathcal{K} (K - z)^{-1} \right\} \quad (5.5)$$

the set $\mathcal{N}(L_\kappa)$ is a linear closed subspace of $\mathcal{K}$. Let us prove that $\mathcal{N}(L_\kappa)$ reduces K . This property is equivalent to

$$(K - \xi)^{-1}\mathcal{N}(L_\kappa) \subseteq \mathcal{N}(L_\kappa), \quad \xi \in \mathbf{C}_\pm. \quad (5.6)$$

For each $z, \xi \in \varrho(L_\kappa)$ and $f \in \mathcal{N}(L_\kappa)$ one has

$$\begin{aligned} (L_\kappa - z)^{-1}P_\mathcal{H}^\mathcal{K}(K - \xi)^{-1}f &= (L_\kappa - z)^{-1}(L_\kappa - \xi)^{-1}P_\mathcal{H}^\mathcal{K}f = \\ &= \frac{1}{z - \xi}\{(L_\kappa - z)^{-1} - (L_\kappa - \xi)^{-1}\}P_\mathcal{H}^\mathcal{K}f = \frac{1}{z - \xi}P_\mathcal{H}^\mathcal{K}\{(K - z)^{-1}f - (K - \xi)^{-1}f\} = \\ &= P_\mathcal{H}^\mathcal{K}(K - z)^{-1}(K - \xi)^{-1}f. \end{aligned} \quad (5.7)$$

However this yields that (5.6) holds for each $z \in \mathbf{C}_\pm$. This last fact immediately implies that

$$\overline{P_\mathcal{H}^\mathcal{K}\mathcal{N}(L_\kappa)} \subseteq \mathcal{H}^{ac}(L_\kappa). \quad (5.8)$$

Let us characterize the subspace $\mathcal{N}(L_\kappa)$. To this end we introduce the bounded analytic operator-valued functions $\Theta_+(z)$,

$$\Theta_+(z) \stackrel{\text{def}}{=} I + \frac{i}{2}(I + \kappa)\alpha(H^* - z)^{-1}\alpha : \mathcal{E} \longrightarrow \mathcal{E}, \quad z \in \mathbf{C}_+, \quad (5.9)$$

and

$$\Theta_-(z) \stackrel{\text{def}}{=} I - \frac{i}{2}(I - \kappa)\alpha(H - z)^{-1}\alpha : \mathcal{E} \longrightarrow \mathcal{E}, \quad z \in \mathbf{C}_-. \quad (5.10)$$

Lemma 5.1 [10] *Let $\ker(\kappa) = \{0\}$. Assume that the operator L_κ is completely non-self-adjoint. One has $g \in P_\mathcal{H}^\mathcal{K}\mathcal{N}(L_\kappa)$ if and only if there are elements $\hat{f}_\pm \in H_\pm^2(\mathbf{R}^1, \mathcal{E})$ such that*

$$\frac{1}{2\sqrt{2\pi}}(I + \kappa)\alpha(H^* - z)^{-1}g = \Theta_+(z)\hat{f}_+(z), \quad z \in \mathbf{C}_+, \quad (5.11)$$

and

$$\frac{1}{2\sqrt{2\pi}}(I - \kappa)\alpha(H - z)^{-1}g = \Theta_-(z)\hat{f}_-(z), \quad z \in \mathbf{C}_-. \quad (5.12)$$

The element $f \in \mathcal{N}(L_\kappa)$ obeying $P_\mathcal{H}^\mathcal{K}f = g$ is uniquely given by $f = (f_-, g, f_+)$, $f_\pm = F^{-1}\hat{f}_\pm$.

Proof: From (5.4) and (3.11) one gets

$$(L_\kappa - z)^{-1}g = (H^* - z)^{-1}g - i\sqrt{2\pi}(H^* - z)^{-1}\alpha\hat{f}_+(z), \quad z \in \varrho(L_\kappa) \cap \mathbf{C}_+, \quad (5.13)$$

where

$$\hat{f}_+(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dy e^{izy} f(y). \quad (5.14)$$

and $f = (f_-, g, f_+)$. Using the identity

$$(L_\kappa - z)^{-1}g = (H^* - z)^{-1}g - \frac{i}{2}(H^* - z)^{-1}\alpha(I + \kappa)\alpha(L_\kappa - z)^{-1}g, \quad z \in \varrho(L_\kappa) \cap \mathbf{C}_+, \quad (5.15)$$

we obtain

$$(H^* - z)^{-1}\alpha\{\hat{f}_+(z) - \frac{1}{2\sqrt{2\pi}}(I + \kappa)\alpha(L_\kappa - z)^{-1}g\} = 0, \quad z \in \varrho(L_\kappa) \cap \mathbf{C}_+. \quad (5.16)$$

Since $\overline{\text{ran}(\alpha(H - z)^{-1})} = \mathcal{E}$, $z \in \mathbf{C}_+$, one has

$$\hat{f}_+(z) = \frac{1}{2\sqrt{2\pi}}(I + \kappa)\alpha(L_\kappa - z)^{-1}g, \quad z \in \varrho(L_\kappa) \cap \mathbf{C}_+. \quad (5.17)$$

Inserting (5.13) into (5.17) we find

$$\hat{f}_+(z) = \frac{1}{2\sqrt{2\pi}}(I + \kappa)\alpha(H^* - z)^{-1}g - \frac{i}{2}(I + \kappa)\alpha(H^* - z)^{-1}\alpha\hat{f}_+(z), \quad z \in \varrho(L_\kappa) \cap \mathbf{C}_+. \quad (5.18)$$

which is equivalent to (5.11). Similarly, we prove (5.12).

Conversely, multiplying on the left (5.15) by $(I + \kappa)\alpha$ we obtain

$$\Theta_+(z)(I + \kappa)\alpha(L_\kappa - z)^{-1}g = (I + \kappa)\alpha(H^* - z)^{-1}g. \quad (5.19)$$

Using (5.11) one gets

$$\Theta_+(z)\{\hat{f}_+(z) - \frac{1}{2\sqrt{2\pi}}(I + \kappa)\alpha(L_\kappa - z)^{-1}g\} = 0. \quad (5.20)$$

Note that

$$\Theta_+(z)^{-1} = I - \frac{i}{2}(I + \kappa)\alpha(L_\kappa - z)^{-1}\alpha, \quad z \in \varrho(L_\kappa) \cap \mathbf{C}_+. \quad (5.21)$$

Since $\Theta_+(z)$ is invertible for each $z \in \varrho(L_\kappa) \cap \mathbf{C}_+$ we find (5.17). Inserting (5.17) into (5.15) we obtain

$$(L_\kappa - z)^{-1}g = (H^* - z)^{-1}g - i\sqrt{2\pi}(H^* - z)^{-1}\alpha\hat{f}_+(z), \quad z \in \varrho(L_\kappa) \cap \mathbf{C}_+, \quad (5.22)$$

which is equivalent to $(L_\kappa - z)^{-1}P_H^\kappa f = (K - z)^{-1}f$, $z \in \varrho(L_\kappa) \cap \mathbf{C}_+$, where $f = (f_-, g, f_+)$, $f_\pm = F^{-1}\hat{f}_\pm$. Similarly we prove that (5.12) yields $(L_\kappa - z)^{-1}P_H^\kappa f = P_H^\kappa(K - z)^{-1}f$, $z \in \varrho(L_\kappa) \cap \mathbf{C}_-$.

It remains to show the uniqueness. Assume that there are two elements $f = (f_-, g, f_+)$ and $\tilde{f} = (k_-, g, k_+)$ of $\mathcal{N}(L_\kappa)$. Taking into account (5.17) this implies $\hat{f}_+(z) = \hat{k}_+(z)$, $z \in \varrho(L_\kappa) \cap \mathbf{C}_+$. Hence $f_+ = k_+$. Similarly, we prove $f_- = k_-$. $\bullet$

From (5.9) and (5.10) we find the representations

$$2\Theta_+(z) = (I - \kappa) + (I + \kappa)\Theta_H(z), \quad z \in \mathbf{C}_+, \quad (5.23)$$

and

$$2\Theta_-(z) = (I + \kappa) + (I - \kappa)\Theta_{H^*}(z), \quad z \in \mathbf{C}_-. \quad (5.24)$$

Since the boundary values $\Theta_H(k) = s\text{-}\lim_{\epsilon \rightarrow +0} \Theta_H(k+i\epsilon)$ and $\Theta_{H^*}(k) = s\text{-}\lim_{\epsilon \rightarrow +0} \Theta_{H^*}(k-i\epsilon)$ exist for a.e. $k \in \mathbf{R}^1$ one gets the existence of the boundary values $\Theta_{\pm}(k) \stackrel{\text{def}}{=} s\text{-}\lim_{\epsilon \rightarrow +0} \Theta_{\pm}(k \pm i\epsilon)$ as well as the relations

$$2\Theta_+(k) = (I - \kappa) + (I + \kappa)\Theta_H(k) \quad (5.25)$$

and

$$2\Theta_-(k) = (I + \kappa) + (I - \kappa)\Theta_{H^*}(k) \quad (5.26)$$

for a.e. $k \in \mathbf{R}^1$.

Lemma 5.1 allows to characterize the elements of the auxiliary subspace $\mathcal{N}(L_\kappa)$ in a new manner. To this end let us introduce the multiplication operator $\hat{\kappa}$,

$$(\hat{\kappa}f)(k) = \kappa f(k), \quad f \in L^2(\mathbf{R}^1, \mathcal{E}). \quad (5.27)$$

Note that $F\hat{\kappa} = \hat{\kappa}F$, i.e. the multiplication operator $\hat{\kappa}$ commutes with the Fourier transformation.

Proposition 5.2 *Let $\ker \kappa = \{0\}$. Assume that the operator L_κ is completely non-selfadjoint. Further, let*

$$G \stackrel{\text{def}}{=} (I + \hat{\kappa})W_-^* + (I - \hat{\kappa})W_+^* : \mathcal{K} \longrightarrow \mathcal{K}_0. \quad (5.28)$$

Then $\ker(G) = \mathcal{N}(L_\kappa)$.

Proof: Using (4.19) and (4.20) we find

$$\begin{aligned} (FGf)(k) &= \\ &= (I + \kappa)\hat{f}_-(k) - (I + \kappa)\frac{1}{\sqrt{2\pi}}\alpha(H^* - k - i0)^{-1}g + (I + \kappa)\Theta_H(k)\hat{f}_+(k) + \\ &\quad + (I - \kappa)\Theta_{H^*}(k)\hat{f}_-(k) - (I - \kappa)\frac{1}{\sqrt{2\pi}}\alpha(H - k + i0)^{-1}g + (I - \kappa)\hat{f}_+(k). \end{aligned} \quad (5.29)$$

By definitions (5.9) and (5.10) one gets

$$\begin{aligned} (FGf)(k) &= \\ &= 2\Theta_-(k)\hat{f}_-(k) - (I - \kappa)\frac{1}{\sqrt{2\pi}}\alpha(H - k + i0)^{-1}g + \\ &\quad + 2\Theta_+(k)\hat{f}_+(k) - (I + \kappa)\frac{1}{\sqrt{2\pi}}\alpha(H^* - k - i0)^{-1}g \end{aligned} \quad (5.30)$$

If $FGf = 0$, then it follows

$$\begin{aligned} \Theta_-(k)\hat{f}_-(k) - \frac{1}{2\sqrt{2\pi}}(I - \hat{\kappa})\alpha(H - k + i0)^{-1}g &= \\ &= -\{\Theta_+(k)\hat{f}_+(k) - \frac{1}{2\sqrt{2\pi}}(I + \kappa)\alpha(H^* - k - i0)^{-1}g\}. \end{aligned} \quad (5.31)$$

The left hand side belongs to $H_-^2(\mathbf{R}^1, \mathcal{E})$ while the right hand side to $H_+^2(\mathbf{R}^1, \mathcal{E})$. Since $H_-^2(\mathbf{R}^1, \mathcal{E}) \perp H_+^2(\mathbf{R}^1, \mathcal{E})$ we obtain (5.11) and (5.12). Hence $f \in \mathcal{N}(L_\kappa)$.

