Gravitational radiation from infall into a black hole: 
Regularization of the Teukolsky equation

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Abstract

The Teukolsky equation has long been known to lead to divergent integrals 
when it is used to calculate the gravitational radiation emitted when a test 
mass falls into a black hole from infinity. Two methods have been used in the 
past to remove those divergent integrals. In the first, integrations by parts 
are carried out, and the infinite boundary terms are simply discarded. In the 
second, the Teukolsky equation is transformed into another equation which 
does not lead to divergent integrals. The purpose of this paper is to show that 
there is nothing intrinsically wrong with the Teukolsky equation when dealing 
with non-compact source terms, and that the divergent integrals result simply 
from an incorrect choice of Green’s function. In this paper, regularization of 
the Teukolsky equation is carried out in an entirely natural way which does 
not involve modifying the equation.

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1. INTRODUCTION

In 1971, Davis, Ruffini, Press, and Price [1] computed for the first time the amount of energy carried off by gravitational waves when a test mass, released from rest at infinity, falls radially into a Schwarzschild black hole. By numerically integrating the recently derived Zerilli equation [2] for black-hole perturbations, they found an amount \( \Delta E = 0.0104\mu^2/M \), where \( \mu \) is the mass of the infalling particle, and \( M \) the mass of the black hole. (Units are such that \( G = c = 1 \).)

This calculation was later generalized to many different situations. In 1973, Ruffini [3] considered a particle released from infinity with nonvanishing initial velocity. In 1979, Detweiler and Szedenits [4] examined infall trajectories with nonzero angular momentum. The first calculation involving a Kerr black hole was carried out in 1982 by Sasaki and Nakamura [5], who considered a particle falling along the symmetry axis. In 1983, Kojima and Nakamura [6] examined the case of a particle moving radially within the equatorial plane. In 1984, the same authors [7] studied infall trajectories with nonzero angular momentum. The infall of spinning test masses was considered for the first time very recently, by Mino, Shibata, and Tanaka [8]. The gravitational waves emitted by a particle in bound motion around a black hole have also been extensively studied [9–26], following the pioneering work of Detweiler in 1978 [27].

Davis \textit{et. al} [1] obtained their classic result by integrating the Zerilli equation [2], which describes in a compact form the (even-parity, or polar) metric perturbations of the Schwarzschild black hole. On the other hand, Detweiler and Szedenits [4] worked with the Teukolsky equation [28], which describes in a compact form the curvature perturbations of the Kerr black hole. (In the limiting case of a nonrotating black hole, and in the absence of a source for the perturbations, the Teukolsky equation reduces to the Bardeen-Press equation [29], which was derived earlier.)

However, when integrating the Teukolsky equation, Detweiler and Szedenits encountered divergent integrals, which they regularized by integrating by parts and then discarding the infinite boundary terms. (The Zerilli formalism does not lead to divergent integrals.) This procedure was also adopted by Simone, Poisson, and Will [30] who, in 1995, reproduced the Davis \textit{et. al} result using the Teukolsky equation, for the purpose of testing the accuracy of post-Newtonian methods. In neither of these papers was a justification given for discarding the infinite boundary terms, and confidence in the procedure came entirely from the comparison with Davis \textit{et. al}.

The first attempt to remove the divergent integrals from the Teukolsky formalism came in 1981 from Tashiro and Ezawa [31], who employed the clever trick of subtracting from the dependent variable a quantity constructed from the source. The resulting equation for the new dependent variable leads to well defined integrals. Also in 1981, Sasaki and Nakamura [32] dealt with the divergent integrals in a different way. In the restricted context of a Schwarzschild black hole, these authors rewrote the Teukolsky equation in a Regge-Wheeler form, with a source term constructed from the source of the Teukolsky equation. (The Regge-Wheeler equation [33] describes the odd-parity, or axial, metric perturbations of the Schwarzschild black hole.) The Sasaki-Nakamura equation, which was later generalized to the case of a Kerr black hole [5], does not lead to divergent integrals. Subsequent studies of infalling particles [6–8] were carried out by integrating this equation.
Those methods of regularization give the impression that when dealing with unbounded particle trajectories, the Teukolsky equation necessarily leads to divergent integrals, and is therefore not well posed. The purpose of this paper is to show that this impression is based on a misconception. Indeed, I wish to show that the Teukolsky equation can be regularized in an entirely natural way which does not involve modifying the equation.

The basic issues are easily summarized.

For perturbations of a Schwarzschild black hole, to which I specialize in this paper, and after separation of the variables (the usual Schwarzschild coordinates are used), the (inhomogeneous) Teukolsky equation [28] takes the form

\[
\left( \frac{d}{dr} p \frac{d}{dr} + p^2 U \right) R = p^2 T.
\]  

Here, \( R(r) \) is the radial function corresponding to a perturbation of frequency \( \omega \) and spherical-harmonic indices \( \ell \) and \( m \), \( p(r) = (r^2 - 2Mr)^{-1} \), \( U(r) \) is the effective potential, whose explicit expression can be found in Eq. (2.15), and \( T(r) \) is the source term, which is constructed from the energy-momentum tensor of the infalling mass.

The inhomogeneous Teukolsky equation is solved with the physical requirement that gravitational waves must be purely ingoing at the black-hole horizon, and purely outgoing at infinity; this is equivalent to a no incoming-radiation initial condition. Mathematically, this translates into two statements. First, that \( R(r) \propto R^H(r) \) near \( r = 2M \). Here, \( R^H(r) \) is a solution to the homogeneous equation, normalized such that \( R^H(r \to 2M) \sim (1 - 2M/r)^2 \exp(-i\omega r^*) \), where \( r^* = r + 2M \ln(r/2M - 1) \). Second, that \( R(r) \propto R^\infty(r) \) near \( r = \infty \), where \( R^\infty(r) \) is also a solution to the homogeneous equation, normalized such that \( R^\infty(r \to \infty) \sim (i\omega r)^3 \exp(i\omega r^*) \).

The most convenient way of integrating Eq. (1.1) is by means of a Green's function, which should be chosen so as to incorporate the specified boundary conditions. The standard theory [34] suggests that the appropriate solution is

\[
R(r) = \frac{R^\infty(r)}{W_T} \int_{2M}^r p^2(r') T(r') R^H(r') \, dr' + \frac{R^H(r)}{W_T} \int_r^\infty p^2(r') T(r') R^\infty(r') \, dr',
\]  

where \( W_T \) is a constant, equal to the conserved Wronskian of \( R^H(r) \) and \( R^\infty(r) \). The misconception is precisely that Eq. (1.2) must be the desired solution. In fact it is not:

In order for the standard theory of Green's functions to guarantee that \( R(r) \) as given by Eq. (1.2) satisfies the specified boundary conditions, the integrals must converge. This is not the case here. Because \( R^H(r) \) and \( R^\infty(r) \) both have a component growing as \( r^3 \) when \( r \to \infty \), and because \( p^2(r) T(r) \) falls off only as \( r^{-3/2} \), those integrals diverge when \( r \to \infty \). The claim that Eq. (1.2) enforces the specified boundary conditions is therefore unjustified, and in fact, is wrong.

The method adopted in this paper for regularizing the Teukolsky equation goes as follows. For simplicity, I consider the specific case of a particle falling radially into a Schwarzschild black hole, having been released from rest at infinity. Generalization to other situations should be straightforward.

