Polynomial deformations of $osp(1/2)$ and generalized parabosons

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Abstract: We consider the algebra $R$ generated by three elements $A, B, H$ subject to three relations $[H, A] = A$, $[H, B] = -B$ and $\{A, B\} = f(H)$. When $f(H) = H$ this coincides with the Lie superalgebra $osp(1/2)$; when $f$ is a polynomial we speak of polynomial deformations of $osp(1/2)$. Irreducible representations of $R$ are described, and in the case $\deg(f) \leq 2$ we obtain a complete classification, showing some similarities but also some interesting differences with the usual $osp(1/2)$ representations. The relation with deformed oscillator algebras is discussed, leading to the interpretation of $R$ as a generalized paraboson algebra.

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I Introduction

Define the algebra $R = \mathbb{C}[A, B, H]$ subject to the following relations:

$$HA - AH = A, \quad HB - BH = -B, \quad AB + BA = f(H),$$

where $f$ is a fixed polynomial function, $f \in \mathbb{C}[x]$. For many of the applications that follow, $f$ can also be another analytic function, but it will be clear that the polynomial case itself is already a rich structure. The purpose of the present paper is to study this algebra and in particular its simple modules (irreducible representations). The algebra $R$ is equal to the enveloping algebra of the Lie superalgebra $osp(1/2)$ when $f(x) = x$, and we shall draw some similarities and differences with the representation theory of $osp(1/2)$ for the general case. In this sense, the algebra (1) can be considered as a polynomial deformation of $osp(1/2)$. If the last relation in (1) is replaced by $AB - BA = f(H)$, the representation theory becomes rather different and in that case one is dealing with polynomial deformations of the enveloping algebra of $sl(2)$. These algebras have been studied by Smith [1] and were shown to have a surprisingly rich theory of representations; from the physical point of view they are non-canonical Heisenberg algebras, and have been investigated by Brodimas et al [2].

There is clearly a relationship between the algebra $R$ and generalized deformed oscillator algebras [3, 4, 5, 6, 7, 8] generated by the operators $a, a^\dagger, N$ subject to the relations

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad a^\dagger a = \tilde{F}(N), \quad aa^\dagger = \tilde{F}(N + 1),$$

where $\tilde{F}$ is called a structure function. If $\tilde{F}(0) = \tilde{F}(p + 1) = 0$ and $\tilde{F}(n) > 0$ for $n \in \{1, 2, \ldots, p\}$, this algebra is referred to as a generalized parafermionic algebra [9]. These algebras are related to nonlinear deformations of $so(3)$ or $sl(2)$ [9, 10, 11, 12]. If $\tilde{F}(\lambda) = 0$ and $\tilde{F}(\lambda + n) > 0$ for $n \in \{1, 2, \ldots\}$ this algebra can be referred to as a generalized parabosonic algebra; another type of generalizations are the $q$-deformed paraboson algebras which have been the topic of many papers [3, 13, 14, 15, 16, 17, 18]. It should be mentioned that for generalized deformed oscillator algebras there is also the condition of unitarity of the representations. When we study the simple modules of $R$, our approach will be more general: first we study all irreducible representations of $R$, and only in a final stage we will determine which of the representations are unitary. The mathematical classification of the irreducible representations (simple modules) is interesting on its own and deserves special attention here.
When studying simple modules of $R$, we shall consider only the highest weight modules. These are modules $V$ generated by a vector $v$ satisfying $Av = 0$ and $Hv = \lambda v$. Then $v$ is highest weight vector of weight $\lambda$ ($\lambda \in \mathbb{C}$).

The structure of the paper is as follows. In section II the algebra $R$ is considered when $\deg(f) \leq 1$, and shown to be familiar. In section III we give a number of general properties of $R$, including a differential realization and the structure of its center. In the following section some general theory of simple modules for $R$ is developed, and in section V this is illustrated for the case $\deg(f) = 2$. The results in section V are very surprising, and show how rich the representation theory is. In section VI we give a number of comments on real and unitary representations. Finally in section VII we indicate the relation with generalized oscillator systems, and show how certain one-dimensional quantum mechanical systems fall into the picture.

To conclude the present section, we fix some notation. For integers we use the common symbol $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$; for the nonnegative integer numbers we use $\mathbb{N} = \{0, 1, 2, \ldots\}$; for the positive integers we use $\mathbb{N}^* = \{1, 2, \ldots\}$. We shall also use the following type of notations: $2\mathbb{N} + 1 = \{1, 3, 5, \ldots\}$, $2\mathbb{N}^* = \{2, 4, 6, \ldots\}$, etc.

II The case $\deg(f) \leq 1$

We have already noticed that when $f(x) = x$ the algebra $R$ coincides with the enveloping algebra of the Lie superalgebra $osp(1/2)$. More generally, we shall see that when $\deg(f) \leq 1$, $R$ is familiar. We shall study the simple modules of $R$ when $\deg(f) \leq 1$.

Let $f(x) = \alpha x + \beta$. There are three cases to be distinguished.

Case (1) : $\alpha = 0$, $\beta \neq 0$. Let $V$ be a highest weight module with highest weight vector $v_\lambda$, thus $Av_\lambda = 0$ and $Hv_\lambda = \lambda v_\lambda$. Define $v_{\lambda-1} = Bv_\lambda$; this vector is nonzero because otherwise the relation $AB + BA = \beta$ is not valid when acting on $v_\lambda$. Moreover, it follows from the defining relations that $Hv_{\lambda-1} = (\lambda - 1)v_{\lambda-1}$ and $Av_{\lambda-1} = \beta v_\lambda$. Next, consider $v_{\lambda-2} = Bv_{\lambda-1}$. One finds that $Av_{\lambda-2} = ABv_{\lambda-1} = (\beta - BA)v_{\lambda-1} = 0$. Hence if $v_{\lambda-2}$ were nonzero, it would be an element of a nonzero submodule of $V$. But since $V$ is simple, this submodule must be zero and $v_{\lambda-2} = 0$. Consequently, all simple modules of $R$ are two-dimensional, and their explicit
matrix representations are given by:

\[ A = \begin{pmatrix} 0 & \beta \\
0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\
1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} \lambda & 0 \\
0 & \lambda - 1 \end{pmatrix}. \]  

(3)

Case (2) : \( \alpha = \beta = 0 \).