Conversely, if $f \in \mathcal{N}(L_\kappa)$, then (5.11) and (5.12) are valid. Hence it holds (5.31) which is equivalent (5.29). Therefore $FGf = 0$, i.e. $f \in \ker(G)$. $\bullet$

With the operator-valued functions $\Theta_\pm(k)$, $k \in \mathbf{R}^1$, we associate the multiplication operators $\hat{\Theta}_\pm : L^2(\mathbf{R}^1, \mathcal{E}) \longrightarrow L^2(\mathbf{R}^1, \mathcal{E})$ defined by

$$(\hat{\Theta}_\pm f)(k) = \Theta_\pm(k)f(k), \quad f \in L^2(\mathbf{R}^1, \mathcal{E}). \quad (5.32)$$

Note that the operators $\hat{\Theta}_\pm$ are bounded ones. Furthermore, we set

$$\Theta_\pm = F^{-1}\hat{\Theta}_\pm F. \quad (5.33)$$

Obviously, the differentiation operator K_0 commutes with $\Theta_\pm$, i.e.

$$e^{itK_0}\Theta_\pm = \Theta_\pm e^{itK_0}, \quad t \in \mathbf{R}^1. \quad (5.34)$$

From (5.25), (5.26) and (4.26) we obtain

$$2\Theta_+ = (I - \hat{\kappa}) + (I + \hat{\kappa})S^* \quad (5.35)$$

and

$$2\Theta_- = (I + \hat{\kappa}) + (I - \hat{\kappa})S. \quad (5.36)$$

Hence we find

$$\Theta_- - \Theta_+ S = \frac{1}{2}(I + \hat{\kappa})(I - S^* S). \quad (5.37)$$

Let us consider the operator Π ,

$$\Pi = W_- - W_+ S : \mathcal{K}_0 \longrightarrow \mathcal{K}. \quad (5.38)$$

A straightforward computation shows that

$$\|\Pi f\|^2 = \|\sqrt{I - S^* S}f\|^2, \quad f \in \mathcal{K}_0. \quad (5.39)$$

Hence there is an isometry $\Gamma : \overline{\text{ran}(\sqrt{I - S^* S})} \longrightarrow \mathcal{K}$ such that

$$\Pi f = \Gamma \sqrt{I - S^* S}f, \quad f \in \mathcal{K}_0. \quad (5.40)$$

Since $(\Pi f, W_+ g) = 0$, $f, g \in \mathcal{K}_0$, one has $\text{ran}(\Gamma) \perp \text{ran}(W_+)$. By the minimality of the self-adjoint dilation K we have

$$\text{ran}(W_-) \vee \text{ran}(W_+) = \mathcal{K} \quad (5.41)$$

which yields

$$\text{ran}(\Gamma) \vee \text{ran}(W_+) = \text{ran}(\Pi) \vee \text{ran}(W_+) = \text{ran}(W_-) \vee \text{ran}(W_+) = \mathcal{K}. \quad (5.42)$$

Hence $\text{ran}(\Gamma) \oplus \text{ran}(W_+) = \mathcal{K}$ or, equivalently,

$$\Gamma\Gamma^* + W_+W_+^* = I_{\mathcal{K}}. \quad (5.43)$$

In particular, one has $\text{ran}(\Gamma) = \mathcal{K}_+$.

In general the operators $\Theta_{\pm}$ are not invertible. However, in the following we assume that for a.e. $k \in \mathbf{R}^1$ the inverse operators $(\Theta_{\pm}(k))^{-1}$ exist and are bounded. We set

$$\Omega_{\pm}(k) \stackrel{\text{def}}{=} (\Theta_{\pm}(k))^{-1}, \quad k \in \mathbf{R}^1. \quad (5.44)$$

Since $\Omega_{\pm}(k)$ are measurable families of bounded operators they generate multiplication operators $\hat{\Omega}_{\pm}$ by

$$\text{dom}(\hat{\Omega}_{\pm}) = \{f \in L^2(\mathbf{R}^1, \mathcal{E}) : \Omega_{\pm}(k)f(k) \in L^2(\mathbf{R}^1, \mathcal{E})\} \quad (5.45)$$

and

$$(\hat{\Omega}_{\pm}f)(k) = \Omega_{\pm}(k)f(k), \quad f \in \text{dom}(\hat{\Omega}_{\pm}), \quad (5.46)$$

which are densely defined and closed but in general not bounded. However they are locally bounded. This means the following. Let $\hat{\delta}$ be the bounded multiplication operator which is induced by $\delta \in L^{\infty}(\mathbf{R}^1)$ and let

$$L_{\Omega}^{\infty} \stackrel{\text{def}}{=} \{\delta \in L^{\infty}(\mathbf{R}^1) : \hat{\Omega}_{\pm}\hat{\delta} \in \mathcal{B}(L^2(\mathbf{R}^1, \mathcal{E}))\}. \quad (5.47)$$

Note that there is always a sequence $\{\delta_n\}_{n=1}^{\infty}$, $\delta_n \in L_{\Omega}^{\infty}(\mathbf{R}^1)$ obeying

$$\|\delta_n\|_{L^{\infty}(\mathbf{R}^1)} \leq 1, \quad n = 1, 2, \dots, \quad (5.48)$$

such that $\lim_{n \rightarrow \infty} \delta_n(k) = 1$ for a.e. $k \in \mathbf{R}^1$. Setting $\Omega_{\pm} = F^{-1}\hat{\Omega}_{\pm}F$ one gets the characterization

$$L_{\Omega}^{\infty}(\mathbf{R}^1) = \{\delta \in L^{\infty}(\mathbf{R}^1) : \Omega_{\pm}\delta(K_0) \in \mathcal{B}(L^2(\mathbf{R}^1, \mathcal{E}))\}. \quad (5.49)$$

Lemma 5.3 *Let $\ker(\kappa) = \{0\}$. Assume that the operator L_{κ} is completely non-selfadjoint. If for a.e. $k \in \mathbf{R}^1$ the inverse operators $(\Theta_{+}(k))^{-1}$ exist and are bounded, then for each $\delta \in L_{\Omega}^{\infty}(\mathbf{R}^1)$ the operator T_{δ} ,*

$$T_{\delta} = \begin{pmatrix} -\frac{1}{2}W_+[\Omega_+\delta(K_0)](1 + \hat{\kappa})\sqrt{I - S^*S}\Gamma^* \\ \delta(K) \end{pmatrix} : \mathcal{K}_+ \longrightarrow \begin{matrix} \mathcal{R}_+ \\ \oplus \\ \mathcal{K}_+ \end{matrix} = \mathcal{K}, \quad (5.50)$$

is bounded. Moreover, if $f \in \ker(G)$, then $\delta(K)f \in \text{ran}(T_\delta)$. Conversely, if $f \in \text{ran}(T_\delta)$, then $f \in \ker(G)$.

Proof: Using (5.35) and (5.38) we find

$$\begin{aligned} Gf &= (I + \hat{\kappa})(W_-^* - S^*W_+^*)f + \{(I - \hat{\kappa}) + (I + \hat{\kappa})S^*\}W_+^*f \\ &= (I + \hat{\kappa})\sqrt{I - S^*S}\Gamma^*f + 2\Theta_+W_+^*f, \quad f \in \mathcal{K}. \end{aligned} \quad (5.51)$$

Hence $Gf = 0$ if and only if

$$\Theta_+W_+^*f = -\frac{1}{2}(I + \hat{\kappa})\sqrt{I - S^*S}\Gamma^*f, \quad f \in \ker(G). \quad (5.52)$$

Note that $G\delta(K) = \delta(K_0)G$, $\delta \in L^\infty(\mathbf{R}^1)$. Hence $Gf = 0$ implies $G\delta(K)f = 0$. Applying (5.52) we find

$$\Theta_+W_+^*\delta(K)f = -\frac{1}{2}\delta(K_0)(I + \hat{\kappa})\sqrt{I - S^*S}\Gamma^*f, \quad (5.53)$$

where we have used $\delta(K)\Gamma = \Gamma\delta(K_0)$. Multiplying on the left by Ω_+ and using $\Omega_+\Theta_+ = I$ one gets

$$W_+^*\delta(K)f = -\frac{1}{2}[\Omega_+\delta(K_0)](I + \hat{\kappa})\sqrt{I - S^*S}\Gamma^*f. \quad (5.54)$$

Finally, multiplying on the left by W_+ we obtain

$$W_+W_+^*\delta(K)f = -\frac{1}{2}W_+[\Omega_+\delta(K_0)](I + \hat{\kappa})\sqrt{I - S^*S}\Gamma^*f. \quad (5.55)$$

By (5.43) one has

$$W_+W_+^*\delta(K)f + \Gamma\Gamma^*\delta(K)f = \delta(K)f. \quad (5.56)$$

Setting $\Gamma\Gamma^*f = h \in \text{ran}(\Gamma)$ from (5.55) we find

$$W_+W_+^*\delta(K)f = -\frac{1}{2}W_+[\Omega_+\delta(K_0)](I + \hat{\kappa})\sqrt{I - S^*S}\Gamma^*h. \quad (5.57)$$

From $\Gamma\Gamma^*\delta(K) = \delta(K)\Gamma\Gamma^*$ we conclude

$$\Gamma\Gamma^*\delta(K)f = \delta(K)h. \quad (5.58)$$

Both relations (5.57) and (5.58) imply $\delta(K)f \in \text{ran}(T_\delta)$.

Conversely, let $f \in \text{ran}(T_\delta)$. Then there is an element $h \in \text{ran}(\Gamma)$ such that

$$W_+W_+^*f = -\frac{1}{2}W_+[\Omega_+\delta(K_0)](I + \hat{\kappa})\sqrt{I - S^*S}\Gamma^*h \quad (5.59)$$

and

$$\Gamma\Gamma^*f = \delta(K)h. \quad (5.60)$$

Multiplying on the left by W_+^* from (5.59) we find

$$W_+^* f = -\frac{1}{2}[\Omega_+ \delta(K_0)](I + \hat{\kappa})\sqrt{I - S^* S} \Gamma^* h. \quad (5.61)$$

Multiplying on the left by Θ_+ we obtain

$$\Theta_+ W_+^* f = -\frac{1}{2}\delta(K_0)(I + \hat{\kappa})\sqrt{I - S^* S} \Gamma^* h. \quad (5.62)$$

Using (5.60) one has

$$\Theta_+ W_+^* f = -\frac{1}{2}(I + \hat{\kappa})\sqrt{I - S^* S} \Gamma^* \delta(K) h = -\frac{1}{2}(I + \hat{\kappa})\sqrt{I - S^* S} \Gamma^* f. \quad (5.63)$$

By (5.52) one gets $f \in \ker(G)$. •

Remark 5.4 *A similar lemma holds assuming that the inverse operators $(\Theta_-(k))^{-1}$ exist and are bounded for a.e. $k \in \mathbf{R}^1$.*

Let us calculate the kernel of T_δ^* .

