Abstract

In this paper, we introduce an $N \times N$ matrix $a^b$ in the quantum groups $SU_q^N$ to transform the conjugate representation into the standard form so that we are able to compute the explicit forms of the important quantities in the bicovariant differential calculus on $SU_q^N$, such as the $q$-deformed structure constants $C^r_{ij}$ and the $q$-deformed transposition operator $\delta$. From the $q$-Gauss constraint condition we define the generalized $q$-deformed Killing form and the $q$-deformed homotopy operator we are able to compute the $q$-deformed Chern-Simons $\Omega^{(N)}_2$ to transform the conjugate representation into the standard form so that we are able to compute the explicit forms of the important quantities in the bicovariant differential calculus on $SU_q^N$. Furthermore, the $q$-deformed Yang-Mills equation is derived.
1. INTRODUCTION

Recently, quantum groups have attracted increasing attention. Since the quantum group is provided by a noncommutative algebra, the noncommutative geometry presented by Connes [Con] plays a basic role like the differential geometry in the usual gauge theory. Following the general ideas of Connes, Woronowicz [Wor1] [Wor2] developed the framework of the noncommutative differential calculus. He introduced the bimodule over the quantum group, and presented various theorems concerning the differential forms and exterior derivative. Manin [Manin] suggested a general construction of quantum groups as linear transformations upon the quantum superplane. The differential calculus on the quantum hyperplane was developed by Wess and Zumino [WZ]. There have been a lot of papers treating the differential calculus on quantum groups and the deformed gauge theories from various viewpoints [Ber] [Jur] [Zum2] [CSWW] [IP] [AC] [Cas] [SW] [FP] [Isa]. Here we would like to emphasize three papers on the noncommutative differential geometry and deformation of BRST algebra.

Aschieri and Castellani [AC] gave a pedagogical introduction to the differential calculus on quantum groups by stressing at all stages the connection with the classical case ($q \to 1$). As an example, they gave the explicit forms of some matrices appearing in the bicovariant differential calculus on $SU_q(2)$. CSWW [CSWW] presented a systematic construction of bicovariant bimodules on the quantum groups $SU_q(N)$ and $SO_q(N)$ by using the $\hat{R}_q$ matrix. They described the conjugate of the fundamental representation for $SU_q(N)$ as antisymmetric product of $(N - 1)$ fundamental representations, and showed the expressions appearing in the bicovariant differential calculus on $SU_q(N)$ both by formulas and by diagrams. On the other hand, the antisymmetrized product makes the calculation of explicit forms very complicated. In the later paper [Wat] Watamura investigated the $q$-deformation of BRST algebra for the quantum group $SU_q(2)$. Its generalization to $SU_q(N)$ depends on the explicit forms of the important quantities in the bicovariant differential calculus on $SU_q(N)$, such as the $q$-deformed structure constant $C_{IJ}^K$, the $q$-deformed transposition operator $\Lambda$, and projection operator $P_{\Lambda q}$. 
Based on the $q$-deformed BRST algebra presented by Watamura, we defined the $q$-deformed Killing form from the $q$-gauge covariant condition, and construct the second $q$-deformed Chern class, $q$-deformed Chern-Simons, and the cocycle hierarchy for $SU_q(2)$ in our previous paper [HIM]. In order to investigate the $SU_q(N)$ gauge theory we have to compute the explicit forms of the quantities in the bicovariant differential calculus on $SU_q(N)$.

In fact, the key for solving this problem is to change the description for the conjugate representation. As everyone knows, the conjugate of the fundamental representation in $SU(N)$ is equivalent to the antisymmetrized product of $(N-1)$ fundamental representations, as used by [CSWW]. However, it is also equivalent to a basic highest weight representation described by the last fundamental dominant weight $\lambda_{N-1}$. The monoid $e^{a\delta}$, that is an $N \times N$ matrix, plays a very important role in the explicit calculations. The monoid $e^{a\delta}$, that seems to have some relation with the $q$-deformed Weyl element [KR], is proportional to the $q$-deformed Clebsch-Gordan matrix reducing the direct representation space of $\lambda_1 \otimes \lambda_{N-1}$ into the identity representation space [Ma], and provides the relations among the relevant $\hat{R}_q$ matrices. On the other hand, it also serves as the similarity transformation from the conjugate of the fundamental representation to the highest weight representation $\lambda_{N-1}$. From those $\hat{R}_q$ matrices and monoid we compute the explicit forms of the quantities appearing in the bicovariant differential calculus on $SU_q(N)$ so that we are able to generalize the $q$-deformed $SU_q(2)$ gauge theory to the quantum groups $SU_q(N)$. In other words, for the quantum groups $SU_q(N)$, we generalize the $q$-deformed BRST algebra, define the $q$-deformed Killing form, and construct the $m$-th $q$-deformed Chern class and $q$-deformed cocycle hierarchy. In [CSWW] and [CW] a similar matrix $e^{[j]}$ was introduced, where $[j]$ is the short form for $(N-1)$ antisymmetrized indices. However, our $e^{a\delta}$ makes the calculations much simpler.

Recently, Isaev [Isa] discussed the $q$-deformed Chern characters where the base manifold is the $q$-deformed coset space $(GL_q(N+1))/(GL_q(N) \otimes GL(1))$. In the present paper, just like in [CSWW] and [AC], the spacetime is taken to be the ordinary commutative Minkowski spacetime, while the $q$-structure resides on the fiber, the gauge
potentials being non-commutating. In the theory there are two nilpotent operators: the BRST transformation $\delta$ and the derivative $d$, such that we can discuss the double cohomology and cocycle hierarchy. In terms of a $q$-deformed homotopy operator we are able to compute the $q$-deformed Chern-Simons $Q_{2m-1}$ by the condition $dQ_{2m-1} = P_m$. As example, we write the explicit forms of the $q$-deformed Chern-Simons $Q_3$ and $Q_5$ explicitly.

The plan of this paper is as follows. In Sec. 2 we calculate the $\hat{R}_q$ matrices in the product representation spaces of $\lambda_1 \otimes \lambda_1$, $\lambda_{N-1} \otimes \lambda_{N-1}$, $\lambda_1 \otimes \lambda_{N-1}$, and $\lambda_{N-1} \otimes \lambda_1$, and discuss their main properties. The monoid $e^{a\tilde{e}}$ is introduced to relate those $\hat{R}_q$ matrices. The algebra of functions on the quantum group $SU_q(N)$ is sketched in Sec. 3. It is proved that the monoid transforms the conjugate of the fundamental representation $\lambda_1$ into the highest weight representation $\lambda_{N-1}$. The generalized $q$-Pauli matrices are defined to separate the singlet from the adjoint components. In Sec. 4 we review the bicovariant differential calculus on $SU_q(N)$, and compute the explicit forms of some important quantities. Generalizing Watamura’s investigation, the $q$-deformed BRST algebra for the quantum group $SU_q(N)$ is constructed in Sec. 5. From the condition $\delta P_m = 0$ and $dP_m = 0$, we define the $q$-deformed generalized Killing form and the $m$-th $q$-deformed Chern class $P_m$ in Sec. 6. In Sec. 7, the $q$-deformed homotopy operator is introduced in $SU_q(N)$ to compute the $q$-deformed Chern-Simons. Furthermore, the $q$-deformed cocycle hierarchy and $q$-deformed Yang-Mills equation for $SU_q(N)$ gauge theory are obtained in Sec. 7 and Sec. 8. Just like in the case of $SU_q(2)$, the components of the identity and the adjoint representations are also separated in the $q$-deformed $SU_q(N)$ gauge theory, although they are mixed in the commutative relations of BRST algebra.

2. $\hat{R}_q$ Matrices and Monoid

In the quantum enveloping algebra $U_q A_{N-1}$ there are $(N-1)$ simple roots $r_n$ and
(\(N-1\)) fundamental dominant weight \(\lambda_m\), that are related by the Cartan matrix \(a_{mn}\):

\[
\begin{align*}
\quad r_n &= \lambda_m \ a_{mn}, \quad n, \ m = 1, 2, \cdots, (N-1) \\
\quad a_{mn} &= d_{m=1}^{-1}(r_m \cdot r_n), \quad d_m = (r_m \cdot r_m)/2
\end{align*}
\]

(2.1)

In the present paper, if without special notification, summation of the repeated indices is understood. To make formulas more symmetrical, we define:

\[
\lambda_0 = \lambda_N = 0
\]

(2.2)

A highest weight representation is denoted by its highest weight, that is a positive integral combination of \(\lambda_j\). The states in the representation are described by their weights that are the integral combinations of \(\lambda_j\). In this paper we are interested in only eight representations: the basic representations \(\lambda_1, \lambda_2, \lambda_{N-2}\) and \(\lambda_{N-1}\), the symmetrical tensor representations \(2\lambda_1\) and \(2\lambda_{N-1}\), the adjoint representation \(\lambda_1 \pm \lambda_{N-1}\), and the identity representation with the highest weight 0. Sometimes, the representations \(\lambda_2\) and \(\lambda_{N-2}\) are also called the antisymmetrical tensor representations.

The fundamental representation \(\lambda_1\) is \(N\)-dimensional. The states in this representation are described by their weight \((\lambda_a - \lambda_{a-1})\). For simplicity, we enumerate the states by one index \(a\) as usual:

\[
a \rightarrow \lambda_a - \lambda_{a-1}, \quad a = 1, 2, \cdots, N
\]

(2.3)

The conjugate of the representation \(\lambda_1\) is equivalent to the representation \(\lambda_{N-1}\), where the states have the weights \(\lambda_{a-1} - \lambda_a\) and are enumerated by one index \(\bar{a}\):

\[
\bar{a} \equiv -a \rightarrow \lambda_{a-1} - \lambda_a, \quad \bar{a} = \bar{N}, \cdots, \bar{2}, \bar{1},
\]

(2.4)

In the present paper, if without special notification, the small Latin letter except for \(n\) and \(m\), such as \(a\) and \(i\), runs over 1, 2, \cdots, \(N\).

The standard method for calculating the solutions \(\hat{R}_q\) of the simple Yang-Baxter equation [Res] [Ma] is to expand it by the projection operators that are the products of two quantum Clebsch-Gordan matrices. The calculation method for \(\hat{R}_q\) was described in the textbook [Ma] in detail. In order to fit the usual notation in the theory of the
quantum groups, the solution $\hat{R}_q$ here is related to the solution $\check{R}_q$ used in the book [Ma] as follows:

$$\hat{R}_q = q \check{R}_q^{-1} \quad (2.5)$$

In the direct product space of $\lambda_1 \otimes \lambda_1$ the solution of the simple Yang-Baxter equation is:

$$\left( \hat{R}_q^{\lambda_1, \lambda_1} \right)_{ab}^{cd} = q \left( P_2^{\lambda_1, \lambda_1} \right)_{ab}^{cd} - q^{-1} \left( P_{\lambda_2}^{\lambda_1, \lambda_1} \right)_{ab}^{cd} \quad (2.6)$$

Similarly, in the direct product spaces of $\lambda_{N-1} \otimes \lambda_{N-1}$, $\lambda_1 \otimes \lambda_{N-1}$, and $\lambda_{N-1} \otimes \lambda_1$, the solutions of the simple Yang-Baxter equation are as follows, respectively:

$$\left( \hat{R}_q^{\lambda_{N-1}, \lambda_{N-1}} \right)_{\tau \delta}^{a \beta} = q \left( P_{\lambda_{N-2}}^{\lambda_{N-1}, \lambda_{N-1}} \right)_{\tau \delta}^{a \beta} - q^{-1} \left( P_{\lambda_{N-2}}^{\lambda_{N-1}, \lambda_{N-1}} \right)_{\tau \delta}^{a \beta} \quad (2.7)$$

The superscripts, for example $\lambda_1 \lambda_1$, have been implied in the super- and sub-scripts $ab$ and $cd$, and can be neglected. Now, through straightforward calculation, we obtain:

$$\left( \hat{R}_q \right)_{ab}^{cd} = \left( \hat{R}_q \right)_{ab}^{cd} = \left( \hat{R}_q \right)_{\tau \delta}^{a \beta} = \left( \hat{R}_q \right)_{\tau \delta}^{a \beta} = \begin{cases} \frac{q}{r} \text{ when } a = b = c = d \\ \frac{r}{p} \text{ when } a = c < b = d \\ \frac{1}{r} \text{ when } a = d \neq b = c \\ 1 \text{ the else cases} \end{cases}$$

$$\left( \hat{R}_q^{-1} \right)_{ab}^{cd} = \left( \hat{R}_q^{-1} \right)_{ab}^{cd} = \left( \hat{R}_q^{-1} \right)_{\tau \delta}^{a \beta} = \left( \hat{R}_q^{-1} \right)_{\tau \delta}^{a \beta} = \begin{cases} \frac{q}{r} \text{ when } a = b = c = d \\ \frac{r}{p} \text{ when } a = c > b = d \\ \frac{1}{r} \text{ when } a = d \neq b = c \\ 1 \text{ the else cases} \end{cases} \quad (2.8)$$

$$\left( \hat{R}_q \right)_{\tau \delta}^{a \beta} = \left( \hat{R}_q \right)_{\tau \delta}^{a \beta} = \begin{cases} \frac{q}{r} \text{ when } a = b = c = d \\ \frac{r}{p} \text{ when } a = b > c = d \\ \frac{1}{r} \text{ when } a = d \neq b = c \\ 1 \text{ the else cases} \end{cases} \quad (2.9)$$
where the diagonal matrix \( D \) is related to double antipode action (see Sec. 3):

\[
\delta_a^b q^{-N+2 s-1} = (-1)^{N-1} c_{a}^{\bar{c}} c_{b}^{d}
\]

Four \( \hat{R}_q \) matrices can be related by \( e^{a b} \) matrices:

\[
\left( \hat{R}_q^{\pm 1} \right)_{\bar{c} d}^{a b} = q^{\pm 1} e_{\bar{c} r} \left( \hat{R}_q^{\mp 1} \right)_{b s}^{r c} e^{c d} = q^{\pm 1} e_{d c} \left( \hat{R}_q^{\mp 1} \right)_{d b}^{c c} e^{c s} \left( \hat{R}_q^{\mp 1} \right)_{a s} \quad (2.13)
\]

\( \hat{R}_q \) matrix satisfies some important relations [Res]:

\[
\left( \hat{R}_q^{\pm 1} \right)_{b d}^{a c} D_{d c}^{d} = q^{\pm N} \delta^a_b \quad (2.14)
\]

\[
D_{r s}^a D_{s d}^c \left( \hat{R}_q^{\pm 1} \right)_{b a}^{r s} D^c_b D^a_d \quad (2.15)
\]
3. Algebra of Functions on the Quantum Group $SU_q(N)$

A quantum group is introduced as the non-commutative and non-cocommutative Hopf algebra $\mathcal{A} = Fun_q(\mathcal{G})$ obtained by continuous deformations of the Hopf algebra of the function of a Lie group. The associative algebra $\mathcal{A}$, the algebra of functions on the quantum group, is freely generated by non-commuting matrix entries $T^a_b$ satisfying the relation:

$$(\hat{R}_q)^{ab}_{rs} T^r_c T^s_d = T^a_r T^b_s (\hat{R}_q)^{rs}_{cd}$$  \hspace{1cm} (3.1)$$

where $(\hat{R}_q)^{ab}_{cd}$ is given in (2.6).