Again, let \( v_\lambda \) be the highest weight vector of a simple module \( V \), thus \( Av_\lambda = 0 \) and \( Hv_\lambda = \lambda v_\lambda \). Consider \( v_{\lambda - 1} = Bv_\lambda \); from \( AB + BA = 0 \), it follows that \( Av_{\lambda - 1} = 0 \), thus \( v_{\lambda - 1} \) belongs to a submodule. Since \( V \) is simple, one concludes that \( v_{\lambda - 1} = 0 \). Thus all simple modules are one-dimensional, and are given by:

\[ Av_\lambda = Bv_\lambda = 0, \quad Hv_\lambda = \lambda v_\lambda. \]  

(4)

Case (3) : \( \alpha \neq 0 \).

Let

\[ \hat{H} = H + \beta / \alpha, \quad \hat{A} = A / \alpha, \quad \text{and} \quad \hat{B} = B. \]  

(5)

The relations (1) with \( f(x) = \alpha x + \beta \) become

\[ \hat{H} \hat{A} - \hat{A} \hat{H} = \hat{A}, \quad \hat{H} \hat{B} - \hat{B} \hat{H} = - \hat{B}, \quad \hat{A} \hat{B} + \hat{B} \hat{A} = \hat{H}. \]  

(6)

These are the defining relations of the enveloping algebra of \( osp(1/2) \). But due to the linear relationship there is a one-to-one correspondence between the simple modules of \( R \) and the simple modules of the algebra generated by \( \hat{A}, \hat{B} \) and \( \hat{H} \), subject to (6). Since all simple modules of \( osp(1/2) \) are known [19, 20], all simple modules of \( R \) follow directly. In particular, for every \( j \in \mathbb{N} \), there exists (up to isomorphism) one simple module of \( R \) with dimension \( 2j + 1 \); and there exist no simple modules with even dimension.

### III Some structure theorems for the general case

First of all, let us give some simple realizations of \( R \), with \( f \) a polynomial of degree \( > 1 \). For \( m, n \in \mathbb{N} \), define the following polynomial:

\[ p_{m, n}(t) = t \prod_{j=1}^{m} (t - j) \prod_{k=1}^{n} (t - k). \]  

(7)

**Lemma 1** With \( x \) as multiplication by \( x \) and \( \partial \) as differentiation with respect to \( x \) on \( \mathbb{C}[x] \), the realizations

\[ A = x^{n+1} \partial^n, \quad B = x^m \partial^{m+1}, \quad H = x \partial \quad (m, n \in \mathbb{N}), \]  

(8)

satisfy the relations (1) with \( f(t) = p_{m, n}(t) + p_{m, n}(t + 1) \).
Proof. First one shows that $AB = p_{m,n}(H)$. Since
\[ x \partial(x^{n+1} \partial^n) = (n+1)x^{n+1} \partial^n + x^{n+2} \partial^{n+1}, \]
we have that $x^{n+2} \partial^{n+1} = (H - n - 1)x^{n+1} \partial^n$, and thus
\[ x^{n+2} \partial^{n+1} x^m \partial^m + 1 = (H - n - 1)x^{n+1} x^m \partial^m + 1. \]
Similarly, one finds
\[ x^{n+1} \partial^n x^m \partial^{n+2} = x^{n+1} \partial^n x^m \partial^m + 1(H - m - 1). \]
Then $AB = p_{m,n}(H)$ follows by induction on $n$ and $m$. The relation $BA = p_{m,n}(H + 1)$ is proved similarly, or follows from the general observation that $BF(H) = F(H + 1)B$.

Next, we consider some general structure results. The three relations (1) allow one to reorder any polynomial in $A, B$ and $H$ unambiguously. Hence we have

**Theorem 2** A basis for $R$ is given by the monomials $A^iB^jH^k$, $(i, j, k \in \mathbb{N})$.

It is easy to see that $[H, A^iB^jH^k] = (i - j)A^iB^jH^k$. Thus we have a weight space decomposition for $R : R = \bigoplus_{\mu \in \mathbb{Z}} R_\mu$, where $R_\mu = \{ x \in R \mid [H, x] = \mu x \}$. From the defining relations one can deduce that $R_0 = \mathbb{C}[A, H]$; in other words, the commutant of $H$ consists of polynomials in $AB$ and in $H$.

For the next theorem, we need an easy lemma.

**Lemma 3** Let $f \in \mathbb{C}[x]$. There exists a unique polynomial $F \in \mathbb{C}[x]$ such that $F(x + 1) + F(x) = f(x)$.

**Proof.** Assume that $\deg(f) = n$. Since $\deg(F(x+1) + F(x)) = \deg(F)$, let $F(x) = \sum_{k=0}^{n} a_k x^k$. Then
\[ F(x + 1) + F(x) = \sum_{k=0}^{n} \left( \sum_{j=k}^{n} \binom{n}{j} a_j a_k \right) x^k. \]
Now $F(x + 1) + F(x) = f(x)$ reduces to a nonsingular upper triangular system in the unknowns $a_0, a_1, \ldots, a_n$, yielding a unique solution for $F$.

Consider the algebra $R$ with relations (1) characterized by $f$, and $F$ determined by the previous lemma.