Lemma 5.5 *Let $\ker(\kappa) = \{0\}$. Assume that the operator L_κ is completely non-selfadjoint. If for a.e. $k \in \mathbf{R}^1$ the inverse operators $(\Theta_\pm(k))^{-1}$ exist and are bounded, then for each $\delta \in L_\Omega^\infty(\mathbf{R}^1)$ the operator D_δ ,*

$$D_\delta \stackrel{\text{def}}{=} [\Omega_- \delta(K_0)]^* W_-^* - [\Omega_+ \delta(K_0)]^* W_+^*, \quad (5.64)$$

is bounded. Moreover, we have $f \in \ker(T_\delta^)$ if and only if $f \in \ker(D_\delta)$.*

Proof: One has $f \in \ker(T_\delta^*)$ if and only if

$$\delta(K)^*[I - W_+ W_+^*]f = \frac{1}{2}\Gamma\sqrt{I - S^* S}(I + \hat{\kappa})[\Omega_+ \delta(K_0)]^* W_+^* f. \quad (5.65)$$

Let $f \in \ker(T_\delta^*)$. By (5.38) and (5.40) we find

$$[I - W_+ W_+^*]\delta(K)^* f = \frac{1}{2}(W_- - W_+ S)(I + \hat{\kappa})[\Omega_+ \delta(K_0)]^* W_+^* f. \quad (5.66)$$

Multiplying on the left by W_-^* we obtain

$$[W_-^* - S^* W_+^*]\delta(K)^* f = \frac{1}{2}(I - S^* S)(I + \hat{\kappa})[\Omega_+ \delta(K_0)]^* W_+^* f \quad (5.67)$$

which yields

$$\delta(K_0)^* W_-^* f = \{S^* \delta(K_0)^* + \frac{1}{2}(I - S^* S)(I + \hat{\kappa})[\Omega_+ \delta(K_0)]^*\} W_+^* f. \quad (5.68)$$

Since $\delta(K_0)^* = \Theta_+^*[\Omega_+\delta(K_0)]^*$ one has

$$\delta(K_0)^*W_-^*f = \{S^*\Theta_+^* + \frac{1}{2}(I - S^*S)(I + \hat{\kappa})\}[\Omega_+\delta(K_0)]^*W_+^*f. \quad (5.69)$$

By (5.37) one gets

$$\delta(K_0)^*W_-^*f = \Theta_-^*[\Omega_+\delta(K_0)]^*W_+^*f. \quad (5.70)$$

Multiplying on the left by Ω_-^* we find

$$[\Omega_-\delta(K_0)]^*W_-^*f = [\Omega_+\delta(K_0)]^*W_+^*f \quad (5.71)$$

which is equivalent to $D_\delta f = 0$.

Conversely, if $D_\delta f = 0$, then (5.71) holds. Starting from (5.71) we prove (5.67) using considerations before. Taking into account (5.40) from (5.67) we obtain

$$\Gamma^*\delta(K)^*f = \frac{1}{2}\sqrt{I - S^*S}(I + \hat{\kappa})[\Omega_+\delta(K_0)]^*W_+^*f. \quad (5.72)$$

Finally, multiplying on the left by Γ and using (5.43) one gets (5.65) which is equivalent to $f \in \ker(T_\delta^*)$. •

The subspace $\mathcal{N}(L_\kappa)$ reduces the self-adjoint operator K . In particular, this yields that for each $\delta \in L^\infty(\mathbf{R}^1)$ and $f \in \mathcal{N}(L_\kappa)$ one has $\delta(K)f \in \mathcal{N}(L_\kappa)$. By this remark, Lemma 5.3 and 5.5 we obtain a characterization of $\mathcal{N}(L_\kappa)^\perp$.

Proposition 5.6 *Let $\ker(\kappa) = \{0\}$. Assume that L_κ is completely non-selfadjoint. If for a.e. $k \in \mathbf{R}^1$ the inverse operators $(\Theta_\pm(k))^{-1}$ exist and are bounded, then*

$$\mathcal{N}(L_\kappa)^\perp = \bigcap_{\delta \in L_\Omega^\infty(\mathbf{R}^1)} \ker(D_\delta). \quad (5.73)$$

Proof. Note that there is a sequence $\{\delta_n\}_{n=1}^\infty$, $\delta_n \in L_\Omega^\infty(\mathbf{R}^1)$, obeying (5.48) and $\lim_{n \rightarrow \infty} \delta_n(k) = 1$ for a.e. $k \in \mathbf{R}^1$. Since K is absolutely continuous we have $s - \lim_{n \rightarrow \infty} \delta_n(K) = I$. Let

$$\mathcal{N}_\delta(L_\kappa) \stackrel{\text{def}}{=} \delta(K)\mathcal{N}(L_\kappa), \quad \delta \in L_\Omega^\infty(\mathbf{R}^1). \quad (5.74)$$

Since $\mathcal{N}(L_\kappa)$ reduces K it holds $\mathcal{N}_\delta(L_\kappa) \subseteq \mathcal{N}(L_\kappa)$. Obviously

$$\overline{\bigcup_{\delta \in L_\Omega^\infty(\mathbf{R}^1)} \mathcal{N}_\delta(L_\kappa)} = \mathcal{N}(L_\kappa). \quad (5.75)$$

By Proposition 5.2 and Lemma 5.3 one gets

$$\mathcal{N}_\delta(L_\kappa) \subseteq \text{ran}(T_\delta) \subseteq \mathcal{N}(L_\kappa). \quad (5.76)$$

Hence

$$\mathcal{N}(L_\kappa) = \overline{\bigcup_{\delta \in L_\Omega^\infty(\mathbf{R}^1)} \text{ran}(T_\delta)}. \quad (5.77)$$

Therefore

$$\mathcal{N}(L_\kappa)^\perp = \bigcap_{\delta \in L_\Omega^\infty(\mathbf{R}^1)} \ker(T_\delta^*). \quad (5.78)$$

Taking into account Lemma 5.5 we obtain (5.73). •

6 Absolutely continuous and singular subspaces

Let us characterize the subspace,

$$\tilde{\mathcal{H}}^s(L_\kappa^*) \stackrel{\text{def}}{=} \mathcal{H} \ominus \overline{P_{\mathcal{H}}^\kappa \mathcal{N}(L_\kappa)} \quad (6.1)$$

The spectrum of a closed operator is called discrete in some region if it consists only of isolated eigenvalues with finite-dimensional Riesz projections there. This is the case if and only if the resolvent is a meromorphic operator-valued function in this region, i.e. an analytic family of bounded operators with isolated singularities such that the coefficients of negative powers of the Laurent expansion in the neighborhoods of the singularities are finite-dimensional operators.

Lemma 6.1 *Let $\ker \kappa = \{0\}$. Assume that L_κ is completely non-selfadjoint and that the spectrum of L_κ is discrete in $\mathbf{C}_\pm$. If for a.e. $k \in \mathbf{R}^1$ there is a $\epsilon(k) > 0$ such that for $\epsilon \in [0, \epsilon(k))$ the inverse operators $(\Theta_\pm(k \pm i\epsilon))^{-1}$ exist and are uniformly bounded, then for each $h, g \in \mathcal{H}$ the meromorphic function $\Xi_{h,g}(z)$, $z \in \varrho(L_\kappa^*)$, admits boundary values $\Xi_{h,g}^\pm(k) = \lim_{\epsilon \rightarrow +0} \Xi_{h,g}(k \pm i\epsilon)$ for a.e. $k \in \mathbf{R}^1$.*

Proof. Since the spectrum of L_κ is discrete in $\mathbf{C}_+$ from the representation (5.21) one easily gets that $\Omega_+(z) \stackrel{\text{def}}{=} \Theta_+(z)^{-1}$ is meromorphic in $\mathbf{C}_+$. Similarly, using

$$(\Theta_-(z))^{-1} = I + \frac{i}{2}(I - \kappa)\alpha(L_\kappa - z)^{-1}\alpha, \quad z \in \varrho(L_\kappa) \cap \mathbf{C}_-, \quad (6.2)$$

we obtain the same for $\Omega_-(z) \stackrel{\text{def}}{=} (\Theta_-(z))^{-1}$ in the lower half plane. Furthermore, we have the representation

$$\Omega_\pm(k \pm i\epsilon) - \Omega_\pm(k) = \Omega_\pm(k \pm i\epsilon)\{\Theta_\pm(k) - \Theta_\pm(k \pm i\epsilon)\}\Omega_\pm(k) \quad (6.3)$$

for a.e. $k \in \mathbf{R}^1$. By the convergence $s - \lim_{\epsilon \rightarrow +0} \Theta_\pm(k \pm i\epsilon) = \Theta_\pm(k)$ and the uniform boundedness $\{\Omega_\pm(k \pm i\epsilon)\}_{\epsilon \in [0, \epsilon(k))}$ we find

$$s - \lim_{\epsilon \rightarrow +0} \Omega_\pm(k \pm i\epsilon) = \Omega_\pm(k) \quad (6.4)$$

for a.e. $k \in \mathbf{R}^1$.

Multiplying the identity

$$(L_\kappa^* - z)^{-1} = (H - z)^{-1} + \frac{i}{2}(H - z)^{-1}\alpha(I + \kappa)\alpha(L_\kappa^* - z)^{-1}, \quad z \in \varrho(L_\kappa^*) \cap \mathbf{C}_-, \quad (6.5)$$

on the left by α we find

$$\Theta_+(\bar{z})^*\alpha(L_\kappa^* - z)^{-1} = \alpha(H - z)^{-1}, \quad z \in \varrho(L_\kappa^*) \cap \mathbf{C}_-. \quad (6.6)$$

Hence one gets

$$\alpha(L_\kappa^* - z)^{-1} = \Omega_+(\bar{z})^*\alpha(H - z)^{-1}, \quad z \in \varrho(L_\kappa^*) \cap \mathbf{C}_-. \quad (6.7)$$

Similarly one proves

$$\alpha(L_\kappa^* - z)^{-1} = \Omega_-(\bar{z})^*\alpha(H^* - z)^{-1}, \quad z \in \varrho(L_\kappa^*) \cap \mathbf{C}_+. \quad (6.8)$$

Using (6.5) we find

$$\begin{aligned} \Xi_{h,g}(z) &= \\ &= ((H - z)^{-1}h, g) + \frac{i}{2}(\alpha(L_\kappa^* - z)^{-1}h, (I + \kappa)\alpha(H^* - \bar{z})^{-1}g), \quad z \in \varrho(L_\kappa^*) \cap \mathbf{C}_- \end{aligned} \quad (6.9)$$

Inserting (6.7) into (6.9) we finally obtain

$$\begin{aligned} \Xi_{h,g}(z) &= \\ &= ((H - z)^{-1}h, g) + \frac{i}{2}(\Omega_+(\bar{z})^*\alpha(H - z)^{-1}h, (I + \kappa)\alpha(H^* - \bar{z})^{-1}g), \quad z \in \varrho(L_\kappa^*) \cap \mathbf{C}_-. \end{aligned} \quad (6.10)$$

By (6.4) we obtain the existence of the boundary value $\Xi_{h,g}^+(k)$, $h, g \in \mathcal{H}$, for a.e. $k \in \mathbf{R}^1$. Similarly we prove the existence of the boundary values $\Xi_{h,g}^-(k)$, $h, g \in \mathcal{H}$, for a.e. $k \in \mathbf{R}^1$. •

Next let us prove that $\Xi_{h,g}^+(k) = \Xi_{h,g}^-(k)$ for each $h \in \tilde{\mathcal{H}}^s(L_\kappa^*)$ and $g \in \mathcal{H}$ and for a.e. $k \in \mathbf{R}^1$.