Instead of the ill-defined particular solution (1.2), the starting point of this analysis is the most general solution to Eq. (1.1), which is written as
\[ R(r) = \frac{R^\infty(r)}{W_T} \left[ A + \int_a^r p^2(r')T(r')R^H(r')\,dr' \right] + \frac{R^H(r)}{W_T} \left[ B + \int_r^b p^2(r')T(r')R^\infty(r')\,dr' \right], \]

(1.3)

where \( a, b, A, \) and \( B \) are constants. This represents the gravitational radiation generated by the infalling particle, plus the free radiation that was initially present in the spacetime. Boundary conditions must be imposed to eliminate this free component.

Now, because the integrals are ill defined when \( r \to \infty \), these conditions cannot be imposed immediately. Instead, integrations by parts are carried out, which are specifically designed to make the integrals well behaved. The boundary terms at \( r' = a \) and \( r' = b \) are then absorbed into the constants \( A \) and \( B \), while the boundary terms at \( r' = r \) are carefully kept. At this stage one is still dealing with the most general solution to the Teukolsky equation, and this is a perfectly valid starting point for the discussion of boundary conditions. Since the integrals are now well behaved, there is no obstacle in setting \( a = 2M \) and \( b = \infty \). The new constants \( A \) and \( B \) are then chosen so that the solution satisfies the specified boundary conditions. By this procedure, regularization of the Teukolsky equation involves no unjustified manipulations nor modifications to the equation, and is entirely natural.

The idea described in this paper appears to be completely trivial, and it is indeed ironic that such a solution to the problem of divergent integrals did not come forth much sooner. However, the work required to carry out the procedure is not, in itself, entirely trivial. The rest of the paper is devoted to a detailed presentation. It is organized as follows:

For the purpose of integrating by parts, it is convenient to write the Teukolsky functions \( R^H(r) \) and \( R^\infty(r) \) in terms of the related, and better behaved, Regge-Wheeler functions \( X^H(r) \) and \( X^\infty(r) \). These functions, and the transformations relating them, are described in detail in Sec. II, which establishes many results used later on.

Regularization of the inhomogeneous Teukolsky equation is carried out in Sec. III for the specific case of a particle falling radially into a Schwarzschild black hole. Generalization to other cases should proceed along similar lines, but I shall not pursue this here.

The main results of this paper are summarized in Sec. IV.

Notation. The following symbols appear frequently throughout the paper: \( f = 1 - 2M/r \), \( r^* = r + 2M \ln(r/2M - 1) \), \( d/dr^* = fd/dr \), \( Q = iM \omega \), \( z = (i\omega r)^{-1} \), and \( x = (r/2M)^{1/2} \).

II. REGGE-WHEELEER, TEUKOLSKY, AND CHANDRASEKHAR

This section is devoted to the derivation of various results which will be used in the following section. Specifically, Sec. II A contains a discussion of the Regge-Wheeler equation [33], and a study of the asymptotic behavior of its solutions near \( r = 2M \) and near \( r = \infty \). Section II B does the same for the homogeneous Teukolsky equation [28]. The Chandrasekhar transformation [35], which relates a solution to the Regge-Wheeler equation to a solution of the homogeneous Teukolsky equation, is the topic of Sec. II C. Finally, the functions \( X^H(r) \), \( X^\infty(r) \), \( R^H(r) \), \( R^\infty(r) \), and the relations between them, are introduced in Sec. II D.
A. Regge-Wheeler equation

The Regge-Wheeler equation [33] compactly describes a metric perturbation of frequency \( \omega \) and spherical-harmonic indices \( \ell \) and \( m \). It reads

\[
\left\{ \frac{d^2}{dr^*} + \omega^2 - f \left[ \frac{\ell(\ell + 1)}{r^2} - \frac{6M}{r^3} \right] \right\} X(r) = 0,
\]

where \( f = 1 - 2M/r \) and \( d/dr^* = fd/dr \), so that

\[
r^* = r + 2M \ln(r/2M - 1).
\]

For two linearly independent solutions \( X_1(r) \) and \( X_2(r) \), the conserved Wronskian is given by

\[
W_{RW}(X_1, X_2) = f(X_1X_2' - X_2X_1'),
\]

where a prime denotes differentiation with respect to \( r \). Because the Regge-Wheeler equation is real, if \( X_1(r) \) is a solution, then \( X_2(r) = \bar{X}_1(r) \) is also a solution, linearly independent from the first; the overbar denotes complex conjugation.

1. Asymptotic behavior near \( r = 2M \)

It can be seen from Eq. (2.1) that near \( r = 2M \), the Regge-Wheeler function must behave as \( \exp(-i\omega r^*) \) or its complex conjugate. To obtain more information, let

\[
X(r) = Y(f) e^{-i\omega r^*}.
\]

Then a short calculation shows that the new function \( Y(f) \) must satisfy

\[
(1 - f)^2 Y'' + [(1 - f)(1 - 3f) - 4Q] Y' - [\ell(\ell + 1) - 3(1 - f)] Y = 0,
\]

where \( Q \equiv iM\omega \) and a prime denotes differentiation with respect to \( f \). Equation (2.5) can be integrated by by writing

\[
Y(f) = 1 + \sum_{n=1}^{\infty} a_n f^n,
\]

where the normalization was chosen arbitrarily. Substituting this into Eq. (2.5) and setting each term of the resulting series to zero gives

\[
a_1 = \frac{\ell(\ell + 1) - 3}{1 - 4Q},
\]

\[
a_2 = \frac{(\ell - 1)\ell(\ell + 1)(\ell + 2) - 12Q}{4(1 - 2Q)(1 - 4Q)}.
\]

The higher-order coefficients can be obtained from the recurrence relation

\[
n(n - 4Q) a_n = [2(n - 1)n + \ell(\ell + 1) - 3] a_{n-1} - (n - 3)(n + 1) a_{n-2},
\]

which holds for \( n \geq 3 \).
2. Asymptotic behavior near \( r = \infty \)

Near \( r = \infty \), the Regge-Wheeler function must also behave as \( \exp(-i\omega r^*) \) or its complex conjugate. As before, let

\[
X(r) = Y(z) e^{-i\omega r^*},
\]

where \( z = (i\omega r)^{-1} \). This leads to the differential equation

\[
z^2(1 - 2Qz) Y'' + 2(1 + z - 3Qz^2) Y' - [\ell(\ell + 1) - 6Q] Y = 0,
\]

where a prime denotes differentiation with respect to \( z \). Equation (2.10) is solved by writing

\[
Y(z) = 1 + \sum_{n=1}^{\infty} b_n z^n,
\]

where once again the normalization was chosen arbitrarily. This gives

\[
b_1 = \frac{1}{2} \ell (\ell + 1),
\]

\[
b_2 = \frac{1}{8} \left[ (\ell - 1) \ell (\ell + 1) (\ell + 2) - 12Q \right].
\]

All other coefficients can be obtained from the recurrence relation

\[
2n b_n = (\ell - n + 1)(\ell + n) b_{n-1} + 2(n - 3)(n + 1)Q b_{n-2},
\]

which holds for \( n \geq 3 \).