**Lemma 4** The element $\Omega = A^2B^2 + AB(F(H) - F(H - 1)) - F(H)^2$ is a central element of $R$. 

5
Proof. By straightforward calculation one finds that

\[ [A, AB F(H)] = A^2 B F(H) - A f(H) F(H + 1) + A^2 B F(H + 1) \]
\[ = A^2 B f(H) - A f(H) F(H + 1), \]
\[ [A, -AB F(H - 1)] = -A^2 B f(H - 1) + A f(H) F(H), \]
\[ [A, -F(H)^2] = -A f(H)^2 + A f(H + 1)^2. \]

Thus \([A, \Omega] = 0\); similarly one finds that \([B, \Omega] = 0\), and with \([H, \Omega] = 0\) it follows that \(\Omega\) is central.

In fact, every other central element of \(R\) is a polynomial expression in \(\Omega\):

**Theorem 5** The center of \(R\) is equal to \(\mathbb{C}[\Omega]\).

Proof. Let \(z\) be a central element; since \(z\) commutes with \(H\) it belongs to \(R_0\), and we have seen that \(R_0 = \mathbb{C}[AB, H]\). But \((AB)^2 = -A^2 B^2 + AB f(H - 1)\), thus

\[ R_0 = \mathbb{C}[\Omega, H] \oplus AB \mathbb{C}[\Omega, H]. \]  \hspace{1cm} (12)

Therefore, \(z\) can be written as

\[ z = \sum_{k=0}^{m} \Omega^k c_k + AB \sum_{k=0}^{m} \Omega^k d_k, \quad c_k, d_k \in \mathbb{C}[H]. \]  \hspace{1cm} (13)

From theorem 2 and the explicit expression of \(\Omega\), it follows that

\[ \Omega^k = \sum_{i=0}^{2k} A^i B^j g_{i,j}^k, \]  \hspace{1cm} (14)

where \(g_{i,j}^k\) is a polynomial in \(H\) and \(g_{2k}^{2k} = 1\). Next, we express that \(z\) is central:

\[ 0 \equiv [A, z] = \sum_{k=0}^{m} A \Omega^k (c_k(H) - c_k(H + 1)) + \]
\[ \sum_{k=0}^{m} 2A^2 B \Omega^k (d_k(H) + d_k(H + 1)) + \sum_{k=0}^{m} A \Omega^k f(H) d_k (H + 1). \]  \hspace{1cm} (15)

Rearranging all terms of (15) according to the basis of theorem 2 using (14), and considered as a polynomial in \(A\) and \(B\) with coefficients in \(\mathbb{C}[H]\), the term of highest degree is \(A^{2m+2} B^{2m+1} (d_m(H) + d_m(H + 1))\). Clearly this term has to vanish (theorem 2), and this can happen only when \(d_m = 0\). Using the fact that \(d_m = 0\), the remaining term of highest
degree in (15) is then $A^{2m+1}B^{2m} (c_m(H) - c_m(H + 1))$. Again, this has to vanish, implying that $c_m$ is just a constant: $c_m \in \mathbb{C}$. But now we can apply the above argument to $z = c_m \Omega^m$, implying that $d_{m-1} = 0$ and $c_{m-1} \in \mathbb{C}$, and so on. So finally all $d_k$ are zero and all $c_k$ constants.

The next result gives the eigenvalue of $\Omega$ acting on a highest weight module.

**Theorem 6** Let $V$ be an $R$ module generated by a highest weight vector $v_\lambda$. Then, for all $v \in V$,

$$\Omega v = -F(\lambda + 1)^2 v,$$

where $F$ is defined by means of Lemma 3.

**Proof.** The expression of $\Omega$ given in Lemma 4 can be rewritten in the following form:

$$\Omega = AB^2 A + BA (F(H + 1) - F(H)) - F(H + 1)^2.$$  \hfill (17)

Thus

$$\Omega v_\lambda = -F(\lambda + 1)^2 v_\lambda,$$ \hfill (18)

and since the module is generated by $v_\lambda$, $\Omega$ has the same eigenvalue for every $v$ in $V$.

**IV Simple modules of $R$**

In this section we shall study some general features of simple modules of $R$, where $f(x)$ is a general polynomial of degree $n$. The Verma module $V(\lambda)$ can be identified with the vector space with basis

$$v_{\lambda-j} = B^j v_\lambda, \quad j \in \mathbb{N},$$ \hfill (19)

where $v_\lambda$ is a highest weight vector satisfying $Av_\lambda = 0$ and $Hv_\lambda = \lambda v_\lambda$. The Verma module $V(\lambda)$ contains a maximal submodule $M$; if $M = \{0\}$ then $V(\lambda)$ is simple, otherwise the quotient module $L(\lambda) = V(\lambda)/M$ is a simple module of highest weight $\lambda$. All simple highest weight modules are obtained this way.

To see whether a vector $v_{\lambda-j}$ is in a submodule one determines $Av_{\lambda-j}$. By induction, it is easy to prove that

$$AB^j v_\lambda = \left(f(\lambda - j + 1) - f(\lambda - j + 2) + f(\lambda - j + 3) - \cdots - (-1)^j f(\lambda)\right)B^{j-1} v_\lambda.$$ \hfill (20)
If \( j \) is even, the coefficient in the rhs of (20) is equal to

\[
F(\lambda + 1 - j) - F(\lambda + 1).
\]

Similarly, using Lemma 3, one finds that for the case \( j \) odd, the rhs of (20) is equal to

\[
F(\lambda + 1) - F(\lambda + 2 - j) + f(\lambda + 1 - j).
\]