Proposition 6.2 *Let $\ker \kappa = \{0\}$. Assume that L_κ is completely non-selfadjoint and that the spectrum of L_κ is discrete in $\mathbf{C}_\pm$. If for a.e. $k \in \mathbf{R}^1$ there is $\epsilon(k) > 0$ such that for $\epsilon \in [0, \epsilon(k))$ the inverse operators $(\Theta_\pm(k \pm i\epsilon))^{-1}$ exist and are uniformly bounded, then for each $h, g \in \mathcal{H}$ the boundary values $\Xi_{h,g}^\pm(k)$ exist for a.e. $k \in \mathcal{H}$. Moreover, if $h \in \tilde{\mathcal{H}}^s(L_\kappa^*)$ and $g \in \mathcal{H}$, then $\Xi_{h,g}^+(k) = \Xi_{h,g}^-(k)$ for a.e. $k \in \mathbf{R}^1$.*

Proof: We note that

$$\tilde{\mathcal{H}}^s(L_\kappa^*) = \mathcal{N}(L_\kappa)^\perp \cap \mathcal{H}. \quad (6.11)$$

Let $g \in \text{dom}(\alpha)$. Then

$$\Xi_{h,\alpha g}(z) = (\alpha(L_\kappa^* - z)^{-1}h, g), \quad z \in \varrho(L_\kappa^*). \quad (6.12)$$

By (6.7) and (6.8) we have

$$\Xi_{h,\alpha g}(z) = (\Omega_+(\bar{z})^* \alpha(H - z)^{-1}h, g), \quad z \in \varrho(L_\kappa^*) \cap \mathbf{C}_-, \quad (6.13)$$

and

$$\Xi_{h,\alpha g}(z) = (\Omega_-(\bar{z})^* \alpha(H^* - z)^{-1}h, g), \quad z \in \varrho(L_\kappa^*) \cap \mathbf{C}_+. \quad (6.14)$$

Let $h \in \mathcal{N}(L_\kappa)^\perp \cap \mathcal{H}$. For each $\delta \in L_\Omega^\infty(\mathbf{R}^1)$ one has

$$D_\delta h = 0. \quad (6.15)$$

However, this yields

$$\delta(k)\Omega_+(k)^* \alpha(H - k + i0)^{-1}h = \delta(k)\Omega_-(k)^* \alpha(H^* - k - i0)^{-1}h \quad (6.16)$$

for a.e. $k \in \mathbf{R}^1$ and each $\delta \in L_\Omega^\infty(\mathbf{R}^1)$. Since the set $L_\Omega^\infty(\mathbf{R}^1)$ is sufficiently large this implies

$$\Omega_+(k)^* \alpha(H - k + i0)^{-1}h = \Omega_-(k)^* \alpha(H^* - k - i0)^{-1}h \quad (6.17)$$

for a.e. $k \in \mathbf{R}^1$. Since

$$\Xi_{h,\alpha g}^-(k) = (\Omega_+(k)^* \alpha(H - k + i0)^{-1}h, g) \quad (6.18)$$

and

$$\Xi_{h,\alpha g}^+(k) = (\Omega_-(k)^* \alpha(H^* - k - i0)^{-1}h, g) \quad (6.19)$$

we obtain $\Xi_{h,\alpha g}^+(k) = \Xi_{h,\alpha g}^-(k)$ for a.e. $k \in \mathbf{R}^1$ and each $g \in \text{dom}(\alpha)$.

The relations (6.4), (6.7) and (6.8) justify the definitions

$$\alpha(L_\kappa^* - k \pm i0)^{-1}h \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow +0} \alpha(L_\kappa^* - k \pm i\epsilon)^{-1}h \quad (6.20)$$

for a.e. $k \in \mathbf{R}^1$ and each $h \in \mathcal{H}$. Moreover, one gets

$$\alpha(L_\kappa^* - k + i0)^{-1}h = \Omega_+(k)^* \alpha(H^* - k + i0)^{-1}h \quad (6.21)$$

and

$$\alpha(L_\kappa^* - k - i0)^{-1}h = \Omega_-(k)^* \alpha(H - k - i0)^{-1}h \quad (6.22)$$

for a.e. $k \in \mathbf{R}^1$ and $h \in \mathcal{H}$. Hence (6.17) is equivalent to

$$\alpha(L_\kappa^* - k + i0)^{-1}h = \alpha(L_\kappa^* - k - i0)^{-1}h, \quad h \in \tilde{\mathcal{H}}^s(L_\kappa^*), \quad (6.23)$$

for a.e. $k \in \mathbf{R}^1$.

Obviously we have

$$\begin{aligned} ((L_\kappa^* - z)^{-1}h, (A - \bar{\xi})^{-1}g) &= \frac{1}{z - \bar{\xi}} \Xi_{h,g}(z) + \\ &- \frac{1}{z - \bar{\xi}} ((A - \xi)^{-1}h, g) + \frac{i}{2} \frac{1}{z - \bar{\xi}} (\alpha(L_\kappa^* - z)^{-1}h, \kappa \alpha(A - \xi)^{-1}g), \quad \xi \in \mathbf{C}_\pm. \end{aligned} \quad (6.24)$$

If $g \in \text{ran}(\alpha)$, then by $\Xi_{h,g}^+(k) = \Xi_{h,g}^-(k)$ for a.e. $k \in \mathbf{R}^1$. Moreover by (6.21) and (6.22) we find that $\Xi_{h,f}^+(k) = \Xi_{h,f}^-(k)$ for $h \in \hat{\mathcal{H}}^s(L_\kappa^*)$ and $f \in \mathcal{L}$, where

$$\mathcal{L} = \left\{ \sum_{j=1}^n c_j (A - \xi_j)^{-1} g_j : 1 \leq n < \infty, c_j \in \mathbf{C}, \xi_j \in \mathbf{C}_\pm, g_j \in \text{ran}(\alpha) \right\}. \quad (6.25)$$

Since L_κ is completely non-selfadjoint and $\ker(\kappa) = \{0\}$ the set $\mathcal{L}$ is dense in $\mathcal{H}$. By (6.9) one gets

$$\Xi_{h,f}^-(k) = \lim_{\epsilon \rightarrow +0} ((H - k + i\epsilon)^{-1}h, f) + \frac{i}{2} (\alpha(L_\kappa^* - k + i0)^{-1}h, (I + \kappa)\alpha(H^* - k - i0)^{-1}f) \quad (6.26)$$

for $h, f \in \mathcal{H}$. Similarly one has

$$\Xi_{h,f}^+(k) = \lim_{\epsilon \rightarrow +0} ((H^* - k - i\epsilon)^{-1}h, f) - \frac{i}{2} (\alpha(L_\kappa^* - k - i0)^{-1}h, (I - \kappa)\alpha(H - k + i0)^{-1}f) \quad (6.27)$$

for $h, f \in \mathcal{H}$. Therefore we obtain

$$\begin{aligned} \Xi_{h,f}^+(k) - \Xi_{h,f}^-(k) &= \\ &= \lim_{\epsilon \rightarrow +0} \left\{ ((H^* - k - i\epsilon)^{-1}h, f) - ((H - k + i\epsilon)^{-1}h, f) \right\} + \\ &- \frac{i}{2} (\alpha(L_\kappa^* - k - i0)^{-1}h, (I - \kappa)\alpha(H - k + i0)^{-1}f) + \\ &- \frac{i}{2} (\alpha(L_\kappa^* - k + i0)^{-1}h, (I + \kappa)\alpha(H^* - k - i0)^{-1}f). \end{aligned} \quad (6.28)$$

Since

$$\begin{aligned} 2\pi i \frac{d}{d\lambda} (E(\lambda)h, f)|_{\lambda=k} &= \\ &= \lim_{\epsilon \rightarrow +0} \left\{ ((K - k - i\epsilon)^{-1}h, f) - ((K - k + i\epsilon)^{-1}h, f) \right\} \\ &= \lim_{\epsilon \rightarrow +0} \left\{ ((H^* - k - i\epsilon)^{-1}h, f) - ((H - k + i\epsilon)^{-1}h, f) \right\} \end{aligned} \quad (6.29)$$

for a.e $k \in \mathbf{R}^1$, where $E(\cdot)$ is the spectral measure of the self-adjoint dilation K , one gets

$$\begin{aligned} \Xi_{h,f}^+(k) - \Xi_{h,f}^-(k) &= \\ &= 2\pi i \frac{d}{d\lambda} (E(\lambda)h, f)|_{\lambda=k} + \\ &- \frac{i}{2} (\alpha(L_\kappa^* - k - i0)^{-1}h, (I - \kappa)\alpha(H - k + i0)^{-1}f) + \\ &- \frac{i}{2} (\alpha(L_\kappa^* - k + i0)^{-1}h, (I + \kappa)\alpha(H^* - k - i0)^{-1}f) \end{aligned} \quad (6.30)$$

for a.e. $k \in \mathbf{R}^1$ and $f \in \mathcal{H}$. The estimate

$$\left| \frac{d}{d\lambda}(E(\lambda)h, f) \right|^2 \leq \frac{d}{d\lambda}(E(\lambda)h, h) \|f\|^2 \quad (6.31)$$

implies

$$\int_{-\infty}^{+\infty} \left| \frac{d}{d\lambda}(E(\lambda)h, f) \right|^2 \leq \|h\|^2 \|f\|^2 \quad (6.32)$$

for $h, f \in \mathcal{H}$. Furthermore, one has

$$\int_{-\infty}^{+\infty} dk \|\alpha(H^* - k - i0)^{-1} f\|^2 \leq 2\pi \|f\|^2 \quad (6.33)$$

and

$$\int_{-\infty}^{+\infty} dk \|\alpha(H - k + i0)^{-1} f\|^2 \leq 2\pi \|f\|^2 \quad (6.34)$$

for each $f \in \mathcal{H}$. If $\{f_n\}_{n=1}^\infty$ tends to f as $n \rightarrow \infty$, then by (6.32) - (6.34) we find

$$\frac{d}{d\lambda}(E(\lambda)h, f_n) \xrightarrow{L^2} \frac{d}{d\lambda}(E(\lambda)h, f), \quad (6.35)$$

$$\alpha(H^* - k - i0)^{-1} f_n \xrightarrow{L^2} \alpha(H^* - k - i0)^{-1} f, \quad (6.36)$$

$$\alpha(H - k + i0)^{-1} f_n \xrightarrow{L^2} \alpha(H - k + i0)^{-1} f. \quad (6.37)$$

Hence there is subsequence $\{n_l\}_{l=1}^\infty$ such that

$$\lim_{l \rightarrow \infty} \frac{d}{d\lambda}(E(\lambda)h, f_{n_l})|_{\lambda=k} = \frac{d}{d\lambda}(E(\lambda)h, f)|_{\lambda=k}, \quad (6.38)$$

$$\lim_{l \rightarrow \infty} \alpha(H^* - k - i0)^{-1} f_{n_l} = \alpha(H^* - k - i0)^{-1} f, \quad (6.39)$$

$$\lim_{l \rightarrow \infty} \alpha(H - k + i0)^{-1} f_{n_l} = \alpha(H - k + i0)^{-1} f \quad (6.40)$$

for a.e. $k \in \mathbf{R}^1$. By (6.30) this yields

$$\begin{aligned} \lim_{l \rightarrow \infty} \left\{ \Xi_{h, f_{n_l}}^+(k) - \Xi_{h, f_{n_l}}^-(k) \right\} &= \\ &= 2\pi i \frac{d}{d\lambda}(E(\lambda)h, f)|_{\lambda=k} + \\ &\quad - \frac{i}{2}(\alpha(L_\kappa^* - k - i0)^{-1} h, (I - \kappa)\alpha(H - k + i0)^{-1} f) + \\ &\quad - \frac{i}{2}(\alpha(L_\kappa^* - k + i0)^{-1} h, (I + \kappa)\alpha(H^* - k - i0)^{-1} f) = \\ &= \Xi_{h, f}^+(k) - \Xi_{h, f}^-(k) \end{aligned} \quad (6.41)$$

for a.e. $k \in \mathbf{R}^1$. If $f_n \in \mathcal{L}$, $n = 1, 2, \dots$, then $\Xi_{h, f_n}^+(k) = \Xi_{h, f_n}^-(k)$ for a.e. $k \in \mathbf{R}^1$. Hence $\Xi_{h, f}^+(k) = \Xi_{h, f}^-(k)$ for a.e. $k \in \mathbf{R}^1$. •

Now we are able to prove the main theorem.