B. Homogeneous Teukolsky equation

The homogeneous Teukolsky equation [28] compactly describes a curvature perturbation of frequency \( \omega \) and spherical-harmonic indices \( \ell \) and \( m \), and is given by

\[
\left[ \frac{d}{dr} p(r) \frac{d}{dr} + p^2(r) U(r) \right] R(r) = 0,
\]

where

\[
p(r) = \frac{1}{r^2 f},
\]

\[
U(r) = f^{-1}[(\omega r)^2 - 4i\omega(r - 3M)] - (\ell - 1)(\ell + 2).
\]

If \( R_1(r) \) and \( R_2(r) \) are two linearly independent solutions, then

\[
W_T(R_1, R_2) = p(R_1 R_2' - R_2 R_1'),
\]

where a prime denotes differentiation with respect to \( r \), is the conserved Wronskian. It should noted that the Teukolsky equation is complex; the complex conjugate of a solution is therefore not a solution. For this reason, an ingoing mode of the Teukolsky equation, which is proportional to \( \exp(-i\omega r^*) \), must be distinguished from an outgoing mode, proportional to \( \exp(i\omega r^*) \).
1. Ingoing mode: Asymptotic behavior near $r = 2M$

It is easy to see from Eq. (2.14) that near the horizon, an ingoing mode must behave as $f^2 \exp(-i\omega r^*)$. This motivates the substitution

$$R(r) = f^2 S^{\text{in}}(f) e^{-i\omega r^*},$$

which implies

$$0 = (1 - f)^3 f S^{\text{in}''} + (1 - f)(7f^2 - 10f + 3 - 4Q) S^{\text{in}'}$$

$$- [8f^2 - (\ell^2 + \ell + 14)f + \ell^2 + \ell + 6 + 4Q] S^{\text{in}}.$$  (2.18)

Here, a prime denotes differentiation with respect to $f$. A solution can be obtained by writing

$$S^{\text{in}}(f) = 1 + \sum_{n=1}^{\infty} p_n^{\text{in}} f^n,$$  (2.19)

which gives

$$p_1^{\text{in}} = \frac{\ell(\ell + 1) + 6 + 4Q}{3 - 4Q},$$

$$p_2^{\text{in}} = \frac{[\ell(\ell + 1) + 4] [\ell(\ell + 1) + 18] + 4[2\ell(\ell + 1) + 33]Q}{8(1 - Q)(3 - 4Q)}.$$  (2.20)

All other coefficients can be obtained from the recurrence relation

$$n(n + 2 - 4Q) p_n^{\text{in}} = [n(3n + 4) + \ell(\ell + 1) - 1 - 4(n - 2)Q] p_{n-1}^{\text{in}}$$

$$- [n(3n + 2) + (\ell - 1)(\ell + 2)] p_{n-2}^{\text{in}} + (n^2 - 1)p_{n-3}^{\text{in}},$$  (2.21)

which is valid for $n \geq 3$ (with $p_0^{\text{in}} = 1$).

2. Outgoing mode: Asymptotic behavior near $r = 2M$

Equation (2.14) implies that near the horizon, an outgoing mode must behave as $\exp(-i\omega r^*)$. This leads the substitution

$$R(r) = S^{\text{out}}(f) e^{i\omega r^*},$$  (2.22)

which implies

$$0 = (1 - f)^3 f S^{\text{out}''} + (1 - f)(3f^2 - 2f - 1 + 4Q) S^{\text{out}'}$$

$$- [(\ell - 1)(\ell + 2)(1 - f) + 12Q] S^{\text{out}} = 0,$$  (2.23)

where a prime denotes differentiation with respect to $f$. This is integrated by writing
\[ S_{\text{out}}(f) = 1 + \sum_{n=1}^{\infty} p_{n}^{\text{out}} f^n, \]  
(2.24)

which gives

\[ p_{1}^{\text{out}} = -\frac{(\ell - 1)(\ell + 2) + 12Q}{1 - 4Q}, \]  
(2.25)

\[ p_{2}^{\text{out}} = -\frac{(\ell - 1)\ell(\ell + 1)(\ell + 2) + 12[2\ell(\ell + 1) - 3]Q + 192Q^2}{8Q(1 - 4Q)}. \]

All other coefficients can be determined with the recurrence relation

\[ n(n - 2 + 4Q)p_{n}^{\text{out}} = [(n - 1)(3n - 5) + (\ell - 1)(\ell + 2) + 4(n - 2)Q] p_{n-1}^{\text{out}} \]
\[ - [(n - 2)(3n - 4) + (\ell - 1)(\ell + 2)] p_{n-2}^{\text{out}} + (n - 3)(n - 1) p_{n-3}^{\text{out}}, \]  
(2.26)

which is valid for \( n \geq 3 \) (with \( p_{0}^{\text{out}} = 1 \)).

### 3. Ingoing mode: Asymptotic behavior near \( r = \infty \)

Near infinity, an ingoing mode of the Teukolsky equation must behave as \( r^{-1} \exp(-i\omega r) \), which suggests the substitution

\[ R(r) = zS_{\text{in}}(z)e^{-i\omega r}, \]  
(2.27)

where \( z = (i\omega r)^{-1} \). The homogeneous Teukolsky equation then implies

\[ 0 = (1 - 2Qz)^2z^2 S_{\text{in}}'' + 2(1 - 2Qz)(1 + 3z - 5Qz^2) S_{\text{in}}' \]
\[ + [-((\ell - 2)(\ell + 3) + 8Q + 2(\ell^2 + \ell - 9)Qz + 12Q^2z^2) S_{\text{in}} = 0, \]  
(2.28)

where a prime denotes differentiation with respect to \( z \). Once again the solution is written as a series,

\[ S_{\text{in}}(z) = 1 + \sum_{n=1}^{\infty} q_{n}^{\text{in}}z^n, \]  
(2.29)

and substitution yields

\[ q_{1}^{\text{in}} = \frac{1}{2}[(\ell - 2)(\ell + 3) - 8Q], \]  
(2.30)

\[ q_{2}^{\text{in}} = \frac{1}{8}\left\{ (\ell - 3)(\ell - 2)(\ell + 3)(\ell + 4) - 4[4\ell(\ell + 1) - 39]Q + 32Q^2 \right\}. \]

The recurrence relation

\[ 2nq_{n}^{\text{in}} = [-n(n + 3) + (\ell - 1)(\ell + 2) + 4(n - 3)Q] q_{n-1}^{\text{in}} \]
\[ + 2[n(2n + 1) - \ell(\ell + 1) - 1]Q q_{n-2}^{\text{in}} - 4n(n + 2)Q^2 q_{n-3}^{\text{in}}, \]  
(2.31)

valid for \( n \geq 3 \) (with \( q_{0}^{\text{in}} = 1 \)), gives the remaining coefficients.
4. Outgoing mode: Asymptotic behavior near \( r = \infty \)

Finally, an outgoing mode of the Teukolsky equation behaves as \( r^3 \exp(i \omega r^*) \) near infinity. This leads to

\[
R(r) = z^{-3} S^\text{out}(z) e^{i \omega r^*}.
\]

Equation (2.14) then implies

\[
(1 - 2Qz)z^2 S^\text{out}^\prime - (2 + 2z - 6Qz^2) S^\text{out} - [(\ell - 1)(\ell + 2) + 6Qz] S^\text{out} = 0,
\]

where a prime denotes differentiation with respect to \( z \). This is integrated by substituting

\[
S^\text{out}(z) = 1 + \sum_{n=1}^{\infty} q_n^\text{out} z^n
\]

into the differential equation, which gives

\[
q_1^\text{out} = -\frac{1}{8}(\ell - 1)(\ell + 2),
\]

\[
q_2^\text{out} = \frac{1}{8}[(\ell - 1)\ell(\ell + 1)(\ell + 2) - 12Q].
\]

The other coefficients are generated by the recurrence relation

\[
2n q_n^\text{out} = [(n - 4)(n - 1) - (\ell - 1)(\ell + 2)] q_{n-1}^\text{out} - 2(n - 3)(n - 5)Q q_{n-2}^\text{out},
\]

which is valid for \( n \geq 3 \).