$T^a_b$, the elements of the fundamental representation of quantum group $SU_q(N)$, satisfy the Hopf algebraic relations:

$$\Delta(T^a_b) = T^a_d \otimes T^d_b, \quad \epsilon(T^a_b) = \delta^a_b$$

$$\kappa(T^a_{rs}) T^r_b = T^a_r \kappa(T^r_b) = \delta^a_b, \quad \kappa^2(T^a_b) = D^a_c T^c_d (D^{-1})^d_b$$  \hspace{1cm} (3.2)$$

where the diagonal matrix $D$ is given in (2.12).

The $q$-determinant $det_q T$ is commutant with any element $T^a_b$. For quantum group $SU_q(N)$ we have:

$$det_q T = \sum_P (-q)^{\ell(P)} T^1_{p1} \cdots T^N_{pN} = 1,$$

$$q^* = q, \quad (T^a_b)^* = \kappa(T^b_a)$$  \hspace{1cm} (3.3)$$

where $\ell(P)$ is the minimal number of inversions in the permutation $P$. $\kappa(T^b_a)$ can be expressed as antisymmetrized product of $(N-1) T^a_b$ [CSWW]. On the other hand, the conjugate of the fundamental representation $\lambda_1$ is equivalent to the representation $\lambda_{N-1}$. Define:

$$T^\pi_{b} = \epsilon_{r} \kappa(T^r_s) \epsilon^{\pi s}, \quad \kappa(T^b_a) = \epsilon^{b r} T^r_s \epsilon_{s a}$$  \hspace{1cm} (3.4)$$

For $SU_q(2)$, $\lambda_1 = \lambda_{N-1}$. In this case the fundamental representation becomes self-conjugate:

$$T^\pi_{b} = T^3_{3-b}$$

From (3.1), (3.2) and (2.13) we can show that $T^\pi_{b} \in \mathcal{A}$ defined in (3.4) belongs to the representation $\lambda_{N-1}$.
Proposition 1. \( T^\alpha_{\beta} \in \mathcal{A} \) defined in (3.4) satisfy:

\[
\begin{align*}
\left( \hat{R}_{\alpha} \right)_{\beta}^{\sigma} T^\sigma_{\gamma} T^{\gamma}_{\delta} = T^\alpha_{\beta} T^\gamma_{\epsilon} \left( \hat{R}_{\gamma} \right)_{\epsilon}^{\sigma} \\
\left( \hat{R}_{\alpha} \right)_{\beta}^{\sigma} T^\sigma_{\epsilon} T^{\epsilon}_{\delta} = T^\alpha_{\beta} T^\sigma_{\gamma} \left( \hat{R}_{\sigma} \right)_{\gamma}^{\sigma} \\
\left( \hat{R}_{\alpha} \right)_{\beta}^{\sigma} T^\sigma_{\gamma} T^{\gamma}_{\delta} = T^\alpha_{\beta} T^\sigma_{\epsilon} \left( \hat{R}_{\sigma} \right)_{\epsilon}^{\sigma}
\end{align*}
\tag{3.5}
\]

**Proof.** From (3.1) and (3.2) we have

\[
\kappa(T^a_{\alpha}) \left( \hat{R}_{\alpha} \right)_{\beta}^{\sigma} T^\epsilon_{\gamma} = T^b_{\alpha} \left( \hat{R}_{\alpha} \right)_{\beta}^{\sigma} \kappa(T^\alpha_{\delta})
\]

In terms of (2.13) and (3.4) we obtain:

\[
\kappa(T^a_{\alpha}) \left( \hat{R}_{\alpha} \right)_{\beta}^{\sigma} T^\epsilon_{\gamma} = e^a T^\alpha_{\delta} \epsilon_{\alpha\delta} \left( \hat{R}_{\alpha} \right)_{\beta}^{\sigma} T^\epsilon_{\gamma}
\]

\[
= q e^a \epsilon_{\alpha\delta} \left( \hat{R}_{\alpha} \right)_{\beta}^{\sigma} T^\epsilon_{\gamma} \epsilon_{\sigma\delta}
\]

\[
T^b_{\alpha} \left( \hat{R}_{\alpha} \right)_{\beta}^{\sigma} \kappa(T^\alpha_{\delta}) = T^b_{\alpha} \left( \hat{R}_{\alpha} \right)_{\beta}^{\sigma} \epsilon_{\alpha\delta} \left( \hat{R}_{\alpha} \right)_{\beta}^{\sigma} T^\epsilon_{\gamma}
\]

\[
= q e^a \epsilon_{\alpha\delta} \left( \hat{R}_{\alpha} \right)_{\beta}^{\sigma} T^\epsilon_{\gamma} \epsilon_{\sigma\delta}
\]

Comparing the quantities in the brackets, we get the first relation in (3.5). The proof of the other two relations in (3.5) can be performed analogously. Q.E.D.

The direct product of \( T^a_{\alpha} \) and \( T^\beta_{\gamma} \) spans the mixed space of the adjoint and identity representations:

\[
T^a_{\alpha} T^\beta_{\gamma} = T^a_{\alpha} \epsilon_{\alpha\beta} \kappa(T^\alpha_{\delta}) \epsilon_{\delta\beta}
\tag{3.6}
\]

In order to separate the singlet and the adjoint components in \( T^a_{\alpha} T^\beta_{\gamma} \), we define the generalized \( q \)-Pauli matrices, that obviously have to be proportional to the quantum Clebsch-Gordan coefficients:

\[
\begin{align*}
\left( \sigma^0 \right)_{a\beta} &= q^2 (\sigma_0)_{a\beta} = -q [N]^{-1/2} e^{a\beta} \\
\left( \sigma^{ij} \right)_{a\beta} &= \left( \sigma_{ij} \right)_{a\beta} = (-1)^{N-a} \delta_{ai} \delta_{bj}, \quad i \neq j \\
\left( \sigma^{ij} \right)_{a\beta} &= \delta_{ab} \quad \begin{cases} (-1)^{N+a} q^{a-j+(N+1)/2} \{ [j-1][i] \}^{-1/2} & \text{when } a < j \\
(-1)^{N+j+1} q^{-j+(N+1)/2} \{ [i-1][j] \}^{1/2} & \text{when } a = j, \quad j \geq 2 \\
0 & \text{when } a > j \end{cases} \\
\left( \sigma^{ij} \right)_{a\beta} &= \delta_{ab} \quad \begin{cases} (-1)^{N+a} q^{a-j-(N+3)/2} \{ [j-1][i] \}^{-1/2} & \text{when } a < j \\
(-1)^{N+j+1} q^{-j-(N+3)/2} \{ [i-1][j] \}^{1/2} & \text{when } a = j, \quad j \geq 2 \\
0 & \text{when } a > j \end{cases}
\end{align*}
\tag{3.7}
\]
where the repeated $j$ is not summed. In the present paper, we describe the adjoint components by a capital Latin, such as $I$, that runs over $(ij)$ where $i \neq j$, and $(jj)$ where $j \geq 2$, and describe both singlet and adjoint components by a capital Latin with a hat, like $\hat{I}$, that runs over $0$ and $I$. In this notation, we have:

$$
(\sigma^I_a^b) (\sigma_j^c) = \delta^I_{j'} \
(\sigma^I_a^a) (\sigma_j^c)^c_d = \delta^c_a \delta^d_b
$$

Sometime, another set of $q$-Pauli matrices are useful:

$$
(\sigma^I_a^b) = (\sigma^I_a^c) \epsilon^b_c, \quad (\sigma_j^a) = (\sigma_j^a) \epsilon^a_{db} \
(\sigma^I_a^b) (\sigma_j^a) = \delta^I_{j'} \quad (\sigma^I_a^b) (\sigma_j^c) = \delta^d_a \delta^c_d 
$$

$$
(\sigma^0_a^b) = -q [N]^{-1/2} D^b_a, \quad (\sigma_0^a) = -q^{-1} [N]^{-1/2} \delta^a_b \
(\sigma^j_a^b) = \delta^a_i \delta^b_i q^{j-2(N+1)/2}, \quad (\sigma^j_a^c) = \delta^a_i \delta^b_j q^{-1} q^{j-1} (N+1)/2, \quad i \neq j 
$$

$$
(\sigma^{ij}_a^b) = \delta^i_a \delta^j_b \begin{cases} 
q^{2a-j} \{[j-1][j]\}^{-1/2} & \text{when } a < j \\
-\{[j-1][j]\}^{-1/2} & \text{when } a = j, \quad j \geq 2 \\
0 & \text{when } a > j 
\end{cases}
$$

$$
(\sigma^{ij}_a^b) = \delta^i_a \delta^j_b \begin{cases} 
q^{-1} \{[j-1][j]\}^{-1/2} & \text{when } a < j \\
-q^{-1} \{[j-1][j]\}^{-1/2} & \text{when } a = j, \quad j \geq 2 \\
0 & \text{when } a > j 
\end{cases}
$$

where the repeated $j$ is not summed. When $q \to 1$, both $(\sigma^I_a^b)$ and $(\sigma_j^c)$ tend to the usual generalized Pauli matrices. Now, it can be seen that the singlet and the adjoint components in $T^a_T^b$ are separated explicitly:

$$
M^I_{\hat{I}} = (\sigma^I_a^b) T^a_T^b (\sigma_j^c) = (\sigma^I_a^d) T^a_T^c \kappa(T^b_d) (\sigma_j^c) \
M^0_{\hat{I}} = 1, \quad M^I_{\hat{I}} = M^0_{\hat{I}} = 0 
$$

$$
\kappa^2(M^I_{\hat{I}}) = D^I_K M^K_L (D^{-1})^L_{\hat{I}} \
D^0_{\hat{I}} = 1, \quad D^I_{\hat{I}} = q^{2a-j}
$$

where $D$ is a diagonal matrix.

The linear functionals $(L^\pm)_a^b$, defined by their values on the entries $T^a_t$, belong to the dual Hopf algebra $\mathcal{A}'$ [FRT]:

$$
(L^+)^a_b (T^c_d) = q^{-1/N} (\hat{F}_q)^a_c d_b, \quad (L^-)^a_b (T^c_d) = q^{1/N} (\hat{F}_q)^a_c d_b
$$
\[ \Delta'(\{(L^\pm)_a\}_b) = (L^\pm)_a \otimes (L^\pm)_b, \quad \epsilon'(\{(L^\pm)_a\}_b) = \delta^a_b \]  
\[ \kappa'(\{(L^\pm)_a\}_c) (L^\pm)_b = \delta^a_b = (L^\pm)_a \kappa'(\{(L^\pm)_c\}_b) \]  
\[ (\hat{R}_q)^{ab}_{rs} (L^\pm)_s (L^\pm)^r_c = (L^\pm)_a (L^\pm)^a_r (\hat{R}_q)^{rs}_{cd} \]  
\[ (\hat{R}_q)^{ab}_{rs} (L^\pm)_s (L^-)^r_c = (L^\pm)_a (L^-)^a_r (\hat{R}_q)^{rs}_{cd} \]  
(3.13)  
(3.14)

From (3.12) we have:
\[ \kappa'(\{(L^\pm)_a\}_b) (T^c_d) = q^{\pm 1/N} (\hat{R}^{\pi 1}_{q})^{c a}_{d b} \]  
\[ (((L^\pm)_a)_b) (T^\pi_{d}) = q^{\pm 1/N} c_d (\hat{R}^{\pi 1}_{q})^{r a}_{b r} c^\pi = q^{\pi (1-1/N)} (\hat{R}^{\pi 1}_{q})^{a \pi}_{d b} \]  
\[ \kappa'(\{(L^\pm)_a\}_b) (T^\pi_{d}) = q^{\pi 1/N} c_d (D^r a) (\hat{R}^{\pi 1}_{q})^{a u}_{v b} (D^{-1})^{v s}_{u r} c^\pi \]  
(3.15)

4. Bicovariant Differential Calculus on Quantum Groups \( SU_q(N) \)

[CSWW] constructed the bimodule, for the quantum groups \( SU_q(N) \) explicitly. In order to avoid confusion with the spacetime derivative, following Watamura’s notation [Wat] in the theory of \( q \)-deformed BRST algebra, we call the first-order differential operator on \( \mathcal{A} \) the BRST transformation operator, denoted by \( \delta \):
\[ \delta : \mathcal{A} \rightarrow , \]  
\[ \rho = \sum \alpha \delta \beta \in , \quad \text{if} \quad \rho \in , \]  
where \( \alpha, \beta \in \mathcal{A} \).