Theorem 6.3 *Let $\ker(\kappa) = \{0\}$. Assume that L_κ is completely non-selfadjoint and that the spectrum of L_κ is discrete in $\mathbf{C}_\pm$. If for a.e. $k \in \mathbf{R}^1$ there is a $\epsilon(k) > 0$ such that for $\epsilon \in [0, \epsilon(k))$ the inverse operators $(\Theta_\pm(k \pm i\epsilon))^{-1}$ exist and are uniformly bounded, then the boundary values $\Xi_{h,g}^\pm(k)$ exist for a.e. $k \in \mathbf{R}^1$ and each $h, g \in \mathcal{H}$. Moreover, the singular subspace of L_κ^* is characterized by*

$$\mathcal{H}^s(L_\kappa^*) = \{h \in \mathcal{H} : \Xi_{h,g}^+(k) = \Xi_{g,h}^-(k) \text{ for a.e. } k \in \mathbf{R}^1, \forall g \in \mathcal{H}\}. \quad (6.42)$$

Proof: It remains to prove (6.42). We set

$$\check{\mathcal{H}}^s(L_\kappa^*) \stackrel{\text{def}}{=} \{h \in \mathcal{H} : \Xi_{h,g}^+(k) = \Xi_{h,g}^-(k) \text{ for a.e. } k \in \mathbf{R}^1 \quad \forall g \in \mathcal{H}\}. \quad (6.43)$$

Let us show that $\check{\mathcal{H}}^s(L_\kappa^*) \perp \mathcal{H}^{ac}(L_\kappa)$. If $g \in M(L_\kappa)$, then $\Omega_{g,h}(z) \in H_\pm^2(\mathbf{R}^1)$ for each $h \in \mathcal{H}$, in particular, for $h \in \check{\mathcal{H}}^s(L_\kappa^*)$. Since $\Omega_{g,h}^\pm(k) = s - \lim_{\epsilon \rightarrow +0} \Omega_{g,h}(k \pm i\epsilon)(k) = \overline{\Xi_{h,g}^\mp(k)}$ one gets $\Omega_{g,h}^+(k) = \Omega_{g,h}^-(k)$ for a.e. $k \in \mathbf{R}^1$ and each $h \in \check{\mathcal{H}}^s(L_\kappa^*)$. Since $\Omega_{g,h}^\pm(k) \in H_\pm^2(\mathbf{R}^1)$ and $H_+^2(\mathbf{R}^1) \perp H_-^2(\mathbf{R}^1)$ we find $\Omega_{g,h}^\pm = 0$ for a.e. $k \in \mathbf{R}^1$ and $h \in \check{\mathcal{H}}^s(L_\kappa^*)$. Hence $\Omega_{g,h}(z) = 0$, $z \in \mathbf{C}_\pm$. By Corollary 2.3 we obtain

$$0 = \lim_{\eta \rightarrow \infty} (-i\eta) \Omega_{g,h}(i\eta) = (g, h), \quad (6.44)$$

which yields $g \perp \check{\mathcal{H}}^s(L_\kappa^*)$. Hence $\mathcal{H}^{ac}(L_\kappa) \perp \check{\mathcal{H}}^s(L_\kappa^*)$. By Proposition 6.2 one has $\check{\mathcal{H}}^s(L_\kappa^*) \subseteq \check{\mathcal{H}}^s(L_\kappa^*)$. Since $\overline{P_{\mathcal{H}}^\kappa \mathcal{N}(L_\kappa)} \subseteq \mathcal{H}^{ac}(L_\kappa)$ and $\overline{P_{\mathcal{H}}^\kappa \mathcal{N}(L_\kappa)} \oplus \check{\mathcal{H}}^s(L_\kappa^*) = \mathcal{H}$ we find $\mathcal{H}^s(L_\kappa^*) = \check{\mathcal{H}}^s(L_\kappa^*) = \overline{\mathcal{H}^s(L_\kappa^*)}$ and $\mathcal{H}^{ac}(L_\kappa) = \overline{P_{\mathcal{H}}^\kappa \mathcal{N}(L_\kappa)}$. •

7 Characteristic function

Assume that L_κ is completely non-selfadjoint. We set

$$\Lambda(z) \stackrel{\text{def}}{=} \alpha(L_\kappa^* - z)^{-1} \alpha, \quad z \in \varrho(L_\kappa^*). \quad (7.1)$$

In general it is not clear whether this is a family of bounded operators. However, using the representation (2.13) this fact immediately follows. Moreover, one gets that $\Lambda(z)$ is an analytic family of bounded operators in $\varrho(L_\kappa^*)$. Therefore

$$\Theta_{L_\kappa}(z) \stackrel{\text{def}}{=} I + i \operatorname{sign}(\kappa) \sqrt{|\kappa|} \alpha(L_\kappa^* - z)^{-1} \alpha \sqrt{|\kappa|}, \quad z \in \varrho(L_\kappa^*). \quad (7.2)$$

defines an analytic family of bounded operators in $\varrho(L_\kappa^*)$. If $\ker(\kappa) = \{0\}$, then this family is called the characteristic function of L_κ [25], [26], [3]. Note that the characteristic function is

$\text{sign}(\kappa)$ -contractive, i.e. $\Theta_{L_\kappa}(z)^* \text{sign}(\kappa) \Theta_{L_\kappa}(z) \leq \text{sign}(\kappa)$, $z \in \varrho(L_\kappa^*) \cap \mathbf{C}_+$. If $\kappa^2 = I$, then the characteristic function simplifies to

$$\Theta_{L_\kappa} = I + i\kappa\alpha(L_\kappa^* - z)^{-1}\alpha, \quad z \in \varrho(L_\kappa^*). \quad (7.3)$$

Furthermore, we set

$$\Gamma_\pm(z) \stackrel{\text{def}}{=} \Theta_\mp(\bar{z})^*, \quad z \in \mathbf{C}_\pm, \quad (7.4)$$

and

$$\Upsilon_\pm(z) \stackrel{\text{def}}{=} \Omega_\mp(\bar{z})^*, \quad z \in \varrho(L_\kappa^*) \cap \mathbf{C}_\pm. \quad (7.5)$$

Obviously one has

$$\Upsilon_\pm(z)\Gamma_\pm(z) = \Gamma_\pm(z)\Upsilon_\pm(z) = I, \quad z \in \varrho(L_\kappa^*) \cap \mathbf{C}_\pm. \quad (7.6)$$

Note that

$$\Lambda(z) = \Upsilon_+(z)\alpha(H^* - z)^{-1}\alpha, \quad z \in \varrho(L_\kappa^*) \cap \mathbf{C}_+, \quad (7.7)$$

and

$$\Lambda(z) = \Upsilon_-(z)\alpha(H - z)^{-1}\alpha, \quad z \in \varrho(L_\kappa^*) \cap \mathbf{C}_-. \quad (7.8)$$

Convention 7.1 Let $\Pi(\cdot) : \mathcal{O} \longrightarrow \mathcal{B}(\mathcal{H})$, $\mathcal{O} \subseteq \mathbf{C}_\pm$, be an analytic family of bounded operators. We say the family $\Pi(\cdot)$ admits an upper boundary value $\Pi^+(k)$ at $k \in \mathbf{R}^1$, if $\Pi^+(k)$ is a bounded operator and there is a $\epsilon(k) > 0$ such that $(0, \epsilon(k)) \subseteq \mathcal{O}$ and $\Pi^+(k) = s - \lim_{\epsilon \rightarrow +0} \Pi^+(k + i\epsilon)$. Similarly we define the lower boundary value $\Pi^-(k)$ at $k \in \mathbf{R}^1$. If the upper and lower boundary values exist at $k \in \mathbf{R}^1$, then $\Pi(\cdot)$ admits boundary values of both sides at $k \in \mathbf{R}^1$.

Lemma 7.2 Let $\ker(\kappa) = \{0\}$. Assume that L_κ is completely non-selfadjoint and that the spectrum of L_κ is discrete in $\mathbf{C}_\pm$. If for a.e. $k \in \mathbf{R}^1$ there is a $\epsilon(k) > 0$ such that for $\epsilon \in [0, \epsilon(k))$ the inverse operator $(\Theta_\pm(k \pm i\epsilon))^{-1}$ exist and are uniformly bounded, then the analytic families of bounded operators $\Upsilon_+(z)$, $\Upsilon_-(z)$ and $\Lambda(z)$ admit upper boundary values $\Upsilon_+(k) \stackrel{\text{def}}{=} \Upsilon_+^+(k)$, lower boundary values $\Upsilon_-(k) \stackrel{\text{def}}{=} \Upsilon_-^-(k)$ and boundary values of both sides $\Lambda^\pm(k)$ for a.e. $k \in \mathbf{R}^1$, respectively.

Proof: From (5.23) and (5.24) one gets

$$2\Gamma_+(z) = (I + \kappa) + \Theta_H(z)(I - \kappa), \quad z \in \mathbf{C}_+, \quad (7.9)$$

and

$$2\Gamma_-(z) = (I - \kappa) + \Theta_{H^*}(z)(I + \kappa), \quad z \in \mathbf{C}_-. \quad (7.10)$$

Hence the boundary values $\Gamma_\pm(k) = s - \lim_{\epsilon \rightarrow +0} \Gamma_\pm(k \pm i\epsilon)$ exist for a.e. $k \in \mathbf{R}^1$.