C. Chandrasekhar transformation

In 1975, Chandrasekhar [35] proved the following theorem: If \( X(r) \) is a solution to the Regge-Wheeler equation (2.1), then there exists a linear differential operator \( C \) such that \( R(r) = CX(r) \) is a solution to the homogeneous Teukolsky equation (2.14). The Chandrasekhar transformation is given by \( C \propto r^3 f \mathcal{L} f^{-1} L r \), where \( \mathcal{L} = \frac{df}{dr} + i \omega \). Since \( X(r) \) satisfies a second-order differential equation, \( C \) can also be written in first-order form as [23]

\[
C = (i \omega r) \left\{ 2(1 - 3M/r + i \omega r) r \mathcal{L} + f[\ell(\ell + 1) - 6M/r] \right\}.
\]

The constant of proportionality was chosen arbitrarily.

Using the results derived in Sec. II A, simple manipulations are required to prove the following statements. First, concerning asymptotic relations near \( r = 2M \):

If \( X \sim e^{-i \omega r^*} \), then \( CX \sim 4Q a_2 f^2 e^{-i \omega r^*} \).

If \( X \sim e^{i \omega r^*} \), then \( CX \sim -8Q^2 (1 - 4Q) e^{i \omega r^*} \).

Next, concerning asymptotic relations near \( r = \infty \):

If \( X \sim e^{-i \omega r^*} \), then \( CX \sim 2b_2 (i \omega r)^{-1} e^{-i \omega r^*} \).

If \( X \sim e^{i \omega r^*} \), then \( CX \sim 4(i \omega r)^3 e^{i \omega r^*} \).

The constants \( a_2 \) and \( b_2 \) are given by Eqs. (2.7) and (2.12), respectively. As they must, the asymptotic relations found here for \( CX(r) \) agree with the relations derived for \( R(r) \) in subsection B.
D. Linearly independent solutions

1. Asymptotic relations

Of all the solutions to the Regge-Wheeler equation, two are preferred. The first describes gravitational waves which are purely ingoing at the black-hole horizon, and is denoted \( X^H(r) \). The other describes waves which are purely outgoing at infinity, and is denoted \( X^\infty(r) \). These solutions satisfy the asymptotic relations

\[
X^H(r) \sim \begin{cases} 
    e^{-i\omega_\pi^r} & r \to 2M \\
    A^\mathrm{in} e^{-i\omega_\pi^r} + A^\mathrm{out} e^{i\omega_\pi^r} & r \to \infty 
\end{cases}, \tag{2.42}
\]

where \( A^\mathrm{in} \) and \( A^\mathrm{out} \) are constants, and

\[
X^\infty(r) \sim \begin{cases} 
    e^{i\omega_\pi^r} & r \to \infty \\
    B^\mathrm{in} e^{-i\omega_\pi^r} + B^\mathrm{out} e^{i\omega_\pi^r} & r \to 2M 
\end{cases}, \tag{2.43}
\]

where \( B^\mathrm{in} \) and \( B^\mathrm{out} \) are also constants. These solutions are linearly independent. Evaluation of their Wronskian in the limit \( r \to \infty \) indeed reveals that

\[
W_{\mathrm{RG}}(X^H, X^\infty) = 2i\omega A^\mathrm{in}, \tag{2.44}
\]

where \( W_{\mathrm{RG}} \) was defined in Eq. (2.3).

Acting with the Chandrasekhar transformation,

\[
\begin{align*}
R^H(r) &= X^H(r), \\
R^\infty(r) &= X^\infty(r),
\end{align*}
\]

where \( \chi^H \) and \( \chi^\infty \) are normalization constants, returns solutions to the homogeneous Teukolsky equation possessing the same physical interpretation. These are normalized so that

\[
R^H(r) \sim \begin{cases} 
    f^2 e^{-i\omega_\pi^r} & r \to 2M \\
    Q^\mathrm{in} (i\omega r)^{-4} e^{-i\omega_\pi^r} + Q^\mathrm{out} (i\omega r)^3 e^{i\omega_\pi^r} & r \to \infty 
\end{cases}, \tag{2.46}
\]

where \( Q^\mathrm{in} \) and \( Q^\mathrm{out} \) are constants, and

\[
R^\infty(r) \sim \begin{cases} 
    (i\omega r)^3 e^{i\omega_\pi^r} & r \to \infty \\
    \mathcal{P}^\mathrm{in} f^2 e^{-i\omega_\pi^r} + \mathcal{P}^\mathrm{out} e^{i\omega_\pi^r} & r \to 2M.
\end{cases} \tag{2.47}
\]

where \( \mathcal{P}^\mathrm{in} \) and \( \mathcal{P}^\mathrm{out} \) are also constants. It follows from Eqs. (2.38) and (2.41) that this normalization is obtained by choosing

\[
\chi^H = \frac{1}{4Qa_2} = \frac{(1 - 2Q)(1 - 4Q)}{Q[(\ell - 1)(\ell + 1)(\ell + 2) - 12Q]}, \tag{2.48}
\]

\[
\chi^\infty = \frac{1}{4},
\]

These solutions are also linearly independent, and

\[
W_T(R^H, R^\infty) = -2i\omega^3 Q^\mathrm{in}, \tag{2.49}
\]

where \( W_T \) was defined in Eq. (2.16).
2. Relations among constants

The constants $A_{\text{in, out}}$, $B_{\text{in, out}}$, $Q_{\text{in, out}}$, and $\mathcal{P}_{\text{in, out}}$ are not all independent. Various relations among them are easily derived.

In Eq. (2.44), the Wronskian $W_{\text{RW}}(X^H, X^\infty)$ was evaluated near $r \to \infty$. It can also be evaluated near $r = 2M$. Since the two values must agree, we have

$$B_{\text{out}} = A_{\text{in}}. \quad (2.50)$$

Similarly, constancy of $W_{\text{RW}}(X^H, \bar{X}^\infty)$ implies

$$B_{\text{in}} = -\bar{A}_{\text{out}}, \quad (2.51)$$

while constancy of $W_{\text{RW}}(X^H, \bar{X}^H)$ gives

$$|A_{\text{in}}|^2 - |A_{\text{out}}|^2 = 1, \quad (2.52)$$

which expresses global conservation of energy.