A left action \( \Delta_L \) and a right action \( \Delta_R \) of the quantum group on , are defined as follows:
\[ \Delta_L : \rightarrow \mathcal{A} \otimes , \]  
\[ \Delta_L(\alpha \delta \beta) = \Delta(\alpha) \text{id} \otimes \delta \Delta(\beta) \]  
\[ \Delta_R : \rightarrow , \otimes \mathcal{A}, \]  
\[ \Delta_R(\alpha \delta \beta) = \Delta(\alpha) \delta \otimes \text{id} \Delta(\beta) \]  
(4.2)

The tensor product between elements \( \rho, \rho' \in , \) is defined to have the properties:
\[ (\rho \alpha) \otimes \rho' = \rho \otimes (\alpha \rho') \]  
\[ \alpha(\rho \otimes \rho') = (\alpha \rho) \otimes \rho', \quad (\rho \otimes \rho')\alpha = \rho \otimes (\rho' \alpha) \]  
(4.3)

The left and right actions on , \( \otimes \), are defined by:
\[ \Delta_L : , \otimes , \rightarrow \mathcal{A} \otimes , \otimes , \]  
\[ \Delta_R : , \otimes , \rightarrow , \otimes , \otimes \mathcal{A} \]  
(4.4)
For example,
\[
\begin{align*}
\Delta_L(\rho_1) &= \alpha_1 \otimes \rho'_1, \quad \Delta_L(\rho_2) = \alpha_2 \otimes \rho'_2 \\
\Delta_L(\rho_1 \otimes \rho_2) &= \alpha_1 \alpha_2 \otimes \rho'_1 \otimes \rho'_2
\end{align*}
\] (4.5)

From the definition we have:
\[
\begin{align*}
(e \otimes id) \Delta_L(\rho) &= \rho, \quad (id \otimes e) \Delta_R(\rho) = \rho \\
(\Delta \otimes id) \Delta_L &= (id \otimes \Delta_L) \Delta_L, \quad (id \otimes \Delta) \Delta_R = (\Delta_R \otimes id) \Delta_R \\
(id \otimes \Delta_R) \Delta_L &= (\Delta_L \otimes id) \Delta_R
\end{align*}
\] (4.6)

[CSWW] constructed the fundamental bicovariant bimodule of \(SU_q(N)\):
\[
\begin{align*}
\Delta_R(\eta^a) &= \eta^a \otimes 1, \quad \Delta_L(\eta^a) = T^a_b \otimes \eta^b \\
\eta^a \alpha &= (\alpha \ast f^a_b) \eta^b, \quad \alpha \eta^a = \eta^b (\alpha \ast f^a_b \circ \kappa)
\end{align*}
\] (4.7)

where \((\alpha \ast f^a_b) \equiv (f^a_b \otimes id) \Delta \alpha\).

Applying the \(*\)-operation on (4.7), [CSWW] obtained
\[
\begin{align*}
(\eta^a)^* &\equiv \bar{\eta}_a, \quad \Delta_R(\bar{\eta}_a) = \bar{\eta}_a \otimes 1, \quad \Delta_L(\bar{\eta}_a) = \kappa(T^a_b) \otimes \bar{\eta}_b \\
\bar{\eta}_a \alpha &= (\alpha \ast \bar{f}^a_b) \bar{\eta}_b, \quad \alpha \bar{\eta}_a = \bar{\eta}_b (\alpha \ast \bar{f}^a_b \circ \kappa)
\end{align*}
\] (4.8)

From the consistency with the commutation relation of the generators and the requirement for \(SU_q(N)\) that the \(q\)-determinant of \(T^a_b\) commutes with any element in \(\sigma_a\), [CSWW] found that there are two independent functionals such that the following constructions are performed in a completely parallel way. Following [CSWW] we choose one of them as follows:
\[
f^a_b = (L^+)^a_b, \quad \bar{f}^a_b = \kappa^{-1}(L^-)^a_b
\] (4.9)

Transforming the bases \(\bar{\eta}_a\) by the monoid \(e^{\bar{L}^a_b}\), we obtain the bases \(\eta^j\) for the mixed representation of identity and adjoint representations:
\[
\begin{align*}
\bar{\eta}^\delta &= \bar{\eta}_a e^{\bar{L}^a_b}, \quad \bar{\eta}_a = \bar{\eta}^\delta e_{\bar{\theta}_a} \\
\eta^j &= (\sigma^j)_{\bar{a}\bar{b}} \eta^\delta \bar{\eta}_b = (\sigma^j)_{\bar{a}\bar{b}} \eta^\delta \bar{\eta}_b
\end{align*}
\] (4.10)

In the commutative relations of the \(q\)-deformed BRST algebra the components of identity and adjoint representations are mixed. For the bases \(\eta^j\) we have
\[
\begin{align*}
\Delta_R(\eta^j) &= \eta^j \otimes 1, \quad \Delta_L(\eta^j) = M^j_K \otimes \eta^K \\
\alpha \eta^j &= \eta^K (\alpha \ast L^j_K), \quad \alpha \in \mathcal{A}, \quad L^j_K \in \mathcal{A}'
\end{align*}
\] (4.11)
where
\[
L^j_i = (\sigma^j)_{\alpha^d} \kappa'((L_+)^\alpha_b) (L_-)^c_d (\sigma_i)^b_c \\
= (\sigma^j)_{\alpha^d} \left\{ e^{\delta_{\alpha^d}} \kappa'((L_+)^\alpha_b) (L_-)^c_d c_{\alpha^d} \right\} (\sigma_i)^b_c
\]  
(4.12)

\[
L^j_i(ab) = L^k_i(a) L^j_k(b), \quad L^j_i(1) = \delta^j_i
\]

\[
(\rho \ast L^j_i) = (L^j_i \otimes i\delta) \Delta_L(\rho)
\]

Note that:
\[
\Delta_L(\eta^0) = 1 \otimes \eta^0, \quad \Delta_R(\eta^0) = \eta^0 \otimes 1
\]  
(4.13)

The bases of the left-invariant element of \(\), are easy to be calculated from \(\eta^j\):
\[
\omega^j = \kappa(M^j_K) \eta^K, \quad \Delta_L(\omega^j) = 1 \otimes \omega^j, \quad \Delta_R(\omega^j) = \omega^K \otimes \kappa(M^j_K)
\]  
(4.14)

As the analogue of the ordinary permutation operator, a bimodule automorphism \(\Lambda\) in \(\otimes\), is defined by:
\[
\Lambda(\omega^j \otimes \eta^K) = \eta^K \otimes \omega^j
\]

\[
\Lambda(\alpha \tau) = \alpha \Lambda(\tau), \quad \Lambda(\tau \alpha) = \Lambda(\tau) \alpha, \quad \alpha \in A, \quad \tau \in \otimes,
\]

Thus, we have
\[
\Lambda(\eta^j \otimes \eta^j) = \Lambda^{ij}_{KL} \eta^j \otimes \eta^j, \quad \Lambda^{ij}_{KL} = L^j_i(M^j_L)
\]  
(4.16)

In terms of (2.13), (3.12) and (3.15) we are able to compute \(\Lambda^{ij}_{KL}\) explicitly:
\[
(\sigma^j)^{\alpha^d} (\sigma_j)^{\alpha^d} \Lambda^{ij}_{KL} (\sigma^K)^{\alpha^d} \eta^j \sigma^L = (\sigma^K)^{\alpha^d} L^j_i(T^a_k) L^j_s(T^a_s) (\sigma_j)^{\alpha^d} \\
= \kappa'((L_+)^\alpha_b) (T^a_k) e^{\alpha^d} \kappa'((L_+)^\alpha_b) (T^a_s) e^{\alpha^d} \\
= q^{1/N} (\hat{R}^{-1})^ar_{iu} \hat{e}_{j}^s q^{1/N} (\hat{R}^{-1})^tu_{kw} e^{w^d} q^{-1/N} e_{ar} D_{r's'} (\hat{R}_r)_{u'r'} (D^{-1})_{u'r'} e^{v^d} \\
\cdot e_{w^d} x^{1-1/N} (\hat{R}^{-1})^ar_{iu} \hat{e}_{j}^s q^{1/N} (\hat{R}^{-1})^tu_{kw} e^{w^d} q^{-1/N} e_{ar} D_{r's'} (\hat{R}_r)_{u'r'} (D^{-1})_{u'r'} e^{v^d}
\]

For given matrices \((P_1)^{\alpha^d}_{ij}\) and \((P_2)^{\beta^d}_{K\ell}\) define an operator \((P_1, P_2)^{ij}_{KL}\) [CSWW]:
\[
(P_1, P_2)^{ij}_{KL} = (\sigma^j)^{\alpha^d} (\sigma_j)^{\alpha^d} (\hat{R}^{-1})^ar_{iu} (X)_{iu} (Y)_{ar} (\hat{R}_r)_{u'r'} (\hat{R}_r)_{u'r'} (\sigma^K)^{\alpha^d} (\sigma^L)^{\alpha^d}
\]  
(4.17)
Hence, we obtain four projection operators, orthogonal to each other:

\[
(P_{2\lambda_1}, P_{2\lambda_{N-1}}), \quad (P_{2\lambda_1}, P_{\lambda_{N-2}}), \quad (P_{\lambda_2}, P_{2\lambda_{N-1}}), \quad (P_{\lambda_2}, P_{\lambda_{N-2}})
\]

\[
P_S \equiv (P_{2\lambda_1}, P_{2\lambda_{N-1}}) + (P_{\lambda_2}, P_{\lambda_{N-2}})
\]

\[
P_A \equiv (P_{2\lambda_1}, P_{\lambda_{N-2}}) + (P_{\lambda_2}, P_{2\lambda_{N-1}})
\]

\[
P_S + P_A = 1
\]

From (4.17), (2.6) and (2.7) we have

\[
\Lambda = \left(\hat{R}^{-1}, \hat{R}\right)
\]

\[
= \left(\hat{R}^{-1}, \hat{R}\right) - q^{-2} (P_{2\lambda_1}, P_{\lambda_{N-2}}) - q^2 (P_{\lambda_2}, P_{2\lambda_{N-1}}) + (P_{\lambda_2}, P_{\lambda_{N-2}})
\]

\[
\Lambda^{-1} = \left(\hat{R}^{-1}, \hat{R}\right)
\]

\[
= \left(\hat{R}^{-1}, \hat{R}\right) - q^2 (P_{2\lambda_1}, P_{\lambda_{N-2}}) - q^{-2} (P_{\lambda_2}, P_{2\lambda_{N-1}}) + (P_{\lambda_2}, P_{\lambda_{N-2}})
\]

Therefore, \(\Lambda^{ij}_{KL}\) satisfy the Yang-Baxter equation:

\[
\Lambda^{ij}_{LS} \Lambda^{KS}_{TR} \Lambda^{ST}_{PQ} = \Lambda^{JK}_{LS} \Lambda^{IL}_{PT} \Lambda^{TL}_{QR}
\]

(4.21)

From (4.20) we know that the eigenvalues of \(\Lambda\) matrix are 1, \(-q^2\) and \(-q^{-2}\):

\[
\left(\Lambda + q^2\right) \left(\Lambda + q^{-2}\right) (\Lambda - 1) = 0
\]

(4.22)

From the symmetry of the quantum Clebsch-Gordan coefficients ([Ma] P.156) we have:

\[
\left(\Lambda^{-1}\right)^{ij}_{(k\ell)} (s\tilde{t}) = \Lambda^{ij}_{(k\ell)(s\tilde{t})}
\]

(4.23)

Through direct calculation of (4.17), we obtain the non-vanishing components of \(\Lambda^{ij}_{KL}\) as follows:

\[
\Lambda^{ij}_{KL} = (\Lambda^{-1})^{ij}_{KL} = \delta^i_K \delta^j_L + f^{ij}_{LP} \tilde{f}^{LP}_{KL}, \quad \Lambda^{00}_{00} = (\Lambda^{-1})^{00}_{00} = 1
\]

\[
\Lambda^{i0}_{JK} = (\Lambda^{-1})^{i0}_{JK} = \lambda f^{iJ}_{JK}, \quad \Lambda^{JK}_{01} = (\Lambda^{-1})^{JK}_{01} = \lambda \tilde{f}^{JK}_i
\]

\[
\Lambda^{0j}_{0J} = (\Lambda^{-1})^{0j}_{0J} = \delta^j_J, \quad \Lambda^{0j}_{0J} = (\Lambda^{-1})^{0j}_{0J} = (\lambda^2 + 1) \delta^j_J
\]