Since $\{(\Theta_{\pm}(k \pm i\epsilon))^{-1}\}_{\epsilon \in [0, \epsilon(k))}$ is uniformly bounded we find

$$\begin{aligned} \sup_{\epsilon \in [0, \epsilon(k))} \|\Upsilon_{\pm}(k \pm i\epsilon)\| &= \\ &= \sup_{\epsilon \in [0, \epsilon(k))} \|\Omega_{\mp}(k \mp i\epsilon)\| = \sup_{\epsilon \in [0, \epsilon(k))} \|(\Theta_{\mp}(k \mp i\epsilon))^{-1}\| < +\infty. \end{aligned} \quad (7.11)$$

Let $\Upsilon_{\pm}(k) \stackrel{\text{def}}{=} \Omega_{\pm}(k)^*$. Note that $\Upsilon_{\pm}(k)\Gamma_{\pm}(k) = \Gamma_{\pm}(k)\Upsilon_{\pm}(k)$ for a.e. $k \in \mathbf{R}^1$. Using (7.6) one gets the representation

$$\Upsilon_{\pm}(k) - \Upsilon_{\pm}(k \pm i\epsilon) = \Upsilon_{\pm}(k \pm i\epsilon)\{\Gamma_{\pm}(k \pm i\epsilon) - \Gamma_{\pm}(k)\}\Upsilon_{\pm}(k) \quad (7.12)$$

and $\Upsilon_{\pm}(k) = s - \lim_{\epsilon \rightarrow +0} \Upsilon_{\pm}(k \pm i\epsilon)$. Hence the analytic families of bounded operators $\Upsilon_{+}(z)$ and $\Upsilon_{-}(z)$ has upper and lower boundary values, respectively, for a.e. $k \in \mathbf{R}^1$. Using the representations (7.7) and (7.8) we finally show that $\Lambda(z)$ has boundary values of both sides for a.e. $k \in \mathbf{R}^1$. •

Lemma 7.3 *Let $\kappa^2 = I$. Assume that L_{κ} is completely non-selfadjoint and that the spectrum of L_{κ} is discrete in $\mathbf{C}_{\pm}$. Then for a.e. $k \in \mathbf{R}^1$ there is a $\epsilon(k) > 0$ such that for $\epsilon \in [0, \epsilon(k))$ the inverse operators $(\Theta_{\pm}(k \pm i\epsilon))^{-1}$ exist and are uniformly bounded if and only if the characteristic function $\Theta_{L_{\kappa}}(z)$ admits boundary values of both sides.*

Proof: By the representation

$$\Theta_{L_{\kappa}}(z) = I + i\kappa\Lambda(z), \quad z \in \varrho(L_{\kappa}^*), \quad (7.13)$$

and by Lemma 7.2 we find the existence of the boundary values $\Lambda^{\pm}(k)$ for a.e. $k \in \mathbf{R}^1$.

Conversely, from (7.13) using $\kappa^2 = I$ we obtain

$$\Lambda(z) = -i\kappa - i\kappa\Theta_{L_{\kappa}}(z), \quad z \in \varrho(L_{\kappa}^*). \quad (7.14)$$

Hence the boundary values $\Lambda^{\pm}(k)$ exist for a.e. $k \in \mathbf{R}^1$. By the representations

$$\Upsilon_{+}(z) = I - \frac{i}{2}\Lambda(z)(I - \kappa), \quad z \in \varrho(L_{\kappa}^*) \cap \mathbf{C}_{+}, \quad (7.15)$$

and

$$\Upsilon_{-}(z) = I + \frac{i}{2}\Lambda(z)(I + \kappa), \quad z \in \varrho(L_{\kappa}^*) \cap \mathbf{C}_{-}, \quad (7.16)$$

the upper and lower boundary values $\Upsilon_{+}(k)$ and $\Upsilon_{-}(k)$ exist for a.e. $k \in \mathbf{R}^1$. Hence for a.e. $k \in \mathbf{R}^1$ there is a $\epsilon(k) > 0$ such that $\{\Upsilon_{\pm}(k \pm i\epsilon)\}_{\epsilon \in [0, \epsilon(k))}$ is uniformly bounded. By (7.5) and $\Omega_{\pm}(z) = (\Theta_{\pm}(z))^{-1}$, $z \in \varrho(L_{\kappa})$, the same holds for $\{\Theta_{\pm}(k \pm i\epsilon)^{-1}\}_{\epsilon \in [0, \epsilon(k))}$. •

Theorem 7.4 *Let $\ker(\kappa) = \{0\}$. Assume that L_κ is completely non-selfadjoint and that the spectrum of L_κ is discrete in $\mathbf{C}_\pm$. If the characteristic function $\Theta_{L_\kappa}(z)$ admits boundary values of both sides $\Theta_{L_\kappa}^\pm(k)$ for a.e. $k \in \mathbf{R}^1$, then for each $h, g \in \mathcal{H}$ the function $\Xi_{h,g}(z)$ is meromorphic in $\mathbf{C}_\pm$ and possesses boundary values $\Xi_{h,g}^\pm(k)$ for a.e. $k \in \mathbf{R}^1$. Moreover, one has $h \in \mathcal{H}^s(L_\kappa)$ if and only if $\Xi_{h,g}^+(k) = \Xi_{h,g}^-(k)$ for a.e. $k \in \mathbf{R}^1$ and each $g \in \mathcal{H}$.*

Proof: Assuming without loss of generality that $\kappa^2 = I$ we have only to combine Theorem 6.3 with Lemma 7.3. •

Applying Theorem 7.4 to completely non-selfadjoint maximal dissipative operators H the conclusions of Theorem 7.4 are valid if the characteristic function $\Theta_H(z)$ admits boundary values of both sides $\Theta_H^\pm(k)$ for a.e. $k \in \mathbf{R}^1$. Since for a.e. $k \in \mathbf{R}^1$ the upper boundary values $\Theta_H^+(k)$ always exist the conclusions of Theorem 7.4 are valid if for a.e. $k \in \mathbf{R}^1$ the lower boundary values $\Theta_H^-(k)$ exist. However, this is the contents of Theorem 4.3 of [1].

8 Existence

Let us answer the question under which conditions the absolutely continuous subspace $\mathcal{H}^{ac}(L_\kappa)$ is not empty. To this end we recall that a maximal dissipative operator H belongs to C_0 or $C_{0.}$ [5] if

$$s - \lim_{t \rightarrow \infty} e^{itH} = 0 \quad \text{or} \quad s - \lim_{t \rightarrow \infty} e^{-itH^*} = 0, \quad (8.1)$$

respectively. Let $\mathcal{K}$ be the dilation space and let $W_\pm$ be the Lax-Phillips wave operators. Then $H \in C_0$ if and only if $\text{ran}(W_-) = \mathcal{K}$. Similarly $H \in C_{0.}$ if only if $\text{ran}(W_+) = \mathcal{K}$.

Theorem 8.1 *Let $\kappa^2 = I$. Assume that L_κ is completely non-selfadjoint and that the spectrum of L_κ is discrete in $\mathbf{C}_\pm$. Furthermore, let us assume that the characteristic function $\Theta_{L_\kappa}(z)$ admits boundary values of both sides $\Theta_{L_\kappa}^\pm(k)$ for a.e. $k \in \mathbf{R}^1$. Then $\mathcal{H}^{ac}(L_\kappa) = \{0\}$ if and only if the associated maximal dissipative operator H belongs to C_0 or $C_{0.}$*

Proof: Let $H \in C_{0.}$. Then $\text{ran}(W_+) = \mathcal{K}$. Hence $\ker G = \{0\}$ if and only if $\ker(GW_+) = \{0\}$. Since

$$GW_+ = (I + \hat{\kappa})S^* + (I - \hat{\kappa}) \quad (8.2)$$

one has

$$FGW_+ = (I + \hat{\kappa})\hat{\Theta}_H + (I - \hat{\kappa}). \quad (8.3)$$

Using (5.9) we obtain

$$FGW_+ = \hat{\Theta}_+. \quad (8.4)$$

Hence $Gh = 0$, $h \in L^2(\mathbf{R}^1, \mathcal{E})$, is equivalent to

$$\Theta_+(k)\hat{h}(k) = 0, \quad \hat{h} = Fh, \quad (8.5)$$

for a.e. $k \in \mathbf{R}^1$. By Lemma 7.3 the inverse operators $(\Theta(k))^{-1}$ exist for a.e. $k \in \mathbf{R}^1$. Moreover, this yields $\hat{h}(k) = 0$. Hence $\ker G = \{0\}$. Therefore $\mathcal{N}(L_\kappa) = \{0\}$. Since under the assumptions of the theorem one has $\mathcal{N}(L_\kappa) = \mathcal{H}^{ac}(L_\kappa)$ we obtain $\mathcal{H}^{ac}(L_\kappa) = \{0\}$. Similarly we prove that $H \in C_0$ implies $\mathcal{H}^{ac}(L_\kappa) = \{0\}$.

Conversely, let us assume $\mathcal{H}^{ac}(L_\kappa) = \{0\}$ and $H \notin C_0$. Then $\mathcal{K}_+ = \text{ran}(W_+)^{\perp}$ is not empty. Applying Lemma 7.3 and Lemma 5.3 we get that $\ker(G) \neq \{0\}$. Therefore $\mathcal{N}(L_\kappa) \neq \{0\}$ which yields $\mathcal{H}^{ac}(L_\kappa) \neq \{0\}$. Hence $\mathcal{H}^{ac}(L_\kappa) = \{0\}$ implies $H \in C_0$. •

Remark 8.2 Using Remark 5.4 it can be shown that $\mathcal{H}^{ac}(L_\kappa) = \{0\}$ yields $H \in C_0$. Hence $\mathcal{H}^{ac}(L_\kappa) = \{0\}$ implies $H \in C_{00} = C_0 \cap C_0$.

In other words L_κ has a non-trivial absolutely continuous subspace if and only if the associated maximal dissipative operator H has a non-trivial $C_{1,-}$ or $C_{1,+}$ -parts (for notation see [5]). By [1] this means that L_κ has a non-trivial absolutely continuous subspace if and only if H or $(-H^*)$ have non-trivial absolutely continuous subspaces

In order to show that H has a non-trivial $C_{1,-}$ or $C_{1,+}$ -parts one can use the scattering theory for maximal dissipative operator [19] - [21]. So H has a non-trivial $C_{1,-}$ or $C_{1,+}$ -parts if the wave operators $W_-(H, A_0)$

$$W_-(H, A_0) = s - \lim_{t \rightarrow \infty} e^{itH} e^{-itA_0} P^{ac}(A_0) \quad (8.6)$$

or $W_+(H, A_0)$,

$$W_+(H, A_0) = s - \lim_{t \rightarrow \infty} e^{-itH^*} e^{itA_0} P^{ac}(A_0) \quad (8.7)$$

exist, respectively. Hence $\mathcal{H}^{ac}(L_\kappa)$ is non-trivial if the absolutely continuous subspace of A is non-trivial and either $W_-(H, A_0)$ or $W_+(H, A_0)$ exist.

More directly the absolutely continuous subspaces $\mathcal{H}^{ac}(L_\kappa)$ and $\mathcal{H}^{ac}(A)$ are related by the following theorem.

Theorem 8.3 Let $\kappa^2 = I$. Assume that L_κ is completely non-selfadjoint and that the spectrum of L_κ is discrete in $\mathbf{C}_\pm$. Furthermore assume that the characteristic function $\Theta_{L_\kappa}(z)$ admits boundary values of both sides $\Theta_{L_\kappa}^\pm(k)$ for a.e. $k \in \mathbf{R}^1$. Let A be the associated pseudo-Friedrichs extension.