Additional relations are a consequence of the Chandrasekhar transformation. Combining Eqs. (2.40)–(2.42), (2.45), (2.46), and (2.48) reveals that

$$Q_{\text{in}} = \frac{b_2}{2Qa_2} A_{\text{in}} = \frac{(1 - 2Q)(1 - 4Q)}{4Q} A_{\text{in}}, \quad (2.53)$$

$$Q_{\text{out}} = \frac{1}{Qa_2} A_{\text{out}}.$$  

Similarly, combining Eqs. (2.38), (2.39), (2.43), (2.45), (2.47), and (2.48) gives

$$\mathcal{P}_{\text{in}} = Qa_2 B_{\text{in}}, \quad (2.54)$$

$$\mathcal{P}_{\text{out}} = -2Q^2(1 - 4Q) B_{\text{out}}.$$  

Finally, combining Eqs. (2.50), (2.51), (2.53), and (2.54) yields

$$\mathcal{P}_{\text{out}} = -\frac{8Q^3}{1 - 2Q} Q_{\text{in}}, \quad (2.55)$$

$$\mathcal{P}_{\text{in}} = Q^2 |a_2|^2 \tilde{Q}_{\text{out}}.$$

This last equation does not follow easily from Wronskian relations.

III. REGULARIZATION OF THE TEUKOLSKY EQUATION

I now proceed with the regularization of the Teukolsky equation. The source function $T(r)$ is constructed in Sec. III A for the specific case of a particle of mass $\mu$ released from rest at infinity and falling with zero angular momentum into a Schwarzschild black hole of mass $M$. The general solution to the inhomogeneous Teukolsky equation is also displayed here. In Sec. III B the regularization procedure is carried out. Then the behavior of the regularized solution is examined near $r = 2M$ in Sec. III C, and near $r = \infty$ in Sec. III D.
A. Source term and general solution

The source function is easily constructed by following the steps spelled out in Poisson and Sasaki [23]. When the motion is purely radial, it is given by

\[
T(r) = 2\sqrt{(\ell - 1)\ell(\ell + 1)(\ell + 2)} r^4 \int dt' d\Omega' T_{\alpha\beta} n^\alpha n^\beta \tilde{Y}_{\ell m}(\theta', \phi') e^{i\omega t'},
\]

(3.1)

where \(d\Omega' = d\cos \theta' d\phi'/n^\alpha = 1/2(1, f, 0, 0)\) is a null vector pointing outward, \(Y_{\ell m}\) are the usual spherical harmonics, and \(T_{\alpha\beta}\) is the particle’s energy-momentum tensor,

\[
T_{\alpha\beta}(x') = \mu \int d\tau u^\alpha u^\beta \delta[x' - x(\tau)].
\]

(3.2)

Here, \(x'\) represents an event in spacetime, labeled by the Schwarzschild coordinates \((t', r', \theta', \phi')\), and \(x(\tau)\) represents the particle’s world line, with four-velocity \(u^\alpha = dx^\alpha/d\tau\), where \(\tau\) is proper time. In Eq. (3.2), the \(\delta\)-function is normalized so that \(\int \delta(x)\sqrt{-g} d^4 x = 1\), where \(g\) is the determinant of the metric.

The geodesic equations for radial motion reduce to \(\theta = \phi = 0\) and

\[
\frac{dt}{d\tau} = -\frac{1}{f} \left(\frac{r}{2M}\right)^{1/2},
\]

(3.3)

which integrates to

\[
t(\tau) = -2M \left(\frac{2}{f} x^3 + 2x + \ln \frac{x - 1}{x + 1}\right),
\]

(3.4)

where \(x \equiv (r/2M)^{1/2}\). The four-velocity has non-vanishing components \(u^t = 1/f\) and \(u^r = -1/x\).

To obtain the source, Eq. (3.2) is first integrated with respect to \(dt\), which returns the factor \(\mu u^\alpha u^\beta /r^2 u^r\) multiplying \(\delta(t' - t(\tau))\delta(\cos \theta' - 1)\delta(\phi')\). Contractions with \(n_\alpha\) are then taken and the result is substituted into Eq. (3.1). After simplification, the result is

\[
p^2(r) T(r) = G\hat{g}(r) e^{i\omega t(r)},
\]

(3.5)

where

\[
G = -\frac{\mu}{8M^2} \left[\frac{(\ell - 1)\ell(\ell + 1)(\ell + 2)(2\ell + 1)}{4\pi}\right]^{1/2}
\]

(3.6)

if \(m = 0\), and \(G = 0\) otherwise (\(m\) is the spherical-harmonic index; that only modes with \(m = 0\) contribute to the full perturbation reflects the axial symmetry of the problem). Also,

\[
\hat{g}(r) = \frac{1}{x(x + 1)^2}.
\]

(3.7)

The general solution to the inhomogeneous Teukolsky equation is obtained by substituting Eq. (3.5) into (1.3). The result is

12
\[
R(r) = \frac{G}{W_T}\left\{ R^\infty(r) \left[ A + \int_a^r \hat{g}(r') R^H(r') e^{i\omega t(r')} \, dr' \right] \\
+ R^H(r) \left[ B + \int_r^\infty \hat{g}(r') R^\infty(r') e^{i\omega t(r')} \, dr' \right] \right\}. \tag{3.8}
\]

Here, \(W_T \equiv W_T(R^H, R^\infty) = -2i\omega^2 \Omega^2\), and \(a, b, A, \) and \(B\) are constants.

Our task now is to see to it that the boundary conditions — waves ingoing at the horizon and outgoing at infinity — are properly imposed. In fact, there is no difficulty in demanding the correct behavior at the black-hole horizon. A short calculation indeed reveals that when \(a = 2M\) and \(A = 0\), the first term of Eq. (3.8) is \(O(f^3)\). Since the second integral is finite, this ensures that \(R(r) \propto R^H(r)\) when \(r \to 2M\), as required. Unfortunately, the behavior at infinity cannot so easily be controlled. This is because both integrals diverge when \(r \to \infty\), due to the fact that \(\hat{g}(r) = O(r^{-3/2})\) while \(R^H, \infty(r) = O(r^3)\). Clearly, the solution (3.8) must be regularized before an attempt is made to impose the correct boundary condition at infinity.

**B. Regularization**

I begin by defining the integrals

\[
I^A(1, 2) = \int_1^2 \hat{g}(r) R^A(r) e^{i\omega t(r)} \, dr, \tag{3.9}
\]

where the index \(A\) stands for either \(”H”\) or \(”\infty”\). For the purpose of regularization, \(R^A(r)\) is conveniently expressed as

\[
R^A(r) = \chi^A C X^A(r), \tag{3.10}
\]

where \(C\) is given in Eq. (2.37). Substitution gives

\[
I^A(1, 2) = \chi^A (I_{\text{conv}} + I_{\text{div}}), \tag{3.11}
\]

where

\[
I_{\text{conv}} = \int_1^2 \Gamma_{\text{conv}}(r) e^{i\omega t(r)} X^A(r) \, dr, \tag{3.12}
\]

\[
I_{\text{div}} = \int_1^2 \Gamma_{\text{div}}(r) e^{i\omega t(r)} \mathcal{L} X^A(r) \, dr, \tag{3.13}
\]

and

\[
\Gamma_{\text{conv}}(r) = (i\omega r) f [\ell(\ell + 1) - 6M/r] \hat{g}(r), \tag{3.14}
\]

\[
\Gamma_{\text{div}}(r) = 2(i\omega r)(1 - 3M/r + i\omega r) r \hat{g}(r). \tag{3.15}
\]