For \( \lambda \in \mathbb{C} \), we define:

\[
S_0(\lambda) = \{ j \in 2\mathbb{N}^* \mid F(\lambda + 1) - F(\lambda + 1 - j) = 0 \},
\]

\[
S_1(\lambda) = \{ j \in (2\mathbb{N} + 1) \mid F(\lambda + 1) - F(\lambda + 2 - j) + f(\lambda + 1 - j) = 0 \},
\]

\[
S(\lambda) = S_0(\lambda) \cup S_1(\lambda).
\]

Note that (21) is of degree \( n - 1 \) in \( \lambda \), and that (22) is of degree \( n \) in \( \lambda \). As equations in \( j \), (21) has the trivial solution \( j = 0 \) (to be excluded from \( S_0(\lambda) \)) so that an equation of degree \( n - 1 \) remains, and (22) is of degree \( n \). Of course, for given \( \lambda \) not all solutions for these equations in \( j \) will yield integer \( j \) so that \( S_0(\lambda) \) and/or \( S_1(\lambda) \) are often empty.

The primitive vectors of \( V(\lambda) \) are those \( v_{\lambda-j} \) such that \( Av_{\lambda-j} = 0 \); they generate the submodules of \( V(\lambda) \). With the given definitions we have the following lemmas:

**Lemma 7** The submodules of \( V(\lambda) \) are \( \mathbb{C}[B]B^jV(\lambda) \) with \( j \in S(\lambda) \).

**Lemma 8** Let \( \lambda \in \mathbb{C} \).

1. If \( S(\lambda) \neq \emptyset \) then \( L(\lambda) = V(\lambda)/B^jV(\lambda) \), where \( j = \min S(\lambda) \), otherwise \( L(\lambda) = V(\lambda) \).

2. For even \( j \) (\( j \in 2\mathbb{N}^* \)) the number of simple modules of dimension \( j \) is equal to \( \#\{ \lambda \in \mathbb{C} \mid F(\lambda + 1) - F(\lambda + 1 - j) = 0 \text{ and } j = \min S(\lambda) \} \).

3. For odd \( j \) (\( j \in 2\mathbb{N} + 1 \)) the number of simple modules of dimension \( j \) is equal to \( \#\{ \lambda \in \mathbb{C} \mid F(\lambda + 1) - F(\lambda + 2 - j) + f(\alpha + 1 - j) = 0 \text{ and } j = \min S(\lambda) \} \).

Because of the previous remarks concerning the degrees of the relevant equations, it follows that for even \( j \) there are at most \( n - 1 \) simple modules of dimension \( j \) and that for odd \( j \) there are at most \( n \) simple modules of dimension \( j \). Generally there will also be \( n - 1 \) simple modules for even dimensions and \( n \) modules for odd dimensions, apart from certain exceptions. This will be illustrated in the following section, where a complete study of \( \deg(f) = n = 2 \) is presented.
V The case \( \deg(f) = 2 \)

In the third relation of (1) \( f(H) \) is of the form \( aH^2 + bH + c \) with \( a \neq 0 \). But due to the fact that the representation theory of \( R \) is equivalent to the representation theory of the algebra generated by (5) (subject to the corresponding relations), there are in fact two degrees of freedom that one can delete. Therefore we shall assume in the rest of this section that the algebra \( R \) is determined by the following relations:

\[
HA - AH = A, \quad HB - BH = -B, \quad AB + BA = 2H^2 + 2H + 1 - c,
\]

where \( c \in \mathbb{C} \). The form of \( f(H) \) in (26) is somewhat arbitrary, but for our purposes the one chosen here seems to be the most appropriate. We emphasize again that for every given \( f \) of degree 2 the algebra can be “rescaled” by means of a transformation of type (5) such that it becomes (26).

With \( f(x) = 2x^2 + 2x + 1 - c \), we have that \( F(x) = x^2 - c/2 \). Then the equations (21) and (22) are easy to calculate and one finds respectively \( j(j-2\lambda-2) = 0 \) and \( (j-2\lambda-2)^2 = 2c-j^2 \). This implies the following:

\[
\text{If } \lambda \in \mathbb{N} \text{ then } S_0(\lambda) = \{2\lambda + 2\} \text{ else } S_0(\lambda) = \emptyset, \quad (27)
\]

\[
S_1(\lambda) = \{ j \in (2\mathbb{N} + 1) \text{ such that } (j - \lambda - 1)^2 = c - (\lambda + 1)^2 \}. \quad (28)
\]

The purpose is now twofold: for given \( \lambda \in \mathbb{C} \), we wish to give (in terms of \( c \)) the dimension of \( L(\lambda) \) (theorem 9); and for given \( j \), we wish to determine the number of (inequivalent) simple modules of dimension \( j \) (theorem 11).

For the rest of this section it will be useful to work with \( \lambda + 1 \) instead of \( \lambda \), and we even introduce a notation for it:

\[
l = \lambda + 1. \quad (29)
\]

Suppose \( \lambda \in \mathbb{C} \) is given.

Consider first the case that \( \lambda \notin \mathbb{N} \), then \( S_0(\lambda) = \emptyset \). The equation determining \( S_1(\lambda) \) is given by

\[
(j - l)^2 = c - l^2. \quad (30)
\]