(4.24)

where \(f^{iJ}_{JK}\) and \(\tilde{f}^{JK}_i\) satisfy:

\[
f^{(rs)}_{(i\tilde{r})(k\ell)} = 0, \quad \tilde{f}^{(ij)}_{(k\ell)} = 0,
\]

\[
\text{if } \lambda_i - \lambda_{i-1} - \lambda_j + \lambda_{j-1} + \lambda_k - \lambda_{k-1} - \lambda_l + \lambda_{l-1} \neq \lambda_r - \lambda_{r-1} - \lambda_s + \lambda_{s-1}
\]

\[
f^{(rs)}_{(i\tilde{r})(k\ell)} = f^{(s\tilde{r})}_{(i\tilde{r})(j\tilde{t})}, \quad \tilde{f}^{(ij)}_{(k\ell)} = \tilde{f}^{(i\ell)}_{(j\tilde{t})}, \quad f^{l}_{RS} \tilde{f}^{RS}_j = -(\lambda^2 + 2) \delta^j_J
\]

(4.25)
The non-vanishing components of \( f_{j \ell \ell}^{jk} \) and \( f_{i \ell \ell}^{jk} \) are listed as follows. In the following (4.26) there is no summation for the repeated indices.

\[
\begin{align*}
\tilde{f}_{(i \ell) (k \ell)}^{(j \ell)} &= [N]^{-1/2} q^{-k-(N-3)/2}, & \tilde{f}_{(i \ell) (j \ell)}^{(k \ell)} &= -[N]^{-1/2} q^{k-(N-1)/2} \\
\tilde{f}_{(i \ell) (k \ell)}^{(i \ell)} &= -[N]^{-1/2} q^{-k+(3N-1)/2}, & \tilde{f}_{(i \ell) (i \ell)}^{(j \ell)} &= [N]^{-1/2} q^{k-(N+3)/2} \quad (4.26a)
\end{align*}
\]

When \( i \neq j \neq k \neq i \)

\[
\begin{align*}
\tilde{f}_{(i \ell) (j \ell)}^{(i \ell)} &= \begin{cases} 
-q^{-N} \left( \frac{[k-1]}{[N][k]} \right)^{1/2} & \text{if } k = i < j \\
-q^{-N} \left( \frac{k}{[N][k-1]} \right)^{1/2} & \text{if } j < i = k \\
q^{2k-N-1} \left( \frac{k}{[N][k-1]} \right)^{1/2} & \text{if } i < j = k \\
q^{2k-N} \left( \frac{[k-1]}{[N][k]} \right)^{1/2} & \text{if } k = j < i \\
q^{k-N} \left( \frac{[N][k][k-1]}{[N][k-1]} \right)^{-1/2} & \text{if } i < k < j \\
q^{-k-N} \left( \frac{[N][k][k-1]}{[N][k-1]} \right)^{-1/2} & \text{if } j < k < i 
\end{cases} \\
\tilde{f}_{(i \ell) (j \ell)}^{(j \ell)} &= \begin{cases} 
q^{2N-2k} \left( \frac{[k-1]}{[N][k]} \right)^{1/2} & \text{if } k = i < j \\
q^{2N-2k+1} \left( \frac{k}{[N][k-1]} \right)^{1/2} - \lambda q^{N-k} \left( \frac{[N]}{[k][k-1]} \right)^{1/2} & \text{if } j < i = k \\
-q^{-1} \left( \frac{k}{[N][k-1]} \right)^{1/2} - \lambda q^{N-k} \left( \frac{[N]}{[k][k-1]} \right)^{1/2} & \text{if } i < j = k \\
-q^{-2} \left( \frac{k}{[N][k-1]} \right)^{1/2} & \text{if } k = j < i \\
-q^{2N-k} \left( \frac{[N][k][k-1]}{[N][k-1]} \right)^{-1/2} & \text{if } i < k < j \\
q^{-k} \left( \frac{[N][k][k-1]}{[N][k-1]} \right)^{-1/2} & \text{if } j < k < i \\
- \lambda q^{N-k} \left( \frac{[N]}{[k][k-1]} \right)^{1/2} & \text{if } i < j < k \\
\end{cases} 
\end{align*}
\]
Substituting $/#28/4/./2/4/#29$ into the Yang-Baxter equation $/#28/4/./2/1/#29$ for two cases:

\[ f^{(k \xi)}_{(i \xi)(j \eta)} = \begin{cases} 
-q^{2-k-j} \left( \frac{[k-1]}{[N][k]} \right)^{1/2} & \text{if } k = i < j \\
-q^{3-k-j} \left( \frac{[k]}{[N][k-1]} \right)^{1/2} & \text{if } j < i = k \\
q^{i-k+1} \left( \frac{[k]}{[N][k-1]} \right)^{1/2} & \text{if } i < j = k \\
q^{i-k+2} \left( \frac{[k-1]}{[N][k]} \right)^{1/2} & \text{if } k = j < i \\
q^{i-j-k+2} \left( \frac{[N][k][k-1]}{[k][k-1]} \right)^{1/2} & \text{if } j < k < i \\
q^{i-j-k+2} \left( \frac{[N][k][k-1]}{[k][k-1]} \right)^{-1/2} & \text{if } i < k < j \\
-q^{i-j-k+2} \left( \frac{[N][k][k-1]}{[k][k-1]} \right)^{-1/2} & \text{if } j < k < i \\
- \lambda q^{i-j-k+2} \left( \frac{[N][k][k-1]}{[k][k-1]} \right)^{1/2} & \text{if } j < k < i \\
\end{cases} \tag{4.26d} \]

\[ f^{(k \xi)}_{(i \xi)(j \eta)} = \begin{cases} 
q^{N+k-j-2} \left( \frac{[k-1]}{[N][k]} \right)^{1/2} & \text{if } k = i < j \\
q^{N+k-j-1} \left( \frac{[k]}{[N][k-1]} \right)^{1/2} - \lambda q^{2k-j-2} \left( \frac{[N]}{[k][k-1]} \right)^{1/2} & \text{if } j < i = k \\
-q^{i-k-N-2} \left( \frac{[k]}{[N][k-1]} \right)^{1/2} - \lambda q^{i-2} \left( \frac{[N]}{[k][k-1]} \right)^{1/2} & \text{if } i < j = k \\
-q^{i+k-N-2} \left( \frac{[k-1]}{[N][k]} \right)^{1/2} & \text{if } k = j < i \\
q^{i-j+k-N-2} \left( \frac{[N][k][k-1]}{[k][k-1]} \right)^{-1/2} & \text{if } i < k < j \\
q^{i-j+k-N-2} \left( \frac{[N][k][k-1]}{[k][k-1]} \right)^{-1/2} & \text{if } j < k < i \\
- \lambda q^{i-j+k-N-2} \left( \frac{[N][k][k-1]}{[k][k-1]} \right)^{1/2} & \text{if } j < k < i \\
- \lambda q^{i-j+k-N-2} \left( \frac{[N][k][k-1]}{[k][k-1]} \right)^{-1/2} & \text{if } j < k < i \\
- \lambda q^{i-j+k-N-2} \left( \frac{[N][k][k-1]}{[k][k-1]} \right)^{1/2} & \text{if } j < k < i \\
\end{cases} \tag{4.26e} \]

\[ f^{(k \xi)}_{(k \xi)(k \xi)} = \lambda q^{k-N} \left( \frac{[k][k-1]}{[N]} \right) \]

\[ f^{(k \xi)}_{(k \xi)(k \xi)} = - \lambda q^{N-k} \left\{ \left( \frac{[N]}{[k][k-1]} \right)^{1/2} - [N-k] \left( \frac{[k-1]}{[N][k]} \right)^{1/2} \right\} \tag{4.26f} \]

\[ f^{(j \xi)}_{(j \xi)(j \xi)} = - \lambda q^{N-2j+k} \left( \frac{[N]}{[k][k-1]} \right)^{1/2}, \text{ if } j < k \]

\[ f^{(j \xi)}_{(j \xi)(j \xi)} = - \lambda q^{N-k} \left( \frac{[N]}{[k][k-1]} \right)^{1/2}, \text{ if } j < k \]

Substituting (4.24) into the Yang-Baxter equation (4.21) for two cases: $\hat{K} = \hat{P} = \hat{Q} = 0,$
and \( J = K = \dot{P} = 0 \), we obtain:

\[
\begin{align*}
\int J^R K^S f^{J^I}_{RS} & = - (\lambda^2 + 1) \int J^J_K \\
\int J^K f^{J^J}_{RS} & = - (\lambda^2 + 1) \int P^J_I
\end{align*}
\] (4.27)

Now, defining the exterior product of the elements in \( ; \),

\[
\rho \wedge \rho' \equiv \rho \otimes \rho' - \Lambda (\rho \otimes \rho')
\] (4.28)

we have:

\[
\eta^I \wedge \eta^J = \left( \delta^I_K \delta^J_L - \Lambda^I_J_{KL} \right) \left( \eta^K \otimes \eta^L \right)
\]

From (4.20) and (4.22) we know that \( \eta^I \wedge \eta^J \) is annihilated by the projection operator \( P_S \):

\[
\begin{align*}
(P_S)_{NK}^{IJ} \left( \eta^K \wedge \eta^L \right) & = 0 \\
P_S = [2]^{-2} (\Lambda + q^2) (\Lambda + q^{-2}) & = [2]^{-2} \{ \Lambda + \Lambda^{-1} + (\lambda^2 + 2) 1 \}
\end{align*}
\] (4.29)

namely,

\[
\begin{align*}
\eta^0 \wedge \eta^0 & = 0 \\
\eta^0 \wedge \eta^I + \eta^I \wedge \eta^0 & = - \frac{\lambda}{\lambda^2 + 2} \int J^I_K \eta^J \wedge \eta^K \\
\int J^I_J f^{P^I}_{RS} & = \int J^I_J, \quad (P_{Adj})_{KL}^{I^I} f^K_{RS} = f^K_{I^I}, \quad (P_{Adj})_{KL}^{I^I} \left( \eta^K \wedge \eta^L \right) = \eta^I \wedge \eta^J
\end{align*}
\] (4.30)

The projection operator \( P_A \) now can be expressed as follows:

\[
P_A = [2]^{-2} \left\{ 2 - \Lambda - \Lambda^{-1} \right\}
\] (4.31)

It is interesting to notice that we can introduce a projection operator \( P_{Adj} \) with only the adjoint components that projects the product space of two adjoint representations into the adjoint representation space:

\[
\begin{align*}
(P_{Adj})_{KL}^{I^I} & = \frac{[2]^2}{2(\lambda^2 + 2)} \left(P_A\right)_{KL}^{I^I} = - (\lambda^2 + 2)^{-1} \int J^I_J f^K_{KL} \\
(P_{Adj})_{RS}^{I^I} (P_{Adj})_{KL}^{I^I} & = (P_{Adj})_{KL}^{I^I} \left(P_{Adj}\right)_{KL}^{R^I} \\
(P_{Adj})_{RS}^{I^I} f^K_{K^I} & = f^K_{I^I}, \quad (P_{Adj})_{I^I}^{R^I} f^K_{RS} = f^K_{I^I} \\
(P_{Adj})_{KL}^{I^I} \left( \eta^K \wedge \eta^L \right) & = \eta^I \wedge \eta^J
\end{align*}
\] (4.32)
Now, we are in the position to define the BRST transformation $\delta$ on $\mathcal{A}$ and $\otimes^m$.