As the names indicate, \(I_{\text{conv}}\) is convergent when \(r \to \infty\), since \(\Gamma_{\text{conv}}(r) = O(r^{-1/2})\), while \(I_{\text{div}}\) is divergent, since \(\Gamma_{\text{div}}(r) = O(r^{3/2})\).
Regularization of $I_{\text{div}}$ can be achieved by integration by parts. To identify what must be done, consider the alternative form

\[
I_{\text{div}} = \int_1^2 \left[ \Gamma_{\text{div}} e^{i\omega t} \mathcal{L} X^A + \frac{d}{dr} \left( h e^{i\omega t} \mathcal{L} X^A \right) \right] dr - h e^{i\omega t} \mathcal{L} X^A \bigg|_1^2 \\
= I_{\text{div}}' + \text{boundary terms,}
\]

where $h(r)$ is a function to be determined. After simplification, the new integral becomes

\[
I_{\text{div}}' = \int_1^2 e^{i\omega t} \left( \Gamma_{\text{div}}' + \Gamma_{\text{conv}}' X^A \right) dr,
\]

where

\[
\Gamma_{\text{div}}' = \frac{dh}{dr} + i\omega \left( \frac{dt}{dr} + \frac{1}{f} \right) h + \Gamma_{\text{div}},
\]

and

\[
\Gamma_{\text{conv}}' = [\ell(\ell + 1) - 6M/r] \frac{h}{r^2}.
\]

The function $h(r)$ must be chosen so that $I_{\text{div}}'$ is well behaved when $r \to \infty$.

The divergence of $I_{\text{div}}$ is caused by the bad behavior of $\Gamma_{\text{div}}(r)$. Happily, its contribution to $I_{\text{div}}'$ can be removed by simply setting $\Gamma_{\text{div}}' = 0$, which gives a differential equation for $h(r)$. One solution to this equation is

\[
h(r) = 8M^2 \frac{1 + x + 2Qx^3}{1 + x},
\]

where $x = (r/2M)^{1/2}$. Substituting this into Eq. (3.19) then reveals that $\Gamma_{\text{conv}}'(r) = O(r^{-1})$, which implies that $I_{\text{div}}'$ is indeed well behaved.

Regularization has thus been achieved. Combining Eqs. (3.11), (3.12), (3.16), (3.17), and (3.19) gives

\[
I^A(1, 2) = \chi^A \int_1^2 g(r) e^{i\omega t(r)} X^A(r) dr - \chi^A h(r) e^{i\omega t(r)} \mathcal{L} X^A(r) \bigg|_1^2 \\
= \chi^A \left[ J^A(1, 2) + \text{boundary terms} \right],
\]

where

\[
g(r) = \frac{2(1 + Qx^3)}{x^4} [\ell(\ell + 1) - 6M/r].
\]

This result will now be put to use.

With the notation introduced above, Eq. (3.8) reads

\[
R(r) = \frac{G}{W_T} \left\{ R^\infty(r)[A + I^H(a, r)] + R^H(r)[B + I^\infty(r, b)] \right\}.
\]
The regularized version of this equation is obtained by substituting Eq. (3.21). The boundary terms at \( r' = a \) and \( r' = b \) can be absorbed into the constants \( A \) and \( B \) by making the replacements

\[
A \rightarrow A + \chi^H h(a)e^{i\omega t(a)}\mathcal{L}X^H(a),
\]

\[
B \rightarrow B - \chi^\infty h(b)e^{i\omega t(b)}\mathcal{L}X^\infty(b).
\]

The boundary terms at \( r' = r \) combine to give

\[
J(r) = \hbar e^{i\omega t}(\chi^\infty R^H \mathcal{L}X^\infty - \chi^H R^\infty \mathcal{L}X^H).
\] (3.25)

This can be simplified by expressing \( R^{H,\infty}(r) \) in terms of \( X^{H,\infty}(r) \), as in Eq. (3.10). After simplification, Eq. (3.25) becomes

\[
J(r) = \chi^H \chi^\infty W_{\text{RW}}(i\omega r) f[\ell(\ell + 1) - 6M/r] h(r)e^{i\omega t(r)},
\] (3.26)

where \( W_{\text{RW}} \equiv W_{\text{RW}}(X^H, X^\infty) = 2i\omega A^\text{in}. \)

Finally, gathering the results yields

\[
R(r) = \frac{G}{W_T} \left\{ R^\infty(r) \left[ A + \chi^H J^H(a, r) \right] + R^H(r) \left[ B + \chi^\infty J^\infty(r, b) \right] + J(r) \right\},
\] (3.27)

where \( J(r) \) is given by Eq. (3.26), and

\[
J^A(1, 2) = \int_I^2 g(r')e^{i\omega t(r')}X^A(r')\,dr',
\] (3.28)

as was first written in Eq. (3.21).

The function \( R(r) \) is now expressed in terms of integrals that are well behaved when \( r \to \infty \). There is therefore no obstacle in setting

\[
a = 2M, \quad b = \infty,
\] (3.29)

which I shall do from now on. It can then be verified that with this choice, the replacements of Eq. (3.24) take the form \( A \to A + 0 \), and \( B \to B + \infty \). The infinite shift in \( B \) reflects the fact that the original expression for \( R(r) \), given by Eq. (1.2), was not well defined. This shows the importance of the procedure carried out here: the integrals must be regularized before \( b \) is set equal to infinity. The fact that \( B \) is then shifted by an infinite amount is of no consequence: Since Eq. (3.27) is a general solution to the inhomogeneous Teukolsky equation, as can be verified by direct substitution, this equation is a perfectly valid starting point for the discussion of boundary conditions, to which I turn next. One might just as well forget how Eq. (3.27) was derived, and proceed afresh from here.
C. Behavior near $r = 2M$

The behavior of $R(r)$, as expressed by Eq. (3.27), must now be examined near $r = 2M$, to ensure that it correctly represents purely ingoing waves at the black-hole horizon. The constants $A$ and $B$ must therefore be chosen so that $R(r \to \infty) \sim (\text{constant}) f^2 \exp(-i\omega r^*)$.

Our first task is to evaluate $J^H(2M, r)$ in the limit $r \to 2M$. Because $R^\infty(r) = O(f^3)$, this calculation must be carried out to second order in $f$. On the other hand, only the leading-order term in $J^\infty(r, \infty)$ is required for the calculation; this is simply given by $J^\infty(2M, \infty)$. Finally, $J(r)$ will have to be computed, also to second order in $f$.

1. Evaluation of $J^H(2M, r)$

The results of Sec. II A imply that near $r = 2M$, $X^H(r)$ can be written as

$$X^H(r) = Y(f) e^{-i\omega r^*}, \quad (3.30)$$

where $Y(f) = 1 + a_1 f + a_2 f^2 + O(f^3)$. The coefficients $a_1$ and $a_2$ are given by Eqs. (2.7). Substituting this into Eq. (3.28) returns an exponential factor of the form $\exp(i\omega u)$, where $u(r)$ is defined by

$$u(r) = t(r) - r^* = -4M \left[ \frac{1}{2} x^3 + \frac{1}{2} x^2 + x + \ln(x - 1) \right], \quad (3.31)$$

where $x = (r/2M)^{1/2}$. Changing the integration variable, what must be evaluated is

$$J^H(2M, r) = \int_u^\infty \frac{g f Y}{x + 1} e^{i\omega u'} du', \quad (3.32)$$

where the integrand is considered to be a function of $u'$.