Can we have two integer solutions \( j_1 \) and \( j_2 \), both odd, for this equation? From the equation it would then follow that \( j_1 + j_2 = 2l \), which would imply that \( l \) or \( \lambda \) is in \( \mathbb{N} \), which is not the case. Hence, we can have at most one integer solution for the equation (30). In order to
have an integer and odd solution for (30) \( l \pm \sqrt{c-l^2} \) must be an element of \( 2N + 1 \). This can only happen when \( c \) is of the form \( 2l^2 - 2lm + m^2 \) with \( m \in (2N + 1) \). If \( c \) is of this form then \( L(\lambda) \) is finite-dimensional with dimension \( m \), otherwise \( L(\lambda) \) is infinite-dimensional. Consider next the case that \( \lambda \in \mathbb{N} \). Then \( S_0(\lambda) = \{2l\} \). Now we must examine whether \( S_1(\lambda) \) contains odd integers less than \( 2l \). In order to have integer solutions for \( j \), (30) implies that \( c - l^2 \) should be a square, or \( c \in \{l^2 + m^2 \mid m \in \mathbb{N} \} \). In that case the solutions to (30) are \( j = l \pm m \). Suppose first that \( l \) is even, and \( c = l^2 + m^2 \). If \( m \) is also even then \( S_1(\lambda) = \emptyset \) and the dimension is given by \( 2l \). If \( m \) is odd we have that \( S_1(\lambda) = \{l + m\} \) or \( S_1(\lambda) = \{l + m, l - m\} \) (if \( m < l \)). So if \( m > l \) then the dimension is \( 2l \) anyway, otherwise the dimension is \( l - m \). Suppose next that \( l \) is odd, and again \( c = l^2 + m^2 \). If \( m \) is odd, \( S_1(\lambda) = \emptyset \) and the dimension is \( 2l \). If \( m \) is even, then, as in the previous case, the dimension is \( 2l \) if \( m > l \) and \( l - m \) is \( m < l \). The conclusion is:

**Theorem 9** Given \( \lambda \in \mathbb{C} \) and \( l = \lambda + 1 \).

(a) If \( \lambda \notin \mathbb{N} \), let \( \Gamma_\lambda = \{2l^2 - 2lm + m^2 \mid m \in (2N + 1)\} \). If \( c \notin \Gamma_\lambda \) then \( \dim L(\lambda) = \infty \); if \( c \in \Gamma_\lambda \) then \( \dim L(\lambda) = m \) (where \( m \) is the unique odd solution to \( c = 2l^2 - 2lm + m^2 \)).

(b) If \( \lambda \in \mathbb{N} \), let \( \Delta_\lambda = \{l^2 + m^2 \mid m \in \mathbb{N} \} \). If \( c \notin \Delta_\lambda \) then \( \dim L(\lambda) = 2l \). If \( c \in \Delta_\lambda \) and \( c \) is even then \( \dim L(\lambda) = 2l \); if \( c \in \Delta_\lambda \) and \( c \) is odd then \( \dim L(\lambda) = 2l \) when \( c \geq 2l^2 \) and \( \dim L(\lambda) = l - \sqrt{c - l^2} \) when \( c < 2l^2 \).

The opposite question is even more interesting: for every given integer \( j \), determine the number of simple modules of dimension \( j \).

Consider first the case that \( j \) is even: \( j \in 2N^* \). If we put \( \lambda = j/2 - 1 \) then \( S_0(\lambda) = \{j\} \). To see whether for this \( \lambda \) also \( \dim L(\lambda) = j \), we have to examine \( S_1(\lambda) \). But according to the previous theorem, the dimension of \( L(\lambda) \) is less than \( j \) only if \( c < 2l^2 \) with \( c = l^2 + m^2 \) odd and \( m \in \mathbb{N} \). It is appropriate to introduce the following set:

\[
I = \{p^2 + q^2 \mid p \in 2\mathbb{N}, \quad q \in 2\mathbb{N} + 1\}.
\] (31)

If \( c \notin I \) then there is one simple module of dimension \( j \), \( j \in 2\mathbb{N}^* \). If \( c \in I \), we define:

\[
K_c = \{(k, m) \mid c = k^2 + m^2 \text{ with } m < k \text{ and } k, m \in \mathbb{N}\}.
\] (32)

If \( c \in I \) then for every even \( j \) there is one module of dimension \( j \), except when \( j \) is of the form \( 2k \) with \( (k, m) \) in \( K_c \): in this case \( L(\lambda) \) has dimension \( k - m < j = 2k \) and there is no
Consider next the case that \( j \) is odd. From the discriminant of (30) one can see that this has two distinct solutions for \( l \) if \( c - j^2/2 \neq 0 \), or if \( c \not\in J \) where

\[
J = \{ k^2/2 \mid k \in 2N + 1 \}.
\]

To see when the solutions for \( l \) are integers, rewrite equation (30) as \( 2c = j^2 + (2l - j)^2 \). So this can have an integer solution for \( l \) only if \( 2c \) is of the form \( u^2 + v^2 \), with both \( u \) and \( v \) odd integers. So, let

\[
I' = \{ u^2 + v^2 \mid u, v \in (2N + 1) \}.
\]

If \( 2c \not\in I' \) then the solutions \( l_1 \) and \( l_2 \) for \( l \) are nonintegers and thus there are two modules of dimension \( j \). If \( 2c \in I' \), write \( 2c = u^2 + v^2 \) with \( u < v \) (the case \( u = v \) implies \( c \in J \) and \( u \) and \( v \) odd. For \( j = u \) the solutions for \( l \) are \( (u + v)/2 \) and \( (u - v)/2 \), and one can verify that both of these yield a simple module of dimension \( j \); for \( j = v \) the solutions for \( l \) are \( l_1 = (u + v)/2 \) and \( l_2 = (v - u)/2 \), and for both of these \( \min S(\lambda) = u < j \) so there are no modules of dimension \( j \). A final step to summarize the situation is given by

**Lemma 10** \( c \in I \) if and only if \( 2c \in I' \).

**Proof.** The proof is rather trivial. If \( c \in I \) then \( c = (2r)^2 + (2s + 1)^2 \) with \( r, s \in \mathbb{N} \). But then \( 2c = (2r + 2s + 1)^2 + (2r - 2s - 1)^2 \), so \( c \in I' \). Conversely, if \( 2c \in I' \), then \( 2c = (2r + 1)^2 + (2s + 1)^2 \) \( (r, s \in \mathbb{N}) \), and then \( c = (r + s + 1)^2 + (r - s)^2 \). Since \( r + s + 1 \) and \( r - s \) have different parity it follows that \( c \in I \).