If there is a dense subset $\mathcal{E}_0 \subseteq \text{ran}(\alpha)$ of $\mathcal{E}$ such that the boundary values

$$\alpha(A - k \pm i0)^{-1}h \stackrel{\text{def}}{=} \lim_{\epsilon \rightarrow +0} \alpha(A - k \pm i\epsilon)^{-1}h \quad (8.8)$$

exist for a.e. $k \in \mathbf{R}^1$ and each $h \in \mathcal{E}_0$, then $\mathcal{H}^{ac}(L_\kappa) = \{0\}$ implies $\mathcal{H}^{ac}(A) = \{0\}$.

Conversely, if the analytic family of bounded operators $A(z)$,

$$A(z) \stackrel{\text{def}}{=} \frac{i}{2} \alpha(A - z)^{-1} \alpha, \quad z \in \mathbf{C}_\pm, \quad (8.9)$$

admits boundary values of both sides $A^\pm(k)$ for a.e. $k \in \mathbf{R}^1$, then $\mathcal{H}^{ac}(A) = \{0\}$ implies $\mathcal{H}^{ac}(L_\kappa) = \{0\}$.

Proof: From the identity

$$(L_\kappa^* - z)^{-1} - (A - z)^{-1} = \frac{i}{2} (L_\kappa^* - z)^{-1} \alpha \kappa \alpha (A - z)^{-1}, \quad z \in \varrho(L_\kappa^*) \cap \varrho(A), \quad (8.10)$$

the relation

$$\alpha(L_\kappa^* - z)^{-1} h = \Pi(z) \alpha(A - z)^{-1} h, \quad h \in \mathcal{H}, \quad z \in \varrho(L_\kappa^*) \cap \varrho(A), \quad (8.11)$$

follows, where

$$\Pi(z) \stackrel{\text{def}}{=} I + \frac{i}{2} \alpha(L_\kappa^* - z)^{-1} \alpha \kappa : \mathcal{E} \longrightarrow \mathcal{E}, \quad z \in \varrho(L_\kappa^*). \quad (8.12)$$

By Lemma 7.2 and 7.3 the analytic family of bounded operators $\Pi(z)$ admits boundary values of both sides $\Pi^\pm(k)$ for a.e. $k \in \mathbf{R}^1$. Hence we find

$$\alpha(L_\kappa^* - k \mp i0)h = \Pi^\pm(k) \alpha(A - k \mp i0)^{-1} h, \quad (8.13)$$

for each $h \in \mathcal{H}$.

If $\mathcal{H}^{ac}(L_\kappa) = \{0\}$, then

$$\alpha(L_\kappa^* - k - i0)^{-1} h - \alpha(L_\kappa^* - k + i0)^{-1} h = 0 \quad (8.14)$$

for a.e. $k \in \mathbf{R}^1$ and $h \in \mathcal{H}$ which yields

$$\Pi^+(k) = \Pi^-(k) \quad (8.15)$$

for a.e. $k \in \mathbf{R}^1$. Hence from (8.13) we find

$$0 = \Pi^+(k) \{ \alpha(A - k - i0)^{-1} h - \alpha(A - k + i0)^{-1} h \} \quad (8.16)$$

for each $h \in \mathcal{E}_0$ and a.e. $k \in \mathbf{R}^1$. Since

$$(\Pi(z))^{-1} = I - \frac{i}{2} \alpha(A - z)^{-1} \alpha \kappa, \quad z \in \varrho(L_\kappa^*) \cap \varrho(A), \quad (8.17)$$

one gets

$$\begin{aligned} & (\{\alpha(A - k - i0)^{-1}h - \alpha(A - k + i0)^{-1}h\}, g) = \\ & = \lim_{\epsilon \rightarrow +0} (\Pi(k + i\epsilon)\{\alpha(A - k - i\epsilon)^{-1}h - \alpha(A - k + i\epsilon)^{-1}h\}, (\Pi(k + i\epsilon)^*)^{-1}g) = 0 \end{aligned} \quad (8.18)$$

for a.e. $k \in \mathbf{R}^1$ and each $h, g \in \mathcal{E}_0$. However, this implies

$$\frac{d}{d\lambda}(E_A(\lambda)h, \alpha g)|_{\lambda=k} = 0 \quad (8.19)$$

for a.e. $k \in \mathbf{R}^1$ and each $h \in \mathcal{E}_0, \text{dom}(\alpha)$, where $E_A(\cdot)$ is the spectral measure of the associated pseudo-Friedrichs extension A . Therefore

$$(E_A(\mathcal{B})h, P^{ac}(A)\alpha g) = 0 \quad (8.20)$$

for each $h \in \mathcal{E}_0, g \in \text{dom}(\alpha)$ and Borel set $\mathcal{B} \subset \mathbf{R}^1$. By Corollary 2.4 the set $\mathcal{E}_0$ is cyclic. Hence $P^{ac}(A)\alpha g = 0$ for each $g \in \text{dom}(\alpha)$ or $\text{ran}(\alpha) \perp \mathcal{H}^{ac}(A)$. Applying again Corollary 2.4 one gets $\mathcal{H}^{ac}(A) = \{0\}$.

Conversely, let $\mathcal{H}^{ac}(A) = \{0\}$. From (8.11) and (8.17) we obtain

$$\{I - \frac{i}{2}\alpha(A - z)^{-1}\alpha\kappa\}\alpha(L_\kappa^* - z)^{-1}h = \alpha(A - z)^{-1}h, \quad h \in \mathcal{H}, \quad z \in \varrho(L_\kappa^*) \cap \varrho(A). \quad (8.21)$$

Taking into account the existence of $A^\pm(k)$ for a.e. $k \in \mathbf{R}^1$ we find

$$\{I - A^\pm(k)\kappa\}\alpha(L_\kappa^* - k \mp i0)^{-1}h = \alpha(A - k \mp i0)^{-1}h \quad (8.22)$$

for a.e. $k \in \mathbf{R}^1$. Since $\mathcal{H}^{ac}(A) = \{0\}$ one has $A^+(k) = A^-(k)$ for a.e. $k \in \mathbf{R}^1$ which yields

$$(\{I - A^+(k)\kappa\}\{\alpha(L_\kappa^* - k - i0)^{-1}h - \alpha(L_\kappa^* - k + i0)^{-1}h\}, g) = 0 \quad (8.23)$$

for a.e. $k \in \mathbf{R}^1$. Hence

$$\begin{aligned} & (\{\alpha(L_\kappa^* - k - i0)^{-1}h - \alpha(L_\kappa^* - k + i0)^{-1}h\}, g) = \\ & = \lim_{\epsilon \rightarrow +0} (\Pi(k + i\epsilon)^{-1}\{\alpha(L_\kappa^* - k - i\epsilon)^{-1}h - \alpha(L_\kappa^* - k + i\epsilon)^{-1}h\}, \Pi(k + i\epsilon)^*g) = 0 \end{aligned} \quad (8.24)$$

for a.e. $k \in \mathbf{R}^1$ and each $h, g \in \mathcal{H}$. Therefore $\Xi_{h, \alpha g}^+(k) = \Xi_{h, \alpha g}^-(k)$ for $h \in \mathcal{H}, g \in \text{dom}(\alpha)$ and a.e. $k \in \mathbf{R}^1$. Repeating the considerations of Proposition 6.2 we complete the proof. $\bullet$

Note that the existence of $A^\pm(k)$ for a.e. $k \in \mathbf{R}^1$ implies condition (8.8). Hence $\mathcal{H}^{ac}(L_\kappa)$ is not trivial if and only if $\mathcal{H}^{ac}(A)$ is not trivial provided $A^\pm(k)$ exists for a.e. $k \in \mathbf{R}^1$.

9 Examples

In the following we denote the Schatten classes of compact operators by $\mathcal{L}_p(\mathcal{H})$, $1 \leq p \leq \infty$, where $\mathcal{L}_\infty(\mathcal{H})$, $\mathcal{L}_2(\mathcal{H})$ and $\mathcal{L}_1(\mathcal{H})$ are the compact, Hilbert-Schmidt and nuclear operators, respectively.

The applications are based on the following proposition and its corollary.

Proposition 9.1 *Let $\ker(\kappa) = \{0\}$. Assume that L_κ is completely non-selfadjoint and*

$$A(z) = \frac{i}{2}\alpha(A - z)^{-1}\alpha \in \mathcal{L}_\infty(\mathcal{H}), \quad z \in \mathbf{C}_\pm. \quad (9.1)$$

if there is an open set $\mathcal{O} \subseteq \mathbf{R}^1$ such that its complement $\mathbf{R}^1 \setminus \mathcal{O}$ is of Lebesgue measure zero and $A(z)$ is norm continuous in $\mathcal{O} \cup \mathbf{C}_\pm$, then for each $h, g \in \mathcal{H}$ the function $\Xi_{h,g}(z)$, $z \in \varrho(L_\kappa^)$ is meromorphic in $\mathbf{C}_\pm$ and possesses boundary values $\Xi_{h,g}^\pm(k)$ for a.e $k \in \mathbf{R}^1$. Moreover, if $h \in \mathcal{H}^s(L_\kappa^*)$, then for each $g \in \mathcal{H}$ we have $\Xi_{h,g}^+(k) = \Xi_{h,g}^-(k)$ for $k \in \mathbf{R}^1$.*

Proposition 9.1 is a simple modification of Proposition 5.3 of [1]. So we omit the proof. The same holds for the following corollary which is similar to Corollary 5.4 of [1].

Corollary 9.2 *Let $\ker(\kappa) = 0$. Assume that L_κ is completely non-selfadjoint. If*

$$A(z) = \frac{i}{2}\alpha(A - z)^{-1}\alpha \in \mathcal{L}_p(\mathcal{H}), \quad 1 \leq p < \infty, \quad z \in \mathbf{C}_\pm, \quad (9.2)$$

and $A(z)$ admits boundary values of both sides $A^\pm(k)$ in operator norm for a.e $k \in \mathbf{R}^1$ and, moreover, $A^\pm(k) \in \mathcal{L}_p(\mathcal{H})$ for a.e. $k \in \mathbf{R}^1$, then the conclusions of Proposition 9.1 hold.

The results follow from Theorem I.8.3 and Theorem I.8.5 of [32], respectively.

Taking into account Proposition 8.3 we find that under the conditions of Proposition 9.1 and Corollary 9.2 the absolutely continuous subspace $\mathcal{H}^{ac}(L_\kappa)$ is not empty if the absolutely continuous subspace $\mathcal{H}^{ac}(A)$ is not empty.

Modifying the considerations of Proposition 5.5 of [1] we can apply the results to the Schrödinger operator.

Proposition 9.3 *Let $A_0 = -\Delta$ be the usual Laplace operator defined on $\mathcal{H} = L^2(\mathbf{R}^1)$, $n = 1, 2, \dots$, and let V be the multiplication operator arising from the complex-valued function $q(\cdot) : \mathbf{R}^n \longrightarrow \mathbf{C}$, which is continuous and obeys*

$$|q(x)| \leq C(1 + |x|)^{-\beta}, \quad 1 < \beta < +\infty, \quad x \in \mathbf{R}^n. \quad (9.3)$$

If $\Im(q) \neq 0$, then the operator L ,

$$Lf = A_0f + Vf, \quad f \in \text{dom}(L) = \text{dom}(A_0), \quad (9.4)$$

is closed and completely non-selfadjoint. Moreover, the spectrum of L is discrete in $\mathbf{C}_\pm$ and for each $h, g \in \mathcal{H}$ the function $\Xi_{h,g}(z)$ is meromorphic in $\mathbf{C}_\pm$ and possesses boundary values $\Xi_{h,g}^\pm(k)$ for a.e. $k \in \mathbf{R}^1$. If $h \in \mathcal{H}^s(L^*)$, then for each $g \in \mathcal{H}$ one has $\Xi_{h,g}^+(k) = \Xi_{h,g}^-(k)$ for a.e. $k \in \mathbf{R}^1$. Moreover, $\mathcal{H}^{ac}(L) \neq \{0\}$.