To compute the integral, Eq. (3.31) must first be inverted in order to express $x$ as a function of $u$. While this cannot be done exactly in closed form, what is required here is a result accurate only to second order in $f = O(x - 1) = O(e^{-u/4M})$. Equation (3.31) implies

$$U \equiv \exp \left( -\frac{u}{4M} - \frac{11}{6} \right) = (x - 1) + 3(x - 1)^2 + O((x - 1)^3), \quad (3.33)$$

which can be inverted to give

$$x - 1 = U - 3U^2 + O(U^3). \quad (3.34)$$

Next, the integrand is expanded in powers of $x - 1$, and Eq. (3.34) is used to write this in terms of $U$. Integration is then straightforward. The final result must be expressed as an expansion in powers of $f$. For this purpose one uses $U = \frac{1}{2} f + \frac{2}{3} f^2 + O(f^3)$, which follows from $x - 1 = \frac{1}{2} f + \frac{2}{3} f^2 + O(f^3)$. The final result is

$$J^H(2M, r) = 4M [\mu_1 f + \mu_2 f^2 + O(f^3)] e^{i\omega u(r)}, \quad (3.35)$$

where
\[ \mu_1 = \frac{(1 + Q)[\ell(\ell + 1) - 3]}{1 - 4Q}, \]  
\[ \mu_2 = \frac{2(\ell^4 + 2\ell^3 - 5\ell^2 - 6\ell + 12) + (2\ell^4 + 4\ell^3 + 11\ell^2 + 9\ell - 63)Q + [6\ell(\ell + 1) - 42]Q^2}{4(1 - 2Q)(1 - 4Q)}. \]  
(3.36)

The evaluation of \( J^H(2M, r) \) is now completed.

2. Evaluation of \( J(r) \)

This calculation is quite straightforward. Equation (3.26), with \( h(r) \) given by Eq. (3.20), can immediately be expanded in powers of \( f \). The result is

\[ J(r) = 16M^2Q\chi^H\chi^\infty W_{RW}[\nu_1 f + \nu_2 f^2 + O(f^3)]e^{i\omega t}, \]  
(3.37)

where

\[ \nu_1 = (1 + Q)[\ell(\ell + 1) - 3], \]  
(3.38)

\[ \nu_2 = \ell(\ell + 1) + \frac{3}{4}[3\ell(\ell + 1) - 5]Q. \]

3. Evaluation of \( R(r) \)

Expansions in powers of \( f \) have been obtained for \( J^H(2M, r) \) and \( J(r) \). These are supplemented by the results of Sec. II B and D, which imply

\[ R^\infty(r) = p_{\text{in}}^f f^2 + O(f^3)e^{-i\omega t} + p_{\text{out}}^f [1 + p_{\text{out}}^f f + O(f^2)]e^{i\omega t}, \]  
(3.39)

where \( p_{\text{out}}^f \) is given by Eq. (2.25). We also have \( R^H(r) = [f^2 + O(f^3)]\exp(-i\omega t) \) and \( J^\infty(r, \infty) = J^\infty(2M, \infty) + O(f) \).

Substituting all this into Eq. (3.27) gives \( R(r) \) as an expansion in powers of \( f \), which contains terms of order \( f^0, f \), as well as the allowed terms of order \( f^2 \) and higher. Each term will be discussed in turn.

The term of order \( f^0 \) can be eliminated by setting

\[ A = 0, \]  
(3.40)

which will be done from here on.

The term of order \( f \) can be simplified using Eqs. (2.44), (2.48), (2.53), (2.55), (3.36), and (3.38). As it must, it vanishes identically.

The term of order \( f^2 \) survives, and contains two contributions. The first is proportional to \( \exp(i\omega t) \) and comes from \( J^H(2M, r) \) and \( J(r) \); the other is proportional to \( \exp(-i\omega t) \) and comes from \( R^H(r) \). The first contribution is simplified using the same equations as
before, in addition to Eq. (2.25); the result is \(4MQ(1 + Q)A^i f^2 e^{i\omega t}\). This is then combined with the second contribution by observing that \(t = -r^* - 2M(\frac{5}{3} - 2\ln 2) + O(f)\).

After simplification, the final result is that near \(r = 2M\),

\[
R(r) = \frac{G}{W_T} \left[ B + C + \chi^\infty J^\infty(2M, \infty) \right] f^2 e^{-i\omega r^*} + O(f^3),
\]

as required. It is recalled that \(J^\infty(2M, \infty)\) was defined in Eq. (3.28), and the constant \(C\) is given explicitly by

\[
C = 4MQ(1 + Q)A^i \exp \left[ -2Q \left( \frac{5}{3} - 2\ln 2 \right) \right].
\]

The constant \(B\) will shortly be set to zero.

**D. Behavior near \(r = \infty\)**

It is much easier to extract the behavior of Eq. (3.27) near \(r = \infty\). I begin with the computation of \(J^\infty(r, \infty)\). In this limit, Eq. (3.22) reduces to

\[
g(r) = \frac{2Q\ell(\ell + 1)}{x} \left[ 1 + O(x^{-2}) \right],
\]

while Eqs. (2.9) and (2.11) imply

\[
X^\infty(r) = [1 + O(x^{-2})] e^{i\omega r^*}.
\]

After substitution into Eq. (3.28), and a change of integration variable to

\[
v(r) = t(r) + r^* = -4M \left[ \frac{1}{2} x^3 - \frac{1}{2} x^2 + x - \ln(x + 1) \right],
\]

one arrives at

\[
J^\infty(r, \infty) = 2\ell(\ell + 1)Q \int_{-\infty}^\infty \frac{1}{x^2} \left[ 1 + O(x^{-1}) \right] e^{i\omega v} dv' = 4\ell(\ell + 1)Q^2 z [1 + O(z^{1/2})] e^{i\omega v},
\]

where \(z = (i\omega r)^{-1}\).

The computation of \(J(r)\) is also straightforward. Equations (3.20) and (3.26) immediately give

\[
J(r) = 8M^2 \chi^H \chi^\infty W_{RW} \ell(\ell + 1) \frac{1}{z^2} \left[ 1 + O(z) \right] e^{i\omega t}.
\]

To finish the job, Eqs. (3.46) and (3.47), together with the relations \(J^H(2M, r) = J^H(2M, \infty) + O(z^{1/2})\) and \(R^\infty(r) = [z^{-3} + O(z^{-2})] \exp(i\omega r^*)\), are substituted into Eq. (3.27). This gives

\[
R(r) = \frac{G}{W_T} \left\{ \chi^H J^H(2M, \infty)[1 + O(z^{1/2})] R^\infty(r) + BR^H(r) + \text{other terms} \right\},
\]

18
where the “other terms” are all proportional to \( \exp[i\omega \ell(r)] \), and are \( O(z^{-2}) \) or higher, and therefore much smaller than the dominant terms of order \( z^{-3} \). Furthermore, because \( t(r) \sim -(4M/3)(r/2M)^{3/2} \) when \( r \to \infty \), their phase increases much more rapidly than \( r^* \sim r \), which means that the “other terms” cannot be combined into a term proportional to \( R^H(r) \), which would represent a free, initially incoming, gravitational wave. The only such term present in \( R(r) \) is \( B R^H(r) \), and the requirement that waves must be purely outgoing at infinity dictates

\[
B = 0. \tag{3.49}
\]

The final result is that near \( r = \infty \),

\[
R(r) = \frac{G}{W_T} \chi^H J^H (2M, \infty) (i\omega r)^3 e^{i\omega r^*} + O(r^{5/2}), \tag{3.50}
\]
as required; \( J^H(2M, \infty) \) was defined in Eq. (3.28).