$\delta$ is a nilpotent operator:

$$
\delta : \mathcal{A} \to \otimes^m, \quad \delta : \otimes^m \to \otimes^{[m+1]}
$$

$$
\delta \alpha = \{i g / \lambda \} (\eta^0 \alpha - \alpha \eta^0), \quad \alpha \in \mathcal{A}
$$

$$
\delta \rho = \{i g / \lambda \} \{\eta^0 \wedge \rho - (-1)^n \rho \wedge \eta^0\}, \quad \rho \in \otimes^m
$$

(4.33)

Introduce a functional $\chi \in \mathcal{A}^2$:

$$
\delta \alpha = \eta^j (\alpha \star \chi_j)
$$

$$
\chi_j = \frac{i g}{\lambda} \left( \epsilon \delta^0 j - L_j^0 \right), \quad \chi_j(1) = 0
$$

$$
\chi_j(\alpha \beta) = \chi_j(\alpha) \epsilon(\beta) + L_j^K(\alpha) \chi_K(b)
$$

$$
\chi_j \chi_K = (\chi_j \otimes \chi_K) \Delta
$$

(4.34)

where $\chi_j$ are the $q$-analogues of the tangent vectors at the identity element of the group, and $(\cdot \star \chi_j)$ are the analogues of right invariant vector fields [AC]. $\chi_j(T^a_b)$ are proportional to the $q$-deformed Pauli matrices:

$$
\chi_j(T^a_b) = -i g \left[ N \right]^{-1/2} q^1 - N + 2/N (\sigma_j)^a_b
$$

$$
\chi_0(T^a_b) = -i g \left[ N \right]^{-1/2} q^1 - N + 2/N (\sigma_0)^a_b + \{i g / \lambda \} \left( 1 - q^{2/3} \right) \delta^a_b
$$

$$
= i g \left\{ q^{-N+2/N} [N]^{-1} + \lambda^{-1} \left( 1 - q^{2/3} \right) \right\} \delta^a_b
$$

(4.35)

The $q$-deformed structure constants can be computed from (4.34) and (4.24):

$$
C^i_{JK} \equiv \chi_j(M^j_{IK}), \quad C^0_{JK} \equiv C^0_{JK} = 0
$$

$$
C^I_{0K} = -i g \lambda \delta_J^i, \quad C^I_{JK} = -i g f^I_{JK}
$$

(4.36)

Since $C^I_{JK}$ are proportional to $f^I_{JK}$, they satisfy the weight conservation condition:

$$
C^{(r \pi)}_{(i)(j) \pi} = 0, \quad i f \quad \lambda_i - \lambda_{i-1} = \lambda_j + \lambda_{j-1} + \lambda_k - \lambda_{k-1} - \lambda_\nu + \lambda_\nu - 1 - \lambda_\nu - \lambda_{\nu-1} - \lambda_\nu - \lambda_{\nu-1}
$$

(4.37)

The $q$-deformed Cartan-Maurer equation can be derived from (4.33):

$$
\delta \eta^0 = \{i g / \lambda \} \{\eta^0 \wedge \eta^0 + \eta^0 \wedge \eta^0\} = 0
$$

$$
\delta \eta^I = \{i g / \lambda \} \{\eta^0 \wedge \eta^I + \eta^I \wedge \eta^0\}
$$

$$
= \{i g / \lambda \} \{\eta^0 \otimes \eta^I + \eta^I \otimes \eta^0 - \left( \Lambda_0^0 \delta^I_{JK} + \Lambda_0^0 \delta^0_{JK} \right) \eta^j \otimes \eta^K \}
$$

$$
= -i g \lambda \eta^0 \otimes \eta^I - i g f^I_{JK} \eta^j \otimes \eta^K
$$

$$
= C^I_{JK} \eta^j \otimes \eta^K
$$

$$
= \eta^j \otimes \left( \eta^I \star \chi_j \right)
$$
namely,
\[ \delta \eta^0 = C_{jK}^{\ 0} \eta^j \otimes \eta^K = 0 \]
\[ \delta \eta^1 = C_{jK}^{\ 1} \eta^j \otimes \eta^K = \{i g / \lambda\} \left\{ \eta^0 \land \eta^j + \eta^j \land \eta^0 \right\} \]
\[ = (\lambda^2 + 2)^{-1} C_{jK}^{\ 1} \left( \eta^j \land \eta^K \right) \]  \hspace{1cm} (4.38)

From the condition \( \delta^2 \alpha = 0 \), the functionals \( \chi_J \) span the "\( q \)-deformed Lie algebra":
\[ \chi_i \chi_j - \Lambda_{i,j}^{KL} \chi_K \chi_L = C_{1,j}^{K} \chi_K \]  \hspace{1cm} (4.39)

Acting (4.39) on \( M^P_S \), we obtain the \( q \)-deformed Jacobi identities satisfied by the \( q \)-deformed structure constants:
\[ C^{P}_{iR} C^{R}_{jS} - \Lambda_{i,j}^{KL} C^{P}_{KR} C^{R}_{LS} = C^{R}_{1,j} C^{P}_{RS} \]  \hspace{1cm} (4.40)

For the adjoint components we obtain from (4.40):
\[ (P_{A_{ij}})_{i,j}^{KL} C^{P}_{KR} C^{R}_{LS} = \frac{\lambda^2 + 1}{\lambda^2 + 2} C^{R}_{1,j} C^{P}_{RS} \]  \hspace{1cm} (4.41)

In fact, (4.41) is the same as the second relation in (4.27). Similarly, acting (4.39) on \( T^a_b \) we obtain following relations:
\[ \chi_i (T^a_d) \chi_j (T^d_b) - \Lambda_{i,j}^{KL} \chi_K (T^a_d) \chi_L (T^d_b) = C^{R}_{1,j} \chi_R (T^a_b) \]  \hspace{1cm} (4.42)
\[ (P_{A_{ij}})_{i,j}^{KL} \chi_K (T^a_d) \chi_L (T^d_b) = \xi (\lambda^2 + 2)^{-1} C^{R}_{1,j} \chi_R (T^a_b) \]  \hspace{1cm} (4.43)

where and hereafter, \( \xi \) denotes a constant:
\[ \xi = q^{2/N} \left\{ 1 - \lambda q^{-N} [N]^{-1} \right\} \]  \hspace{1cm} (4.44)

5. \( q \)-deformed BRST Algebra

Watamura [Wat] investigated the \( q \)-deformed BRST algebra \( \mathcal{B} \) for \( SU_q(2) \). The investigation can be generalized into the quantum groups \( SU_q(N) \) straightforwardly. We sketch the main results in our notation.

\( \eta^j \) in the bimodule , is defined as the ghost field in the BRST algebra, that has the ghost number 1, but the degree of form 0. The gauge potential \( A^j \) has the degree of form
1, but the ghost number 0. There are two nilpotent operators in the BRST algebra: the operator $\delta$ increases the ghost number by one, and the operator $d$ increases the degree of form by one. Neglecting the matter field, that is irrelevant to our following discussion, we are only interested in four fields in the BRST algebra $\mathcal{B}$: $\eta$, $d\eta$, $A$, and $dA$, which satisfy the following algebraic relations.

Firstly, we introduce an index $n$ that is equal to the difference between the degree of form and the ghost number. The indices $n$ for $\eta$, $d\eta$, $A$ and $dA$ are -1, 0, 1, and 2, respectively. Both nilpotent operators $\delta$ and $d$ satisfy the Leibniz rule in the graded sense for the index $n$:

$$\delta^2 = 0, \quad d^2 = 0, \quad d \delta + \delta d = 0$$

$$d(XY) = (dX)Y + (-1)^{n_x} X(dY)$$

(5.1)

$$\delta(XY) = (\delta X)Y + (-1)^{n_x} X(\delta Y)$$

where $X$, $Y \in \mathcal{B}$, and $n_x$ is the index of $X$. Both $d$ and $\delta$ are covariant for the left and right actions: For any element $X \in \mathcal{B}$ they satisfy:

$$\Delta_L(\delta X) = (id \otimes \delta)\Delta_L(X), \quad \Delta_L(dX) = (id \otimes d)\Delta_L(X)$$

$$\Delta_R(\delta X) = (\delta \otimes id)\Delta_R(X), \quad \Delta_R(dX) = (d \otimes id)\Delta_R(X)$$

(5.2)

Secondly, the gauge potentials $A^I$ are assumed [Wat] to have similar properties like $\eta^I$. Hereafter, we neglect the wedge sign $\wedge$ for simplicity.

$$\{P_{\eta}^{ij} \}_{KL} \left( A^K A^L \right) = 0$$

$$A^0 A^0 = 0, \quad \{P_{A\delta}^{ij} \}_{KL} \left( A^K A^L \right) = A^I A^J$$

$$\{ig/\lambda \} \left( A^0 A^I + A^I A^0 \right) = (\lambda^2 + 2)^{-1} C_{JK}^I A^J A^K$$

(5.3)

From the consistent conditions [Wat], $d\eta^J$ and $dA^J$ have to satisfy another relation:

$$\{P_{\eta}^{ij} \}_{KL} \left( d\eta^K d\eta^L \right) = 0, \quad \{P_{A}^{ij} \}_{KL} \left( dA^K dA^L \right) = 0$$

(5.4)

namely,

$$d\eta^I d\eta^0 = d\eta^0 d\eta^I = -\lambda^{-1} f^I_{JK} d\eta^J d\eta^K$$

$$dA^I dA^0 = dA^0 dA^I = -\lambda^{-1} f^I_{JK} dA^J dA^K$$
Thirdly, the gauge potential is introduced in the covariant derivative. The covariant condition of the covariant derivative in the BRST transformation requires:

$$\delta A^0 = 0, \quad \delta A^I = d\eta^I + \frac{ig}{\chi} \left( \eta^0 A^I + A^I \eta^0 \right)$$  \hspace{1cm} (5.5)$$

Fourthly, for two different fields $X^J$ and $Y^K$ in $\mathcal{B}$ with indices $n_x$ and $n_y$, $n_x > n_y$, respectively, the consistent condition requires the following commutative relations:

$$(-1)^{n_x n_y} X^I Y^J = Y^K \left( X^I * I^J_K \right) = \Lambda^{ij}_{KL} Y^K X^L$$  \hspace{1cm} (5.6)$$

From (4.24) we have

$$(-1)^{n_x n_y} X^0 Y^J = Y^J X^0$$

$$\{ig/\lambda\} \left( Y^0 X^I - (-1)^{n_x n_y} X^I Y^0 \right)$$

$$= - ig\lambda Y^0 X^I - ig f^I_{JK} Y^J X^K = Y^J X^K C^I_{JK}$$

At last, the gauge fields $F^J$ satisfy:

$$F^J = dA^J + \{ig/\lambda\} \left( A^0 A^J + A^J A^0 \right)$$

$$F^0 = dA^0, \quad F^I = dA^I + (\lambda^2 + 2)^{-1} C^I_{JK} A^J A^K$$

$$F^I \eta^J = \eta^K F^I \Lambda^{ij}_{KL}$$

$$\delta F^I = \{ig/\lambda\} \left( \eta^0 F^I - F^I \eta^0 \right) = \eta^J F^K C^I_{JK}$$

$$dF^I = - \{ig/\lambda\} \left( A^0 F^I - F^I A^0 \right)$$

$$= - \{ig/\lambda\} \left( A^0 dA^I - dA^I A^0 \right)$$

$$= - A^J dA^K C^I_{JK}$$

The commutative relation (5.6) can be rewritten as follows.

**Proposition 2.** The components $X^a_b$ and $Y^a_b$ of two different fields $X^J$ and $Y^K$ in $\mathcal{B}$ with indices $n_x$ and $n_y$, $n_x > n_y$, respectively, satisfy:

i) 

$$(-1)^{n_x n_y} X^a_r \left( \tilde{R}^{-1}_{s_k} \right)^{ri}_{sk} Y^a_t \left( \tilde{R}^{-1}_{b_j} \right)^{tk}_{bj} = \left( \tilde{R}^{-1}_{s_k} \right)^{ai}_{rk} Y^a_s \left( \tilde{R}^{-1}_{t_j} \right)^{sk}_{tj} X^r_b$$  \hspace{1cm} (5.11)$$
\[ \{i g/\lambda\} \left( Y^o X^I - (-1)^{n_x n_y} X^I Y^o \right) (\sigma_I)_b^a = -i g q^{1-N} [N]^{-1/2} \left( Y^a_d X^d_b - (-1)^{n_x n_y} X^a_d Y^d_b \right) \] (5.12)

where

\[ X^a_b = X^i (\sigma_I)_b^a, \quad Y^a_b = Y^i (\sigma_I)_b^a \]

Proof. (5.6) can be rewritten in terms of the explicit form (4.17) of \( \Lambda^{ij}_{KL} \):

\[ (-1)^{n_x n_y} (\sigma_I)_b^a (\sigma_j)_l^p X^I Y^J = \left( \hat{R}^{-1}_q \right)_{i u}^{ar} \left( \hat{R}_b \right)^{a \tilde{a}} \left( \hat{R}^{-1}_q \right)^{\tilde{a} \sigma} \left( \hat{R}_c \right)^{\sigma \tilde{b}} \left( \sigma_I \right)^{c \tilde{c}} \left( \sigma_K \right)^{k \tilde{d}} \left( \sigma_L \right)^{l \tilde{e}} Y^K X^L \] (5.13)

Moving two factors \( \left( \hat{R}^{-1}_q \right)^{\tilde{a} \sigma} \left( \hat{R}_c \right)^{\sigma \tilde{b}} \) from the right hand side of (5.13) to the left, and left multiplying (5.13) by \( q^{-1} \epsilon_{s t} c_{\tilde{r} \tilde{r}'} \), we obtain the left hand side of the equation as follows:

\[ (-1)^{n_x n_y} q^{-1} \epsilon_{s t} c_{\tilde{r} \tilde{r}'} \left( \hat{R}^{-1}_q \right)^{\tilde{a} \sigma} \left( \hat{R}_c \right)^{\sigma \tilde{b}} \left( \sigma_I \right)^{c \tilde{c}} X^I Y^J = (-1)^{n_x n_y} q^{-1} \epsilon_{s t} \left( \hat{R}^{-1}_q \right)^{br} \left( \hat{R}_c \right)^{ru} X^a_b Y^c_d \epsilon^{\tilde{a} \tilde{b}} \]

The right hand side becomes:

\[ q^{-1} \epsilon_{s t} c_{\tilde{r} \tilde{r}'} \left( \hat{R}^{-1}_q \right)^{ar} \left( \hat{R}_c \right)^{uj} \left( \sigma_L \right)^{k \tilde{d}} X^k \]

Comparing two sides of (5.13) we obtain (5.11). From (5.7) and (4.42) we have:

\[ \{i g/\lambda\} \left( Y^o X^I - (-1)^{n_x n_y} X^I Y^o \right) (\sigma_I)_b^a = (\sigma_I)_b^a C_{JK}^{ij} X^J X^K \]

\[ = (i g q^{1-N+2/N} [N]^{-1/2})^{-1} \chi_j(T^q) C_{JK}^{ij} Y^J X^K \]

\[ = (i g q^{1-N+2/N} [N]^{-1/2})^{-1} \left\{ Y^J X_j(T^q) X^K \chi_K(T^q) \right\} \]

\[ = (i g q^{1-N+2/N} [N]^{-1/2}) \left( \hat{R}^{-1}_q \right)^{ar} \left( \hat{R}_c \right)^{uj} \left( \sigma_L \right)^{k \tilde{d}} X^k \]

(5.12) was proved. Q.E.D.
6. \(q\)-Deformed Chern Class and \(q\)-Trace

In our previous paper [HHM], omitting the possible constant factor, we assume that the second \(q\)-deformed Chern class \(P\) for the quantum group \(SU_q(2)\) has the following form:

\[
P = \langle F, F \rangle \equiv F^I F^J g'_{IJ}
\]  

(6.1)

where from the condition:

\[
\delta P = 0, \quad dP = 0
\]

(6.2)

we defined the \(q\)-deformed Killing form \(g'_{IJ}\) as:

\[
g'_{IJ} = D^R_S C^T_{IT} C^T_{JR}
\]

(6.3)

For \(SU_q(N)\) the non-vanishing components are:

\[
g'_{(k \bar{k})(k \bar{k})} = -g^2 q^{2k-2N} \{[2N]/[N]\}
\]

\[
g'_{(j \bar{k})(k \bar{j})} = -g^2 q^{j-k-N+2} \{[2N]/[N]\}, \quad j \neq k
\]

where the repeated indices are not summed.