We can now state the result.

**Theorem 11** Let \( c \in \mathbb{C} \) and \( I, J, K_c \) be defined by (31), (33), (32) respectively.

(a) When \( c \not\in I \) and \( c \not\in J \) then for every even dimension \( j \) there is one simple module \( V(\lambda) \) [determined by \( l = \lambda + 1 = j/2 \)] and for every odd dimension \( j \) there are two simple modules \( V(\lambda_1) \) and \( V(\lambda_2) \) [determined by the solutions \( l_1 \) and \( l_2 \) of \( 2l^2 - 2jl + j^2 - c = 0 \)].

(b) When \( c \in J \), i.e. \( c = k^2/2 \) with \( k \in (2N + 1) \), then for every even dimension \( j \) there is one simple module [determined by \( l = j/2 \)], and for every odd dimension there are two simple modules [again determined by the solutions of (30)] except when \( j = k \) : then there is only one simple module [determined by \( l = k/2 \)].
(c) When \( c \in I \) then for every even dimension \( j \) there is one simple module [determined by \( l = j/2 \)] except for those \( j \) of the form \( j = 2k \) with \( (k, m) \in K_c \); and for every odd dimension \( j \) there are two simple modules [determined by \( (30) \)] except when \( j \) is of the form \( j = k + m \) with \( (k, m) \in K_c \), in which case there are no simple modules.

To illustrate the theorem, consider two examples.

When \( c = 9/2 \), \( R \) has one simple module in every even dimension, two simple modules in every odd dimension different from 3, and one simple module in dimension 3.

When \( c = 1105 \), we see that \( c \in I \) and in fact

\[
K_{1105} = \{(33,4),(32,9),(31,12),(24,23)\}.
\]

Then \( R \) has no modules of dimension 37, 41, 43, 47, 48, 62, 64 and 66. For the remaining even dimensions there is one simple module, and for the remaining odd dimensions there are two simple modules.

All this illustrates that the structure of simple modules of \( R \) is very rich when \( \deg(f) > 1 \).

We conclude this section with a remark. When \( v_{\lambda-j} \) is a primitive vector in \( V(\lambda) \), it follows from the proof of theorem 6 that \( \Omega v_{\lambda-j} = -F(\lambda - j + 1)^2 v_{\lambda-j} \). This implies the relation \( F(\lambda + 1)^2 = F(\lambda - j + 1)^2 \). On the other hand when \( F(\lambda + 1)^2 = F(\lambda - j + 1)^2 \) for some integer \( j \), it does not necessarily imply that \( v_{\lambda-j} \) is a primitive vector. An example is given by \( c = 9/2 \) and \( \lambda = 3/2 \); then \( F(5/2) = F(-5/2) \), but \( v_{\lambda-3/2} \) is not primitive in \( V(5/2) \). In fact, \( V(5/2) \) has no primitive vectors and is simple.

**VI Real and unitary representations**

In the previous sections we considered complex representations, and in this section we shall consider \( f \in \mathbb{R}[x] \) and examine the real simple modules of \( R \). This is actually rather easy: one simply takes the classification of complex simple modules of \( R \), and keeps only those with real matrix elements of \( A, B \) and \( H \). For the modules \( L(\lambda) \), the matrix elements of \( A \) and \( B \) are real anyway (if \( f \in \mathbb{R}[x] \)), so the only remaining condition is that also \( \lambda \) must be real.

To illustrate this, consider again the case \( \deg(f) = 2 \), and suppose \( R \) is now defined by (26) with \( c \in \mathbb{R} \). Then the results of theorem 11 yield also the real representations of \( R \), as long as the roots to be taken are real, in other words (in the following module stands for real module):
(a) When \( c \notin I \) and \( c \notin J \) then for every even dimension \( j \) there is one simple module \( V(\lambda) \) and for every odd dimension \( j < \sqrt{2c} \) there are two simple modules; for odd dimensions \( j > \sqrt{2c} \) there are no simple modules.

(b) When \( c \in J \), i.e. \( c = k^2/2 \) with \( k \in (2\mathbb{N} + 1) \), then for every even dimension \( j \) there is one simple module. For every odd dimension \( j < k \) there are two simple modules; for \( j = k \) there is one simple module; for \( j > k \) there are no simple modules.

(c) When \( c \in I \) then for every even dimension \( j \) there is one simple module except for those \( j \) of the form \( j = 2k \) with \( (k, m) \in K_c \). For every odd dimension \( j < \sqrt{2c} \) there are two simple modules except when \( j \) is of the form \( j = k + m \) with \( (k, m) \in K_c \) (in which case there are no simple modules); for \( j > \sqrt{2c} \) there are no simple modules.

We can consider the same example \( c = 1105 \) as in the previous section. So \( R \) has one simple real module in every even dimension different from 48, 62, 64 and 66. For the odd-dimensional modules, note that \( \sqrt{2c} \approx 47.0106 \). Thus, using also the results of (35), \( R \) has two simple real modules in dimensions 1, 3, 5, \ldots, 33, 35 and also in 39 and 45. In the remaining odd dimensions, \( R \) has no simple real modules.

In order to define “unitary representations” for \( R \), one first considers a Hermitian operation on \( R \). This is an operation \( \ast : R \to R \) which satisfies \((rs)^\ast = s^\ast r^\ast\), \((\lambda r)^\ast = \bar{\lambda} r^\ast\) and \((r^\ast)^\ast = r\), for all \( r, s \in R \) and \( \lambda \in \mathbb{C} \) (with \( \bar{\lambda} \) the complex conjugate). The Hermitian operation introduced here is as follows:

\[
A^\ast = B, \quad B^\ast = A, \quad H^\ast = H. \quad (36)
\]

This induces indeed a Hermitian operation on the whole of \( R \) provided it leaves the defining relations invariant, that is provided \( f \) is a real polynomial.