Proof: By (9.3) the operator V is bounded. Hence the operator L is closed. Let V_0 , α and $\tilde{\kappa}$ the multiplication operators induced by $q_0(x) = \Re(q(x))$, $\alpha(x) = \sqrt{2|\Im(q(x))|}$ and $\kappa(x) = \text{sign}(\Im(q(x)))$. We set $\kappa = \tilde{\kappa}|_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{E}$, $\mathcal{E} = \overline{\text{ran}(\alpha)}$. Note that $\ker(\kappa) = \{0\}$.

From Theorem XIII 57 of [30] it follows that the associated maximal dissipative operator $Hf = A_0f + V_0f + \frac{i}{2}\alpha^2f$, $f \in \text{dom}(H) = \text{dom}(A_0)$, is completely non-selfadjoint. For details see [22] and [29]. By Corollary 2.4 the operators H and L are simultaneously completely non-selfadjoint. Hence the operator L is completely non-selfadjoint.

The remaining part of the theorem is similar to the proof of Proposition 5.5 of [1].

Under the assumption (9.3) from [19] it follows that the wave operators $W_\pm(H, A_0)$ exist and are complete. Hence $\mathcal{H}^{ac}(H) \neq \{0\}$. Applying Theorem 8.1 gets that $\mathcal{H}^{ac}(L) \neq \{0\}$. •

Very often nuclear perturbations are considered.

Proposition 9.4 *Let A_0 be a self-adjoint operator and let $V \in \mathcal{L}_1(\mathcal{H})$. Then the operator L ,*

$$Lf = A_0f + Vf, \quad f \in \text{dom}(L) = \text{dom}(A_0), \quad (9.5)$$

is closed and the spectrum of L is discrete in $\mathbf{C}_\pm$. For each $h, g \in \mathcal{H}$ the function $\Xi_{h,g}(z)$ is meromorphic in $\mathbf{C}_\pm$ and possesses boundary values $\Xi_{h,g}^\pm(k)$ for a.e. $k \in \mathbf{R}^1$. If $h \in \mathcal{H}^s(L^)$, then for each $g \in \mathcal{H}$ one has $\Xi_{h,g}^+(k) = \Xi_{h,g}^-(k)$ for a.e. $k \in \mathbf{R}^1$. If $\mathcal{H}^{ac}(A_0) \neq \{0\}$, then $\mathcal{H}^{ac}(L) = \{0\}$.*

Proof: We set $V_0 = \frac{1}{2}(V + V^*) \in \mathcal{L}_1(\mathcal{H})$ and $W = \frac{1}{2i}(V - V^*) \in \mathcal{L}_1(\mathcal{H})$, $A = A_0 + V_0$, $\alpha = \sqrt{2|W|} \in \mathcal{L}_2(\mathcal{H})$ and $\kappa = \text{sign}(W)|_{\mathcal{E}} : \mathcal{E} \rightarrow \mathcal{E}$, $\mathcal{E} = \overline{\text{ran}(W)}$. Note that $\ker(\kappa) = \{0\}$. Hence we have the representation

$$Lf = L_\kappa f = (A_0 + V_0)f + \frac{i}{2}\alpha\kappa\alpha f, \quad f \in \text{dom}(L) = \text{dom}(L_\kappa) = \text{dom}(A_0). \quad (9.6)$$

Let $L_{c.n.s.}$ be the completely non-selfadjoint part of L . By Corollary 2.4 one has

$$L_{c.n.s.} = L|_{\text{dom}(L) \cap \mathcal{H}_{c.n.s.}(L)}, \quad (9.7)$$

where

$$\mathcal{H}_{c.n.s.}(L) = \bigvee_{z \in \varrho(A)} (A - z)^{-1} \text{ran}(\alpha). \quad (9.8)$$

Note that $\mathcal{H}_{c.n.s.}(L)$ reduces A . Setting $\hat{A} = A|_{\text{dom}(A) \cap \mathcal{H}_{c.n.s.}(L)}$ the completely non-selfadjoint part $L_{c.n.s.}$ of L admits the representation

$$L_{c.n.s.}f = \hat{A}f + \frac{i}{2}\alpha\kappa\alpha f, \quad f \in \text{dom}(A_{c.n.s.}) = \text{dom}(\hat{A}). \quad (9.9)$$

Since for the self-adjoint part L_s of L the conclusions of the proposition are valid one can assume without loss of generality that L is completely non-selfadjoint, i.e.

$$\mathcal{H} = \bigvee_{z \in \varrho(A)} (A - z)^{-1} \text{ran}(\alpha). \quad (9.10)$$

By Proposition I.14 of [2] the boundary values $A^\pm(k)$ exist and belong to $\mathcal{L}_2(\mathcal{H})$ for a.e. $k \in \mathbf{R}^1$. Moreover, the convergence $A^\pm(k) = \lim_{\epsilon \rightarrow +0} \alpha(A - k \mp i\epsilon)^{-1}\alpha$ is in the Hilbert-Schmidt norm. Applying Corollary 9.2 we finish the proof.

Applying Theorem 8.3 we immediately find that the absolutely continuous subspace cannot disappear under a nuclear perturbation. •

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References

- [1] V.M.Adamyan, H.Neidhardt: On the absolutely continuous subspace for contractions and dissipative operators. To appear
- [2] H.Baumgärtel, M.Wollenberg: *Mathematical scattering theory*. Akademie-Verlag Berlin 1983.
- [3] M.S.Brodskii: *Triangular and Jordan representations of linear operators*. Nauka, Moscow 1966 (in Russian).
- [4] E.Durszt: On the unitary part of an operator on Hilbert space. *Acta Sci. math.* **31**, 87-89, 1970.

- [5] C.Foias, B.Sz.-Nagy: *Harmonic analysis of operators on Hilbert spaces*, Akadémiai Kiadó and North-Holland Publishing Company, Budapest-Amsterdam-London 1970.
- [6] T.Kato: *Perturbation theory of linear operators*. Springer-Verlag, Berlin-Heidelberg-New York 1966.
- [7] T.Kato: Some results on potential scattering. In *Proc. Int. Conf. Funct. Analysis and Related Topics*, 206 - 215, Tokyo 1969.
- [8] P.Koosis: *Introduction to H_p spaces*. London Mathematical Society Lecture Note Series 40. Cambridge University Press, Cambridge-London-New York-New Rochelle-Melbourne-Sydney 1980.
- [9] P.D.Lax, R.S.Phillips: *Scattering theory*. Academic Press, New York-London 1967.
- [10] N.G.Makarov, V.I.Vasjunin: A model for non-contractions and stability of the continuous spectrum. *Lecture Notes in Mathematics* **Vol. 864**, 365 - 412, Springer-Verlag 1981.
- [11] S.N.Naboko: The absolutely continuous spectrum of a non-dissipative operator, and a functional model. I. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **65**, 90 - 102, 1976 (in Russian).
- [12] S.N.Naboko: The absolutely continuous spectrum of a non-dissipative operator, and a functional model. II *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI)* **73**, 118 - 135, 1977 (in Russian).
- [13] S.N.Naboko: Wave operators for non-selfadjoint operators and the functional model. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **69**, 129 - 135, 1977 (in Russian).
- [14] S.N.Naboko: A functional model of perturbation theory, and its application to scattering theory. *Trudy Mat. Inst. Steklov. (LOMI)* **147**, 86 - 114, 1980 (in Russian).
- [15] S.N.Naboko: On the singular spectrum of a non-selfadjoint operator. *Zap. Naučn. Sem. Leningrad. Otdel. Inst. Steklov. (LOMI)* **113**, 149 - 177, 1981 (in Russian).
- [16] S.N.Naboko: Similarity problem and the structure of the singular spectrum of non-dissipative operators. *Lecture Notes in Math.* **Vol. 1043**, 147 - 151, Springer-Verlag 1984.
- [17] S.N.Naboko: On existence conditions of wave operators in the non-selfadjoint case. *Problemy. Mat. Fiz.*, **vyp. 12**, Izdat. Leningrad. Univ., 132 - 155, Leningrad 1987 (in Russian).
- [18] S.N.Naboko: The determinant of the characteristic function and the singular spectrum of a non-selfadjoint operator. *Matem. Sbornik SSSR* **57**, No.1, 21 - 41, 1987.

- [19] H.Neidhardt: *Scattering theory of contraction semigroups*. Report R-Math-05/81, Akad. Wiss. DDR, Berlin 1981.
- [20] H.Neidhardt: A dissipative scattering theory. In: *Operator Theory: Advances and Applications*. Vol. **14**, 197 - 212, 1984.
- [21] H.Neidhardt: A nuclear dissipative scattering theory. *J. Operator Theory* **14**, 57-66, 1985.
- [22] H.Neidhardt: Some remarks on the scattering theory for dissipative Schrödinger operators. In *Schrödinger Operators, Standard and Non-Standard*. World Scientific, 273-292, Singapore 1989.
- [23] V.A.Ryzhov: Locally absolutely continuous and singular subspaces of non-selfadjoint operators. *Dep. VINITI 03.11.1993, No. 2732 - V 93*, S.-Petersburg 1993 (in Russian)
- [24] V.A.Ryzhov: Rigged absolutely continuous subspaces and the stationary construction of wave operators in non-selfadjoint scattering. *Zap. Nauchn. Sem. S.-Petersburg. Otdel. Mat. Inst. Steklov. (POMI)* **217**, 144-171, 1994 (in Russian).
- [25] A.V.Shtraus: Characteristic functions of a linear operator. *Dokl. Akad. Nauk SSSR* **126**, 514-516, 1959 (in Russian)
- [26] A.V.Straus: Characteristic function of linear operators. *Izv. Akad. Nauk SSSR, ser. matemat.* **24**, No.1, 43-74, 1960 (in Russian)
- [27] A.S.Tikhonov: Invariance principle of wave operators for non-selfadjoint and non-unitary operators. *Dep. Ukr. NIINTI 25.10.1988, No. 2727 - Uk. 88*, Simferopol 1988 (in Russian).
- [28] A.S.Tikhonov: Absolutely continuous spectrum of linear operators. *Dep. Ukr. NIINTI 27.09.1988, No. 2471 - Uk 88*, Simferopol 1988 (in Russian).
- [29] B.S.Pavlov: Self-adjoint dilation of the dissipative Schrödinger operator and its resolution in terms of eigenfunctions. *Math. Sbornik SSSR* **102**, No. 4, 511-536, 1977 (in Russian).
- [30] M.Reed, B.Simon: *Methods of Modern Mathematical Physics IV: Analysis of Operators*. Academic Press, New York-San Francisco-London 1978
- [31] B.M.Solomyak: Scattering theory for almost unitary operators and functional model. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **178**, 92 - 119, 1989 (in Russian).
- [32] D.R.Yafaev: *Mathematical Scattering Theory: General Theory*. Translations of Mathematical Monographs Volume 105. American Mathematical Society, Providence, Rhode Island, 1992.

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