Recalling (4.35) and (4.36), we may define another \(q\)-deformed Killing form \(g_{IJ}\):

\[
g_{IJ} = D^a_b \chi_I(T^b_a) \chi_J(T^c_a)
\]

\[
g_{(k \bar{k})(k \bar{k})} = -g^2 [N] q^{2k-3N+4/N}
\]

\[
g_{(j \bar{k})(k \bar{j})} = -g^2 [N] q^{j-k+2-2N+4/N}, \quad j \neq k
\]

(6.4)

where the repeated indices are not summed. The else components of \(g_{JK}\) are vanishing.

Both for \(SU_q(2)\) and for \(SU_q(N)\) two \(q\)-deformed Killing forms are proportional to each other, namely, just like the Killing form in a Lie algebra, the \(q\)-deformed Killing form is also independent of the representation in which it is calculated.

Now, we are going to define the higher \(q\)-deformed Chern class \(P_m\) for the quantum group \(SU_q(N)\) from the covariant condition (6.2). Generalizing (6.4) we define the "generalized \(q\)-deformed Killing forms" and the \(q\)-trace as follows:

\[
g_{l_1 l_2 \ldots l_m} = D^{a_0}_{a_1} \chi_l(T^{a_1}_{a_2}) \chi_l(T^{a_2}_{a_3}) \ldots \chi_l(T^{a_m}_{a_0})
\]

(6.5)

\[
\langle X_1, X_2, \ldots, X_m \rangle = X_1^{l_1} X_2^{l_2} \ldots X_m^{l_m} g_{l_1 l_2 \ldots l_m}
\]

(6.6)
where $X_i$ are fields $\eta$, $d\eta$, $A$ or $dA$ in the BRST algebra $\mathcal{B}$. In (6.6) the fields can also be replaced by, for example, $XY^0$, $Y^0X$ or $F$. From the properties of $q$-Pauli matrices, the sum of all subscripts of nonvanishing components $g_{i_1\cdots i_m}$ as weights, has to be zero. The following theorem is easily proved from (4.43) and the definition (6.5).

**Proposition 3.** The generalized $q$-deformed Killing forms satisfy the following relations:

$$
(P_A)_j^{RS}_{JK} g_{l_1\cdots l_{n-1} R S l_{n+2} \cdots l_m} = - i g \xi (\lambda^2 + 2)^{-1} f^{R}_{JK} g_{l_1\cdots l_{n-1} R l_{n+2} \cdots l_m} \quad (6.7)
$$

where $\xi$ was given in (4.44).

(6.7) can be rewritten in another form by removing a factor $f^{R}_{JK}$:

$$
\tilde{f}_{JK}^{RS} g_{l_1\cdots l_{n-1} R S l_{n+2} \cdots l_m} = i g \xi g_{l_1\cdots l_{n-1} K l_{n+2} \cdots l_m} \quad (6.8)
$$

The $m$-th $q$-deformed Chern class $P_m$ for the quantum group $SU_q(N)$ is defined as follows:

$$
P_m = \langle F_1, F_2, \ldots, F_m \rangle
$$

$$
= F^{l_1} F^{l_2} \cdots F^{l_m} g_{l_1 l_2 \cdots l_m} \quad (6.9)
$$

From (5.9) and (5.12) we have:

$$
\delta P_m = q^{2m/N} \left(-i g q^{1-N} [N]^{-1/2}\right)^{m+1} D^{a_0}_{a_1} F^{a_0}_{a_2} F^{a_0}_{a_3} \cdots F^{a_m}_{a_0} \eta^{a_0}_{b} \quad (6.10)
$$

By (2.14), (2.15) and (5.11) the second term cancels the first term:

$$
D^{b}_{a_1} F^{a_1}_{a_2} F^{a_2}_{a_3} \cdots F^{a_m}_{a_0} \eta^{a_0}_{b} = D^{b}_{a_1} F^{a_1}_{a_2} F^{a_2}_{a_3} \cdots F^{a_m}_{a_0} \eta^{a_0}_{b}
$$

$$
= q^N D^{b}_{a_1} F^{a_1}_{a_2} F^{a_2}_{a_3} \cdots F^{a_m}_{a_0} \{ D^{i}_{j} q^{N} \left( \hat{R}^{-1}_q \right)_{a_0}^{a_0 j} \} \eta^{d}_{b}
$$

$$
= q^N D^{b}_{a_1} F^{a_1}_{a_2} F^{a_2}_{a_3} \cdots F^{a_m}_{a_0} \left( \hat{R}^{-1}_q \right)_{a_0}^{a_0 j} \eta^{d}_{b} \{ D^{i}_{j} q^{N} \left( \hat{R}^{-1}_q \right)_{a_0}^{a_0 j} \}
$$

$$
= q^N D^{b}_{a_1} F^{a_1}_{a_2} F^{a_2}_{a_3} \cdots F^{a_m}_{a_0} \left( \hat{R}^{-1}_q \right)_{a_m j} \eta^{d}_{b} \{ D^{i}_{j} q^{N} \left( \hat{R}^{-1}_q \right)_{a_0}^{a_0 j} \}
$$

$$
= q^N D^{b}_{a_1} F^{a_1}_{a_2} F^{a_2}_{a_3} \cdots F^{a_m}_{a_0} \left( \hat{R}^{-1}_q \right)_{a_m j} \eta^{d}_{b} \{ D^{i}_{j} q^{N} \left( \hat{R}^{-1}_q \right)_{a_0}^{a_0 j} \}
$$

$$
= D^{o}_{a_1} \eta^{a_1}_{r} F^{r}_{a_2} F^{a_2}_{a_3} \cdots F^{a_m}_{a_0} F^{a_m}_{a_0}
$$

25
Thus, $\delta P_m = 0$. This technique of proof was firstly used by Isaev [Isa]. The proof of $dP_m = 0$ can be performed analogously:

$$
\begin{align*}
dP_m &= \left(g q^{1-N+1/N} [N]^{-1/2}\right)^{2m} \left\{ D_{a_1}^{a_0} A_{a_2}^{a_1} dA_{a_3}^{a_2} \ldots dA_{a_m}^{a_{m-1}} \right. \\
&\quad \left. - D_{a_1}^{b} dA_{a_2}^{a_1} dA_{a_3}^{a_2} \ldots dA_{a_m}^{a_{m-1}} A^{a_0}_b \right\} \\
&= 0
\end{align*}
$$

Note that the components of the identity and the adjoint representations are separated in the $q$-deformed Chern class, although they are mixed in the commutative relations of BRST algebra.

This technique can be used to prove more general relations. Remind (6.6) and that the condition $\delta P_m = 0$ can be rewritten as follows:

$$
\begin{align*}
\delta P_m &= \{ig/\lambda\} \left\{ \eta^0 F^{l_1} F^{l_2} \ldots F^{l_m} g_{l_1 l_2 \ldots l_m} - F^{l_1} F^{l_2} \ldots F^{l_m} \eta^0 g_{l_1 l_2 \ldots l_m} \right\} \\
&= 0
\end{align*}
$$

Now, the following theorem can be proved straightforwardly.

**Proposition 4.** Let $Y \in \mathcal{B}$ with index $n_y$, and $X_i$, that are not the field $Y$, be any fields in $\mathcal{B}$ with the indices $n_i$, $n_i > n_y$. Then:

$$
\begin{align*}
(-1)^n \langle X_1 , \ldots , X_m , Y \rangle &= \langle Y , X_1 , \ldots , X_m \rangle \\
(-1)^n \langle X_1 , \ldots , X_m \rangle Y^0 &= Y^0 \langle X_1 , \ldots , X_m \rangle \\
n &= \sum_{i=1}^{m} n_i n_y
\end{align*}
$$

(6.11)

Some corollaries can be derived from Proposition 4. First of all, substituting the commutative relation (5.6) into (6.11) we obtain some constraints for the "generalized $q$-deformed Killing forms":

$$
\begin{align*}
g_{l_1 l_2 \ldots l_m} A_{j_0 j_1}^{l_1} A_{j_2 j_1}^{l_2} \ldots A_{j_{m-1} j_{m-1}}^{l_{m-1}} A_{j_{m-1} j_{m-1}}^{l_{m-1}} &= g_{j_0 j_1 \ldots j_{m}} \\
g_{l_1 l_2 \ldots l_m} A_{0 j_1}^{l_1} A_{j_2 j_1}^{l_2} \ldots A_{j_{m-1} j_{m-1}}^{l_{m-1}} A_{j_{m-1} j_{m-1}}^{l_{m-1}} &= 0 \\
g_{l_1 l_2 \ldots l_m} A_{K_1 j_1}^{l_1} A_{K_2 j_1}^{l_2} \ldots A_{K_{m-1} j_{m-1}}^{l_{m-1}} A_{K_{m-1} j_{m-1}}^{l_{m-1}} &= \delta_{K_0}^{0} g_{j_1 j_2 \ldots j_{m}}
\end{align*}
$$

(6.12)

(6.13)

(6.12) describes the cyclic property of the $q$-trace. (6.13) is equivalent to the following form:

$$
\sum_{n=1}^{m} g_{j_1 \ldots j_n j_{n+1} \ldots j_{m}} A_{K_1 j_1}^{l_1} A_{K_2 j_1}^{l_2} \ldots A_{K_{m-1} j_{m-1}}^{l_{m-1}} C_{K_{n+1} j_n}^{l_n} = 0
$$

(6.14)
namely,

\[ g_{i_1 i_2 \cdots i_m} \chi_{K_0} \left( M_{i_1}^{j_1} \cdots M_{i_m}^{j_m} \right) = 0 \quad (6.15) \]

Bernard [Ber] gave the special form \( (m = 2) \) of (6.14):

\[ g_{i_1 j_1} C_{K_0, j_1}^{j_1} + g_{i_1 j_2} \Lambda_{K_0, j_1}^{j_1} C_{K_1, j_2}^{j_2} = 0 \quad (6.16) \]

In terms of Proposition 3, it can be proved by direct calculation that the constraint (6.12) is equivalent to (6.13). It is not surprising because they come from the same source. In fact, by making use of the explicit form (4.24) we obtain the same relations from (6.13) as from (6.12). For example, when \( m = 1 \) we obtain:

\[ g_i = 0 \quad (6.17) \]

It implies the orthogonality (3.9) of \( q \)-Pauli matrices. When \( m = 2 \), we obtain:

\[
\begin{align*}
(\lambda^2 + 2) g_{i_1 j_1} + g_{R S} f_{1 i_1}^{R T} f_{j_1}^{S T} &= 0 \\
(\lambda^2 + 1) g_{R K} f_{1 i_1}^{R K} + g_{1 R} f_{j_1}^{R K} + g_{R S} f_{i_1}^{R T} f_{j_1}^{S K} &= 0
\end{align*}
\]

Thus,

\[ f_{i_1 j_1}^{R K} = g_{1 R} f_{i_1 j_1}^{S K} \quad (6.18) \]

When \( m = 3 \) we have:

\[
\begin{align*}
(\lambda^4 + 3\lambda^2 + 3) g_{i_1 j_1 k_1} + (\lambda^2 + 1) \left\{ f_{i_1}^{R T} f_{j_1}^{S K} g_{R S K} + f_{i_1}^{R T} f_{j_1}^{T K} g_{R S R} \right\} \\
+ f_{i_1}^{R S} f_{j_1}^{T K} g_{R S T} + f_{i_1}^{R P} f_{j_1}^{P S} f_{j_1}^{P K} g_{R S S} = 0 \\
(\lambda^2 + 2) f_{i_1}^{R L} g_{R L K} - (\lambda^2 + 1) f_{i_1 j_1}^{R L} g_{L R K} - f_{i_1 j_1}^{R K} g_{L R R} \\
= f_{L i_1}^{R T} f_{i_1 j_1}^{R S} f_{K R S} + f_{L i_1}^{R S} f_{i_1 j_1}^{R P} f_{R P K} g_{R S T}
\end{align*}
\]