Such a Hermitian operation \( \ast \) induces a Hermitian form \( \langle \cdot | \cdot \rangle \) on the \( R \) modules \( L(\lambda) \) by:

\[
\langle v_\lambda | v_\lambda \rangle = 1, \quad (37)
\]

\[
\langle rv | sw \rangle = \langle v | r^\ast sw \rangle, \quad r, s \in R, \quad v, w \in L(\lambda). \quad (38)
\]

In particular, such a form is non-degenerate on \( L(\lambda) \) and vectors of different weights are orthogonal with respect to this form. If this form is positive definite, then it yields an inner product on \( L(\lambda) \), and then the representation is called unitary.
Consider a module \( L(\lambda) \) with highest weight vector \( v_\lambda \). In order to see when \( L(\lambda) \) is unitary, we use (20):

\[
\langle v_{\lambda-j} | v_{\lambda-j} \rangle = \left( f(\lambda - j + 1) - f(\lambda - j + 2) + \cdots - (-1)^j f(\lambda) \right) \langle v_{\lambda-j+1} | v_{\lambda-j+1} \rangle.
\]

So, if \( \dim L(\lambda) \geq 1 \), the first condition is \( f(\lambda) > 0 \). If the dimension of \( L(\lambda) \) is larger than 2, the next matrix element yields \( f(\lambda - 1) - f(\lambda) > 0 \). In the most general case one continues like this and requires that all matrix elements in (39) must be positive.

Here we shall again examine the case \( \deg(f) = 2 \), and \( R \) defined by (26) with \( c \) real.

The condition \( f(\lambda - 1) - f(\lambda) > 0 \) reduces to \( \lambda < 0 \), and then automatically the remaining matrix elements in (39) are also positive when \( f(\lambda) > 0 \), since:

\[
\langle v_{\lambda-j} | v_{\lambda-j} \rangle = j(j - 2\lambda - 2) \quad \text{for } j \in 2\mathbb{N},
\]

\[
\langle v_{\lambda-j} | v_{\lambda-j} \rangle = (j - 1)(j - 2\lambda - 1) + f(\lambda) \quad \text{for } j \in 2\mathbb{N} + 1.
\]

Thus, for the degree 2 case there are two types of unitary representations: if \( L(\lambda) \) is 2-dimensional then the only condition is \( f(\lambda) > 0 \); else there are two conditions: \( f(\lambda) > 0 \) and \( \lambda < 0 \).

Consider first the 2-dimensional (real simple) modules. There is at most one such module, namely \( L(0) \). From theorem 9 we deduce that \( L(0) \) is 2-dimensional unless \( c = 1 \). On the other hand, \( f(\lambda) > 0 \) implies \( c < 1 \). The conclusion is as follows: for \( c < 1 \) there is precisely one 2-dimensional unitary simple module of \( R \); the explicit matrix elements are in fact given by:

\[
A = \begin{pmatrix} 0 & 1 - c \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},
\]

with \( \langle v_0 | v_0 \rangle = 1 \) and \( \langle v_{-1} | v_{-1} \rangle = 1 - c \) (note that by going to an orthonormal basis the nonzero matrix elements of \( A \) and \( B \) would become \( \sqrt{1-c} \) and in matrix notation they would indeed be each other’s Hermitian conjugate).

Consider next the remaining modules \( L(\lambda) \) that must satisfy both the conditions \( f(\lambda) > 0 \) and \( \lambda < 0 \). From theorem 9(a) one deduces that there are no such finite-dimensional modules but only infinite-dimensional modules \( L(\lambda) \). Then, by examining the roots of the equation 

\[
2\lambda^2 + 2\lambda + 1 - c = 0,
\]

one finds:

- if \( c < 1/2 \), the unitary simple modules are \( L(\lambda) = V(\lambda) \) with \( \lambda < 0 \);
- if \( 1/2 \leq c \leq 5/8 \), the unitary simple modules are \( L(\lambda) = V(\lambda) \) with either \( \lambda < -1/2 - \sqrt{2c-1} \) or else \( -1/2 + \sqrt{2c-1} < \lambda < 0 \).
• if \( c > 5/8 \) the unitary simple modules are \( L(\lambda) = V(\lambda) \) with \( \lambda < -1/2 - \sqrt{2c-1} \).

Of course, apart from the ones given above (which are all infinite-dimensional), there is also the unique finite-dimensional unitary real simple module \( L(0) \) when \( c < 1 \).

This leads to an interesting remark. For the undeformed paraboson algebra with \( f(x) = x \), one is referred to the representations of \( osp(1/2) \), and thus there are no finite-dimensional unitary representations; the only unitary representations are infinite-dimensional [20]. In the present case, the generalized paraboson algebra related to \( R \) can allow an isolated number of unitary finite-dimensional representations (apart from the infinite-dimensional ones). In the case \( \deg(f) = 2 \) there can be one such isolated example with dimension 2. We have come across other such examples for \( \deg(f) > 2 \), e.g. when \( f(x) = x^3 + x^2 - 2x + 2 \), the module \( L(0) \) is 3-dimensional and it is also a unitary representation with matrix elements given (after normalization) explicitly by

\[
A = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \quad (43)
\]

VII Deformed oscillator algebras

A generalized deformed oscillator algebra is defined in terms of three generators \( a, a^\dagger, N \) subject to the relations

\[
[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad a^\dagger a = \tilde{F}(N), \quad aa^\dagger = \tilde{F}(N + 1). \quad (44)
\]