**IV. SUMMARY AND CONCLUSION**

I now summarize. The regularization of the Teukolsky equation was successful. The procedure consisted of two stages. In the first stage, the most general solution to the inhomogeneous Teukolsky equation was written in terms of integrals that are well behaved when \( r \to \infty \). This was accomplished in Eq. (3.27). In the second stage, an ingoing-wave boundary condition was imposed at the black-hole horizon, and an outgoing-wave boundary condition was imposed at infinity. While the correct behavior at infinity could not be verified with the original form of the solution given by Eq. (3.8), it was quite straightforward to do so with the regularized form (3.27).

The regularized solution is written as

\[
R(r) = \frac{G}{W_T} \left[ \chi^H J^H (r) R^\infty (r) + \chi^\infty J^\infty (r) R^H (r) + J(r) \right], \tag{4.1}
\]

where

\[
G = -\frac{\mu}{8 M^2} \sqrt{\frac{2\ell + 1}{4\pi}} \sqrt{(\ell - 1)\ell(\ell + 1)(\ell + 2)}, \tag{4.2}
\]

\[
W_T = -2i\omega^3 \frac{(1 - 2Q)(1 - 4Q)}{4Q} A^{\text{in}}, \tag{4.3}
\]

\[
\chi^H = \frac{(1 - 2Q)(1 - 4Q)}{Q[(\ell - 1)\ell(\ell + 1)(\ell + 2) - 12Q]}, \tag{4.4}
\]

\[
\chi^\infty = \frac{1}{4}, \tag{4.5}
\]

\[
J^H(r) = \int_{2M}^r g(r') e^{i\omega \ell(r')} X^H(r') \, dr', \tag{4.6}
\]

\[
J^\infty(r) = \int_{r}^{\infty} g(r') e^{i\omega \ell(r')} X^\infty(r') \, dr', \tag{4.7}
\]

\[
J(r) = -2\omega^2 \chi^H \chi^\infty A^{\text{in}} \tau f [\ell(\ell + 1) - 6M/r] h(r) e^{i\omega \ell(r)}. \tag{4.8}
\]
The functions \( X^H(r) \) and \( X^\infty(r) \) are linearly independent solutions of the Regge-Wheeler equation (2.1), with normalizations determined by Eqs. (2.42) and (2.43). The constant \( A^\infty \) is also defined by these equations. Also, \( f = 1 - 2M/r, \ \ x = (r/2M)^{1/2}, \ \ Q = iM\omega, \) and

\[
g(r) = \frac{2(1 + Qx^3)}{x^4} [\ell(\ell + 1) - 6M/r], \tag{4.9}
\]

\[
h(r) = 8M^2 \frac{1 + x + 2Qx^3}{1 + x}, \tag{4.10}
\]

\[
t(r) = -2M \left( \frac{\frac{3}{2}x^3 + 2x + \ln \frac{x-1}{x+1}}{x+1} \right). \tag{4.11}
\]

Equation (4.1) is obtained directly from (3.27) by setting \( a = 2M, \ \ b = \infty, \ \ A = B = 0, \ \ J^H(r) \equiv J^H(2M, r), \) and \( J^\infty(r) \equiv J^\infty(r, \infty) \). Equation (4.2) is the same as (3.6). Equation (4.3) follows from (2.49) and (2.53). Equations (4.4) and (4.5) are the same as (2.48). Equations (4.6) and (4.7) are the same as (3.28). And finally, Eqs. (4.8)–(4.11) are the same as (3.26), (3.22), (3.20), and (3.4), respectively.

Equation (4.1) implies that near \( r = 2M, \ \ R(r) \) behaves as

\[
R(r) \sim \frac{G}{W_T} \left[ 2^{2+4Q}e^{-10Q/3}MQ(1 + Q)A^\infty + \chi^\infty J^\infty(2M) \right] f^2 e^{-i\omega r}. \tag{4.12}
\]

This follows from Eqs. (3.41) and (3.42). It also implies that near \( r = \infty, \)

\[
R(r) \sim \frac{G}{W_T} \chi^H J^H(\infty) (i\omega r)^3 e^{i\omega r}, \tag{4.13}
\]

which is the same statement as in Eq. (3.50). This expression agrees precisely with the one derived by Simone, Poisson, and Will [30], who obtained it by “throwing away the infinite boundary term”.

The radial function \( R(r) \), found here to be a solution of the inhomogeneous Teukolsky equation, represents a gravitational perturbation of frequency \( \omega \) and spherical-harmonic indices \( \ell \) and \( m \). A better notation for it (which I did not adopt in order to keep all symbols simple) would be \( R_{\ell m}(\omega; r) \). The full perturbation is obtained by summing over all these modes. More precisely, the perturbation in the Riemann tensor caused by the infalling particle is represented by the complex function \( \Psi_4 \) [36] given by

\[
\Psi_4(x) = \frac{1}{r^4} \int \sum_{\ell m} R_{\ell m}(\omega; r) _{2} Y_{\ell m}(\theta, \phi) e^{-i\omega t} d\omega. \tag{4.14}
\]

Here, \( x \) is the event in spacetime labeled by the Schwarzschild coordinates \( (t, r, \theta, \phi) \), and the functions \( _{2} Y_{\ell m}(\theta, \phi) \) are spherical harmonics of spin-weight \(-2\) [37]. For an axially symmetric problem, such as the one considered in this paper, modes with \( m \neq 0 \) vanish identically, so the sum over \( m \) reduces to the single term \( m = 0 \). From \( \Psi_4(x) \) one may obtain many relevant quantities, such as the gravitational-wave field \( h_{\alpha \beta}^{TT}(x) \) and the fluxes of energy at the black-hole horizon and at infinity. Additional details are provided by Refs. [23,30].

The considerations of this paper were limited to the simplest case of an infall into a black hole: the hole was assumed to be nonrotating, and the particle was assumed to have
zero angular momentum and a vanishing initial velocity. There is, however, no reason to believe that the methods used here could not be extended to more complicated situations. Of course, the amount of labor involved, already considerable here, would increase, but there is no issue of principle.

To conclude, I would like to stress the main message of this paper. The standard choice of Green’s function for solving the inhomogeneous Teukolsky equation leads to divergent integrals, and contrary to naive expectations, fails to enforce the correct boundary conditions at the black-hole horizon and at infinity. The regularization procedure amounts to nothing more — and nothing less — than finding an adequate Green’s function. Contrary to what may have been believed, there is nothing intrinsically wrong with the Teukolsky equation when dealing with non-compact source terms.

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