At last, from Propositions 3 and 4 we can prove:

\[ \langle A^m \rangle \equiv A^{i_1} A^{i_2} \cdots A^{i_m} g_{i_1 i_2 \cdots i_m} = 0, \quad \text{when} \ m \neq 3 \quad (6.20) \]

In fact, define:

\[
\begin{align*}
\langle A^m \rangle \phi_n &= g_{T_{i_1} T_{i_2} \cdots T_{i_m} i_1 \cdots i_m} \Lambda_{K_{i_1} J_{i_1}}^{i_1} \Lambda_{K_{i_2} J_{i_2}}^{i_2} \cdots \Lambda_{K_{i_m} J_{i_m}}^{i_m} \Lambda_{K_{i_1} J_{i_1} \cdots J_{i_m}}^{i_1} \Lambda_{K_{i_1} J_{i_1} \cdots J_{i_m}}^{i_m} \\
&\cdot A^{i_1} \cdots A^{i_m} A^{i_1} \cdots A^{i_n} A^{i_n} \cdots A^{i_m}, \quad 2 \leq n \leq m \\
\langle A^m \rangle \psi_n &= g_{T_{i_1} T_{i_2} \cdots T_{i_m} i_1 \cdots i_m} \Lambda_{K_{i_1} J_{i_1}}^{i_1} \Lambda_{K_{i_2} J_{i_2}}^{i_2} \cdots \Lambda_{K_{i_m} J_{i_m}}^{i_m} \Lambda_{K_{i_1} J_{i_1} \cdots J_{i_m}}^{i_1} \Lambda_{K_{i_1} J_{i_1} \cdots J_{i_m}}^{i_m} \\
&\cdot A^{i_1} \cdots A^{i_m} A^{i_1} \cdots A^{i_n} A^{i_n} \cdots A^{i_m}, \quad 3 \leq n \leq m
\end{align*}
\]

\[ (6.21) \]
Since the factor \( \langle A^m \rangle \) has been abstracted, the functions \( \phi_n \) and \( \psi_n \) are independent of \( m \). From (4.24), (4.27) and (5.3) we obtain the recurrence relations and some explicit values for the functions:

\[
\phi_n = - (\lambda^2 + 1) \phi_{n-1} + \psi_n, \quad n \geq 3
\]

\[
\psi_n = (\lambda^2 + 1) \psi_{n-1} - \lambda^2 (\lambda^2 + 1) \psi_{n-2}
\]

\[+ \sum_{s=1}^{n-5} \left\{ (-1)^{s+1} \lambda^2 \left( (\lambda^2 + 1)^2 - a(\lambda^2 + 2) \right) \psi_{n-s-2} \right\}
\]

\[+ (-1)^n \lambda^2 (\lambda^2 + 2)(\lambda^2 + 1)^{n-4}(\lambda^2 - n + 4), \quad n \geq 5
\]

\[
\phi_2 = -\lambda^2 - 1, \quad \phi_3 = 1, \quad \phi_4 = -\lambda^2 (\lambda^2 + 3) - 1
\]

\[
\phi_5 = \lambda^2 (\lambda^2 + 2)^2 + 1, \quad \phi_6 = -\lambda^2 (\lambda^2 + 3)(\lambda^4 + 3\lambda^2 + 3) - 1
\]

The limit values of the functions are as follows:

\[
\lim_{q \to -1} \phi_{2n} = -1, \quad \lim_{q \to -1} \phi_{2n+1} = 1, \quad \lim_{q \to -1} \psi_n = 0
\]

\[
\lim_{q \to -1} \lambda^2 (\phi_{2n+1}) = -2n^2 + 4n - 3
\]

\[
\lim_{q \to -1} \lambda^2 (\phi_{4n+1} - 1) = 4n(2n - 1), \quad \lim_{q \to -1} \lambda^2 (\phi_{4n+3} + 1) = 4n(2n + 1)
\]

\[
\lim_{q \to -1} \lambda^2 \psi_{2n} = -2(n - 1), \quad \lim_{q \to -1} \lambda^2 \psi_{2n+1} = 2(n - 2)
\]

Now, substituting (6.12) into (6.21) we have:

\[
(\phi_m - 1) \langle A^m \rangle = 0 \quad (6.22)
\]

It leads to (6.20). It is interesting to notice that in the classical case \( (q = 1) \) only \( \langle A^{2n} \rangle = 0 \) is well known. (6.20) also holds for \( \eta \) instead of \( A \) due to (4.32).

Generalize the \( q \)-trace (6.6) as follows:

\[
\langle \cdots, Z_1, [X, Y], Z_2, \cdots \rangle = \cdots Z_1^1 X^J Y^K Z_2^1 \cdots (P_{Adj})^{RS}_{JK} g_{-1,RSI_2,\ldots}
\]

(6.23)

where \( X, Y \) and \( Z \) are fields in the BRST algebra \( \mathcal{B} \). The fields can also be replaced by, for example, \( XY^0 \), \( Y^0X \) or \( F \). From Propositions 3 and 4, we have:

\[
\langle \cdots, X, [\eta, \eta], Y, \cdots \rangle = \langle \cdots, X, \eta, \eta, Y, \cdots \rangle
\]

\[
\langle \cdots, X, [A, A], Y, \cdots \rangle = \langle \cdots, X, A, A, Y, \cdots \rangle
\]

\[
\langle \cdots, Z_i, [X, Y], Z_2 \cdots \rangle
\]

\[= \cdots Z_1^1 \left\{ -ig\xi(\lambda^2 + 2)^{-1} \int_{\mathcal{K}} X^J Y^K \right\} Z_2^b \cdots g_{-1,TL_2,\ldots}
\]

28
\[(P_{Adj})^{RS}_{JK} g_{RS} = 0, \quad \int^{RS}_{J} g_{RS} = 0, \quad \Lambda^{RS}_{JK} g_{RS} = g_{JK} \quad (6.25)\]

\[(P_{Adj})^{RS}_{IJ} g_{RS} = g_{RS} (P_{Adj})^{RS}_{JK}, \quad \langle X, [Y, Z] \rangle = \langle [X, Y], Z \rangle \quad (6.26)\]

where \(X, Y\) and \(Z\) are fields in the BRST algebra \(\mathcal{B}\).

**Proposition 5.** Assume \(X^J\) and \(Y^J\) are two different fields in \(\mathcal{B}\) with the indices \(n_x > n_y\). \(Z^J\) denotes a field in \(\mathcal{B}\), or a field multiplying by the zero component of this or another field. Since the following relations are linear for the field \(Z^J, Z^J\) can also be replaced by their linear combination, for example, \(F^J\). Then,

\[(-1)^{n_x n_y} \langle \cdots, Z_1, X Y^0, Z_2, \cdots \rangle \]
\[= (\lambda^2 + 1) \langle \cdots, Z_1, Y^0 X, Z_2, \cdots \rangle - \frac{\lambda (\lambda^2 + 2)}{i g \xi} \langle \cdots, Z_1, [Y, X], Z_2, \cdots \rangle \quad (6.27)\]
\[\{i g / \lambda \} \langle \cdots, Z_1, (A^0 A + A A^0), Z_2, \cdots \rangle = \xi^{-1} \langle \cdots, Z_1, A, A, Z_2, \cdots \rangle \quad (6.28)\]
\[\{i g / \lambda \} \langle \cdots, Z_1, (\eta^0 \eta + \eta \eta^0), Z_2, \cdots \rangle = \xi^{-1} \langle \cdots, Z_1, \eta, \eta, Z_2, \cdots \rangle \]
\[(-1)^{n_x n_y} \langle \cdots, Z_1, X, Y, Z_2, \cdots \rangle \]
\[= \langle \cdots, Z_1, Y, X, Z_2, \cdots \rangle - i g \lambda^{-1} \xi \langle \cdots, Z_1, (Y^0 X - (-1)^{n_x n_y}XY^0), Z_2, \cdots \rangle \quad (6.29)\]
\[(-1)^{n_x n_y} \langle \cdots, Z_1, X, Y, Z_2, \cdots \rangle \]
\[= i g \lambda \xi \langle \cdots, Z_1, Y^0 X, Z_2, \cdots \rangle - (\lambda^2 + 2) \langle \cdots, Z_1, [Y, X], Z_2, \cdots \rangle + \langle \cdots, Z_1, Y, X, Z_2, \cdots \rangle \quad (6.30)\]

**Proof.** From (5.7) and (4.36) we have

\[(-1)^{n_x n_y} X^J Y^0 = (\lambda^2 + 1) Y^0 X^J + \lambda Y^J X^K f^J_{JK} \]

Due to (4.43),

\[(-1)^{n_x n_y} X^J Y^0 \chi_I (T^*_K) = (\lambda^2 + 1) Y^0 X^J \chi_I (T^*_K) \]
\[- \frac{\lambda (\lambda^2 + 2)}{i g \xi} Y^I X^L (P_{Adj})^{JK}_{IL} \chi_J (T^*_Q) \chi_K (T^*_P) \]
According to the definition (6.6) we obtain (6.27).

From (5.3) we have:

\[
\{ig/\lambda\} \left( A^0 A^I + A^I A^0 \right) \chi_I (T^a_b) = (\lambda^2 + 2)^{-1} A^j A^K \chi_J (T^a_d) \\
= \xi^{-1} A^j A^K \chi_J (T^a_d) \chi_K (T^d_b)
\]

This relation leads to (6.28). The relation also holds for \( \eta^I \) due to (4.32).

From (4.42) and (5.7) we obtain:

\[
(-1)^{n_x n_y} X^I X^J \chi_i (T^a_d) \chi_j (T^d_b) = Y^K X^L \Lambda_{K L}^{IJ} \chi_i (T^a_d) \chi_j (T^d_b)
\]

Then, noting (4.35) and (5.7) we have:

\[
\left\{ (-1)^{n_x n_y} X^I X^J - Y^I X^J \right\} \chi_i (T^a_d) \chi_j (T^d_b)
\]

\[
= \left\{ (-1)^{n_x n_y} X^I X^J - Y^I X^J \right\} \chi_i (T^a_d) \chi_j (T^d_b)
\]

\[
- ig \left\{ q^{-2N+2}/N \right\}^{-1} + \lambda^{-1} \left( 1 - q^{2/N} \right) \left\{ (-1)^{n_x n_y} X^J Y^0 - Y^0 X^j \right\} \chi_J (T^a_d)
\]

\[
= - ig\lambda^{-1}\xi \left\{ Y^0 X^J - (-1)^{n_x n_y} X^J Y^0 \right\} \chi_J (T^a_d)
\]

It leads to (6.29). From (6.27) and (6.29) we obtain (6.30). Q.E.D.

From Proposition 5 we obtain:

\[
\langle F , A \rangle = \langle A , F \rangle = \langle A , dA \rangle + \{ig/\lambda\} \langle A , (A^0 A + AA^0) \rangle
\]

\[
= \langle A , dA \rangle + \xi^{-1} \langle A , A , A \rangle
\]

(6.31)

**Proposition 6.** For any field \( X \in \mathcal{B} \),

\[
\langle \eta , \eta , [\eta , X] \rangle = 0, \quad \langle A , A , [A , X] \rangle = 0
\]

(6.32)

**Proof.** From (4.27) and (4.32) we have:

\[
f^I_{KL} f^J_{LK} \eta^I \eta^J X^K = - (\lambda^2 + 2)^{-1} f^I_{KL} f^J_{LK} f^J_{PR} \eta^R \eta^S X^K
\]

\[
= \frac{\lambda^2 + 1}{\lambda^2 + 2} f^P_{RS} f^T_{PR} \eta^R \eta^S X^K
\]

(6.33)
Noting (6.7) and (6.18) we have:

\[
\left( \frac{\lambda^2 + 2}{-i g \xi} \right)^2 \left\{ \eta^I \eta^J \eta^K X^L f^T_{IJ} f^Q_{KL} g_{TQ} \right. \\
= \eta^I \eta^J \eta^K X^L f^T_{IJ} f^Q_{KL} g_{QL} \\
= \frac{\lambda^2 + 2}{\lambda^2 + 1} \eta^I \eta^J \eta^K X^L f^Q_{JT} f^T_{JK} g_{QI} \\
= \left( \frac{\lambda^2 + 2}{\lambda^2 + 1} \right)^2 \eta^I \eta^J \eta^K X^L f^Q_{JT} f^T_{KL} g_{QI} \\
= \left( \frac{\lambda^2 + 2}{\lambda^2 + 1} \right)^2 \eta^I \eta^J \eta^K X^L f^Q_{JT} f^T_{KL} g_{QI} \\
\left. \right\}
\]

Thus,

\[
\left\{ 1 - \left( \frac{\lambda^2 + 2}{\lambda^2 + 1} \right)^2 \right\} \eta^I \eta^J \eta^K X^L f^T_{IJ} f^Q_{KL} g_{TQ} = 0
\]

The first relation in (6.32) is now proved. The proof for the next relation can be performed analogously. Q.E.D.