There is a close relationship with the algebra generated by the same generators but subject to

\[
[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad aa^\dagger + a^\dagger a = \tilde{f}(N), \quad (45)
\]

where \( \tilde{f}(t) = \tilde{F}(t) + \tilde{F}(t + 1) \). From Lemma 3 we know that \( \tilde{f} \) and \( \tilde{F} \) determine each other uniquely. As is well known, the operators \( a, a^\dagger \) and \( N \) can under certain restrictions be realized in terms of ordinary boson operators \( b = (x + ip)/(\sqrt{2}) \) and \( b^\dagger = (x - ip)/(\sqrt{2}) \) as follows

\[
N = b^\dagger b, \quad a = \sqrt{\frac{\tilde{F}(N + 1)}{N + 1}} b, \quad a^\dagger = b^\dagger \sqrt{\frac{\tilde{F}(N + 1)}{N + 1}}. \quad (46)
\]

For the ordinary boson operator with \( [b, b^\dagger] = 1 \) there exists the Fock representation with states \( \psi_n \) (\( n = 0, 1, 2, \ldots \)), where \( b\psi_n = \sqrt{n}\psi_{n-1} \) and \( b^\dagger\psi_n = \sqrt{n+1}\psi_{n+1} \). This becomes a
representation of the deformed algebra if $\tilde{F}(n) > 0$ for $n = 1, 2, \ldots$ – this could be called the Fock representation of the generalized oscillator algebra. Note that its basis vectors are the states of the ordinary Fock space. If we consider
\[ \mathcal{H} = \frac{1}{2}(aa^\dagger + a^\dagger a) = \frac{1}{2} \tilde{f}(N) \] (47)
as the Hamiltonian of a system associated with the generalized oscillator algebra, then its energy spectrum is given by
\[ E_n = \frac{1}{2} \tilde{f}(n), \quad n = 0, 1, 2, \ldots \] (48)
The interesting question is whether there exist physical systems with such an energy spectrum. In fact, in certain cases one can find physically realizable systems using the above boson realization. One just has to substitute in $\tilde{f}(N)$ the boson realization for $N$, namely
\[ N = b^\dagger b = \frac{1}{2}(p^2 + x^2 - 1), \] (49)and see whether $\frac{1}{2} \tilde{f}(N)$ can correspond to the Hamiltonian of any physically realizable system.

For example, let $\tilde{f}(t) = 2(t + 1)^2$. Then
\[ \mathcal{H} = (N + 1)^2 = (p^2 + x^2 + 1)^2/4 = \frac{1}{2} p^2 + V(x, p), \] (50)
with $V(x, p) = \frac{1}{2} x^2 + \frac{1}{4}(x^4 + p^4 + x^2 p^2 + p^2 x^2) + \frac{1}{4}$. (51)
This is a physical system with a potential $V(x, p)$ depending on both the position and the velocity, and its energy eigenvalues are given by $E_n = (n + 1)^2$, $n = 0, 1, \ldots$. Furthermore, the eigenfunctions of this system would be the same as the eigenfunctions of the ordinary harmonic oscillator of unit mass and unit circular frequency. Hence one can distinguish the ordinary harmonic oscillator from the above system not by their eigenfunctions but only by the quantum jumps! Another example of $\tilde{f}(t)$ is available in [2].

To see the connection between the above Fock representation and the unitary modules discussed in the previous section, one can make for the above example the following identification:
\[ A = a, \quad B = a^\dagger, \quad H = -N - 3/2. \] (52)
Then we find the usual structure (1) for $R$ with $f(t) = 2t^2 + 2t + 1/2$. Hence, $f$ is in the standard form (26) with $c = 1/2$. The Fock representation with $N$-lowest weight 0
corresponds to the module $L(\lambda)$ with $H$-highest weight $\lambda = -3/2$. For $c = 1/2$ the module $L(-3/2)$ is indeed unitary. All the other unitary modules, including the 2-dimensional one, will be consistent with the relations $aa^\dagger + a^\dagger a = \tilde{f}(N) = 2(N + 1)^2$ and thus all correspond to unitary representations of (45). For example, the 2-dimensional module (42) would yield (after normalizing):

$$a = \begin{pmatrix} 0 & \sqrt{1/2} \\ 0 & 0 \end{pmatrix}, \quad a^\dagger = \begin{pmatrix} 0 & 0 \\ \sqrt{1/2} & 0 \end{pmatrix}, \quad N = \begin{pmatrix} -3/2 & 0 \\ 0 & -1/2 \end{pmatrix}. \quad (53)$$

These satisfy indeed $[N, a] = -a$, $[N, a^\dagger] = a^\dagger$ and $aa^\dagger + a^\dagger a = 2(N + 1)^2$. But among all the unitary representations, only in $L(-3/2)$ the extra relations $a^\dagger a = \tilde{f}(N)$ and $aa^\dagger = \tilde{f}(N+1)$ are valid. So $L(-3/2)$ is the only unitary representation of (45) that is also a unitary representation of (44). This is the property that makes the Fock representation different from the others.

This difference between (44) and (45) can be considered as the generalization of the difference between ordinary bosons and parabosons. For ordinary bosons there is only the relation $bb^\dagger - b^\dagger b = 1$, and only the Fock space. For parabosons, the relations are

$$[[B^-, B^+], B^\pm] = \pm 2B^\pm, \quad (54)$$

and now all unitary representations of $osp(1/2)$ are consistent with these relations. Among all these, the one corresponding to the Fock space is the only representation where the extra relation $B^- B^+ - B^+ B^- = 1$ is valid. In this sense (44) could be called a generalized boson algebra, and (45) could be called a generalized paraboson algebra.

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