Let \( Z^J \) be a field in \( B \) with the index \( n_z \), and let \( X^J \) denote the field \( \eta^J \) or \( A^J \) with the index \( n_x = -1 \) or 1, respectively. From Propositions 4 and 5 we obtain that if \( n_z < n_x \):

\[
\langle X, X, Z \rangle = \langle Z, X, X \rangle \\
(-1)^{n_x n_z} \langle X, Z, X \rangle = - (\lambda^2 + 1) \langle Z, X, X \rangle \\
\langle X, X, Z \rangle - (-1)^{n_x n_z} \langle X, Z, X \rangle + \langle Z, X, X \rangle = [3] \langle Z, X, X \rangle \tag{6.34}
\]

If \( n_z > n_x \), we have:

\[
\langle Z, X, X \rangle = i g \lambda \xi (\lambda^2 + 2) X^0 \langle X, Z \rangle + (\lambda^2 + 1) \langle X, X, Z \rangle \\
(-1)^{n_x n_z} \langle X, Z, X \rangle = - i g \lambda \xi X^0 \langle X, Z \rangle - \langle X, X, Z \rangle \\
= - (\lambda^2 + 2)^{-1} \left\{ \langle X, X, Z \rangle + \langle Z, X, X \rangle \right\} \tag{6.35}
\]

\[
\langle X, X, Z \rangle - (-1)^{n_x n_z} \langle X, Z, X \rangle + \langle Z, X, X \rangle \\
= - (\lambda^2 + 2)^{-1} [3] \langle X, Z, X \rangle
\]

7. \( q \)-Deformed Chern-Simons and Cocycle Hierarchy
In the classical case Zumino [Zum1] [MSZ] introduced a homotopy operator $k$ to compute the Chern-Simons. Generalizing his method we compute the second $q$-deformed Chern-Simons for $SU_q(2)$ in our previous paper [HHM]. Now, we compute the $m$-th $q$-deformed Chern-Simons for $SU_q(N)$.

Introduce a $q$-deformed homotopy operator $k$ that is nilpotent and satisfies the Leibniz rule in the graded sense for the index $n$:

$$k^2 = 0, \quad dk + kd = 1 \quad (7.1)$$

In the following we are going to show the existence of $k$, and compute the $q$-deformed Chern-Simons $Q_{2m-1}(A)$ from the $m$-th $q$-deformed Chern class by the operator $k$:

$$P_m = (dk + kd) P_m = d(k P_m) = d Q_{2m-1}(A)$$

$$Q_{2m-1}(A) = k P_m \quad (7.2)$$

where we used $dP_m = 0$.

Introduce a real parameter $t$, $0 \leq t \leq 1$. When $t$ changes from 0 to 1, the gauge potentials $A^I$ change from 0 to $A^J$:

$$A^I_t = t A^I$$

$$F^J_t = t dA^J + \{ig t^2/\lambda\} \left( A^0 A^J + A^J A^0 \right)$$

$$= t F^J + \{ig (t^2 - t)/\lambda\} \left( A^0 A^J + A^J A^0 \right) \quad (7.3)$$

We choose the symmetrized definition for the $q$-deformed derivative and the $q$-deformed integral [GR] [Jackson] to fit our definition (2.9) for $q$-number. Define the $q$-deformed derivative along $t$ by:

$$\frac{\partial}{\partial t} f(t) = \frac{f(qt) - f(q^{-1}t)}{t (q - q^{-1})} \quad (7.4)$$

satisfying the $q$-deformed Leibniz rule:

$$\frac{\partial}{\partial t} f(t) g(t) = \frac{\partial f(t)}{\partial t} g(qt) + f(q^{-1}t) \frac{\partial g(t)}{\partial t} \quad (7.5)$$

The $q$-deformed integral is defined by:

$$\int_0^{t_0} d_t f(t) = t_0(1 - q^2) \sum_{k=0}^{\infty} q^{2k} f(q^{2k+1} t_0) \quad (7.6)$$
At least for a polynomial, the $q$-deformed integral is the inverse of $q$-deformed derivative. For example,

$$\frac{\partial}{\partial t} t^m = [m] t^{m-1}, \quad \int_0^t d_q t \ t^{m-1} = t_0^m / [m]$$

Now, define the $q$-deformed Lie derivative $\hat{\ell}_q$ along $t$ in the gauge space:

$$\hat{\ell}_q \equiv d_q t \frac{\partial}{\partial q_t}$$

and the $q$-deformed operator $\ell_i$ that satisfies the $q$-deformed Leibniz rule in the graded sense for the index $n$:

$$\ell_i A^J_t = 0, \quad \ell_i F^J_t = \hat{\ell}_q A^J_t = d_q t \ A^J_t$$

$$\ell_i \left\{ X^J_t Y^K \right\} = \left\{ \ell_i X^J \right\} Y^K + (-1)^n X^J_{i-1} \left\{ \ell_i Y^K \right\}$$

where $X^J_t$ and $Y^K$ are the fields in $\mathcal{B}$, and $X^J_t$ has the index $n$.

It is easy to check that for all formal polynomials (vanishing at $F^J_t = 0$ and $A^J_t = 0$) we have

$$\ell_i \ell_i = 0$$

$$\ell_i d + d \ell_i = \hat{\ell}_q = d_q t \frac{\partial}{\partial q_t}$$

Comparing it with (7.1) we obtain:

$$k = \int_0^1 \ell_i$$

The $(2m - 1)$-th $q$-deformed Chern-Simons can be computed from (7.2) straightforwardly. In the following we give some examples. For $m = 2$ we have:

$$\ell_i (P_2)_t = \langle \ell_i F_i , \ F_{q_t} \rangle + \langle F_{q_t} , \ \ell_i F_i \rangle$$

$$= d_q t \left\{ \langle A , \ F_{q_t} \rangle + \langle F_{q_t} , \ A \rangle \right\}$$

$$= d_q t \left\{ t [2] \langle A , \ dA \rangle + i g \lambda^{-1} \lambda^2 (\lambda^2 + 2) \langle A , \ (A^0 A + A A^0) \rangle \right\}$$

$$= d_q t \left\{ t [2] \langle A , \ dA \rangle + t^2 \xi^{-1} (\lambda^2 + 2) \langle A , \ A , \ A \rangle \right\}$$

where we have used (6.28).

$$Q_3(A) = k (P_2)_t$$

$$= \langle A , \ dA \rangle + \xi^{-1} \left\{ \left[ 4 / [3] [2] \right] \langle A , \ A , \ A \rangle \right\}$$

$$= \langle A , \ F \rangle - \left\{ \xi [3] \right\}^{-1} \langle A , \ A , \ A \rangle$$

33
For $m = 3$ we have:

$$
\ell_t (P_3)_t = \langle \ell_t F_t, F_{\gamma t}, F_{\gamma t} \rangle + \langle F_{\gamma -1 t}, \ell_t F_t, F_{\gamma t} \rangle + \langle F_{\gamma -1 t}, F_{\gamma -1 t}, \ell_t F_t \rangle
$$

$$
= d_\gamma t \langle A, F_{\gamma t}, F_{\gamma t} \rangle + \langle F_{\gamma -1 t}, A, F_{\gamma t} \rangle + \langle F_{\gamma -1 t}, F_{\gamma -1 t}, A \rangle
$$

$$
= d_\gamma t^2 \{ q^2 \langle A, dA, dA \rangle + \langle dA, A, dA \rangle + q^{-2} \langle dA, dA, A \rangle \}
$$

$$
+ d_\gamma t^3 \xi^{-1} \{ (q^3 + q^{-1}) \langle A, A, dA \rangle + q^{-3} \langle A, A, dA, A \rangle
$$

$$
+ q^3 \langle A, dA, A, A \rangle + (q + q^{-3}) \langle dA, A, A, A \rangle \}
$$

$$
Q_5(A) = k (P_3)_t
$$

$$
= [3]^{-1} \{ q^2 \langle A, dA, dA \rangle + \langle dA, A, dA \rangle + q^{-2} \langle dA, dA, A \rangle \}
$$

$$
+ \{ \xi [4] \}^{-1} \{ (q^3 + q^{-1}) \langle A, A, A, dA \rangle + q^{-3} \langle A, A, dA, A \rangle
$$

$$
+ q^3 \langle A, dA, A, A \rangle + (q + q^{-3}) \langle dA, A, A, A \rangle \}
$$

$$
= [3]^{-1} \{ q^2 \langle A, F, F \rangle + \langle F, A, F \rangle + q^{-2} \langle F, F, A \rangle
$$

$$
- ig \{ \xi [4] [3] \}^{-1} \{ (q^3 + q^{-1}) \langle A, A, F \rangle + q \langle A, A, F, A \rangle
$$

$$
+ q^{-1} \langle A, F, A, A \rangle + (q + q^{-3}) \langle F, A, A, A \rangle \}
$$

(7.13)

Just like those in the classical case [HZ], the gauge fields $F^J$ are invariant under the transformation:

$$
A^J \rightarrow A^J - \eta^J, \quad d \rightarrow d + \delta
$$

(7.14)

In fact,

$$
F^J \rightarrow \tilde{F}^J
$$

$$
= (d + \delta) (A^J - \eta^J)
$$

$$
+ \frac{ig}{\lambda} \left\{ (A^0 - \eta^0)(A^J - \eta^J) + (A^J - \eta^J)(A^0 - \eta^0) \right\}
$$

$$
= F^J + \left\{ \delta A^J - d\eta^J - \frac{ig}{\lambda} (\eta^0 A^J + A^0 \eta^J) \right\}
$$

$$
- \left\{ \delta \eta^J - \frac{ig}{\lambda} (\eta^0 \eta^J + \eta^J \eta^0 - \eta^J A^0) \right\}
$$

$$
= F^J
$$

Now, transforming (7.2) and expanding it by the ghost number, we obtain:

$$
P_m = (d + \delta) Q_{2m-1} (A - \eta)
$$

$$
Q_{2m-1}(A - \eta) = \sum_{n=0}^{2m-2} \omega_n^{2m-n-1}
$$

$$
P_m = d\omega_0^{2m-1} + \sum_{n=0}^{2m-2} \left\{ \delta \omega_n^{2m-n-1} + d\omega_n^{n+1} \right\} + \delta \omega_0^{2m-1}
$$

(7.15)
where the subscripts denote the degrees of form of the quantities, and the superscripts denote the ghost numbers. In two sides of (7.15) the quantities with the same degree of form and the same ghost number should be equal to each other, respectively:

\[
P_m = d\omega^0_{2m-1}, \quad \delta\omega^1_{2m-1} = 0
\]

\[
\delta\omega^n_{2m-n-1} + d\omega^{n+1}_{2m-n-2} = 0, \quad n = 0, 1, \ldots, (2m - 2)
\]

(7.16)

For example, for \(m = 2\) we have:

\[
Q_3(A - \eta) = \langle A - \eta, F \rangle = \{\xi[3]\}^{-1} \langle A - \eta, A - \eta \rangle
\]

Simplifying them by the formulas given in Section 6, we obtain:

\[
\omega^0_3 = Q_3(A) = \langle A, F \rangle - \{\xi[3]\}^{-1} \langle A, A, A \rangle
\]

\[
= \langle A, dA \rangle + \frac{[4]}{\xi[3][2]} \langle A, A, A \rangle
\]

\[
\omega^1_2 = - \langle \eta, dA \rangle - \{ig/\lambda\} \langle \eta, (A^0 A + AA^0) \rangle
\]

\[
+ \{\xi[3]\}^{-1} \{ \langle \eta, A, A \rangle + \langle A, \eta, A \rangle + \langle A, A, \eta \rangle \}
\]

\[
= - \langle \eta, dA \rangle
\]

\[
\omega^2_1 = - \{\xi[3]\}^{-1} \{ \langle \eta, \eta, A \rangle + \langle \eta, A, \eta \rangle + \langle A, \eta, \eta \rangle \}
\]

\[
= - \xi^{-1} \langle \eta, A, \eta \rangle
\]

\[
\omega^3_0 = \{\xi[3]\}^{-1} \langle \eta, \eta, \eta \rangle
\]

(7.17)

It is easy to check by the formulas in Section 6 that (7.17) satisfies (7.16).

8. \(q\)-deformed Lagrangian and Yang-Mills equation

In the present paper the spacetime is the ordinary commutative Minkowski spacetime. Explicitly writing down the spacetime indices, we have:

\[
A^J = A^J_\mu dx^\mu, \quad F^J = \frac{1}{2} F^J_{\mu\nu} dx^\mu \wedge dx^\nu
\]

(8.1)

It is well known that the metric \(g^{\mu\nu}\) in the Minkowski spacetime can change the covariant index to contravariant index, or vice versa.

Now, the \(q\)-deformed Lagrangian that is covariant both in the Lorentz transformation and the \(q\)-gauge transformation is:

\[
L = - \frac{1}{4} \langle F_{\mu\nu}, F^{\mu\nu} \rangle = - \frac{1}{4} (F^I)_{\mu\nu} (F^J)^{\mu\nu} g_{IJ}
\]

(8.2)
We have known that the components of the identity and the adjoint representations are separated in the $q$-deformed Chern class, and obviously in the $q$-deformed Lagrangian. Here we only discuss the $q$-deformed Lagrangian constructed by the adjoint components.

The $q$-deformed Yang-Mills equation is just the $q$-deformed Lagrangian equation:

\[
\partial_{\nu} (F_{\nu}^J)^{\mu\nu} \left( g_{KJ} + g_{JK} \right) = (\lambda^2 + 2)^{-1} \left( C_{K,R}^J \right) \left\{ g_{1J}(A^R)^{\mu}(F^J)^{\mu\nu} + g_{JI}(F^J)^{\mu\nu}(A^R)^{\nu} \right\}
\]  (8.3)

Acknowledgments. This work was supported by the National Natural Science Foundation of China and Grant No. IWTZ-1298 of Chinese Academy of Sciences.

References


