The elliptic genus of Calabi-Yau 3- and 4-folds, product formulae and generalized Kac-Moody algebras.

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ABSTRACT

In this paper the elliptic genus for a general Calabi-Yau fourfold is derived. The recent work of Kawai calculating N=2 heterotic string one-loop threshold corrections with a Wilson line turned on is extended to a similar computation where K3 is replaced by a general Calabi-Yau 3- or 4-fold. In all cases there seems to be a generalized Kac-Moody algebra involved, whose denominator formula appears in the result.
1 Introduction.

In this paper, I extend the work of Kawai [4], calculating N=2 heterotic string one-loop threshold corrections with a Wilson line turned on, to Calabi-Yau three- and fourfolds. (See also [10] for an alternative interpretation of Kawai’s result.) In full generality, this calculation provides a map from a certain class of Jacobi functions (including elliptic genera) to modular functions of certain subgroups of $Sp_4(\mathbb{Q})$, in a product form. In a number of cases, these products turn out to be equal to the denominator formula of a generalized Kac-Moody algebra. It seems natural to assume that this algebra is present in the corresponding string theory, and indeed in [9] it is argued that this algebra is formed by the vertex operators of vector multiplets and hypermultiplets.

2 Elliptic genus.

In this section, I recall some basic facts about elliptic genera for Calabi-Yau manifolds, mostly from [5], and I explicitly derive it for 4-folds. Let $C$ be a complex manifold of complex dimension $d$, with $SU(d)$ holonomy. Then its elliptic genus is a function $\phi(\tau,z)$ with the following transformation properties

$$
\phi \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = e^{\left[ \frac{mcz^2}{c\tau + d} \right]} \phi(\tau,z), \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z})
$$

(1)

$$
\phi(\tau, z + \lambda\tau + \mu) = (-1)^{2m(\lambda + \mu)} e^{-m(\lambda^2 \tau - 2\lambda z)} \phi(\tau, z), \quad \lambda, \mu \in \mathbb{Z}
$$

(2)

where $m = \frac{d}{2}$, and it has an expansion of the form

$$
\phi(\tau,z) = \sum_{n \geq 0, r \in \mathbb{Z} + m} c(n,r) q^n y^r
$$

(3)

I use here the notations $e[x] = e^{2\pi ix}$, $q = e[\tau]$, $y = e[z]$. The coefficients $c(0, -m + p)$ for $0 \leq p \leq c$ have the following geometrical meaning

$$
c(0, -m + p) = \chi_p = \sum_{q=0}^{c} (-1)^{p+q} h^{p,q}
$$

(4)

where $h^{p,q}$ are the Hodge-numbers of $C$. Furthermore

$$
\phi(\tau,0) = \chi
$$

(5)
is the Euler number of $C$. An important feature is that the elliptic genus can be decomposed as

$$
\phi(\tau, z) = \sum_{\mu=-m+1}^{m} h_\mu(\tau) \theta_{m,\mu}(\tau, z)
$$

for functions $h_\mu$ and $\theta_{m,\mu}$ defined by

$$
h_\mu(\tau) = \sum_{N \equiv -\mu^2 (\text{mod} 4m)} c_\mu(N) q^{N/4m}
$$

$$
\theta_{m,\mu}(\tau, z) = \sum_{r \equiv \mu (\text{mod} 2m)} (-1)^{r-\mu} q^{r^2/4m} y^r
$$

Note that the $c_\mu(N)$ are only defined for $-m + 1 \leq \mu \leq m$, but since $\theta_{m,\mu+2m} = (-1)^{2m} \theta_{m,\mu}$, it is useful to define

$$
c_r(N) = (-)^{r-\mu} c_\mu(N)
$$

for all $r \equiv \mu \text{mod} 2m$. The relation between the coefficients of $h_\mu$ and $\phi$ is then given by

$$
c(n, r) = c_r(4mn - r^2)
$$

Finally, the transformation properties of the $h_\mu$ can be derived to be

$$
h_\mu(\tau + 1) = e^{-\frac{\mu^2}{4m}} h_\mu(\tau)
$$

$$
h_\mu(-1/\tau) = \sqrt{i/2m\tau} \sum_{\nu=-m+1}^{m} e^{-\frac{\mu\nu}{2m}} h_\nu(\tau)
$$

Now if $m$ is integer, the elliptic genus satisfies the defining properties of what is called a weak Jacobi form of index $m$ and weight 0. The ring $J_{2*,*}$ of weak Jacobi forms of even weight and all indices is well known \cite{1}. It is a polynomial algebra over $M_*$ (the ring of ordinary modular forms) with two generators

$$
A = \frac{\phi_{10,1}(\tau, z)}{\Delta(\tau)}, \quad B = \frac{\phi_{12,1}(\tau, z)}{\Delta(\tau)}
$$

Here $\Delta(\tau) = \eta^{24}(\tau)$ and $\phi_{10,1}$ and $\phi_{12,1}$ are unique cusp forms of index 1 and weights 10 and 12 respectively. The generators have an expansion

$$
A = y^{-1} - 2 + y + O(q)
$$
\[ B = y^{-1} + 10 + y + O(q) \]

It immediately follows that the space \( J_{0,1} \) is one dimensional with basis \( B \), which implies that the elliptic genus of a Calabi-Yau 2-fold is

\[ \frac{\chi}{12} B \]

So for \( K3 \), with \( \chi = 24 \), it should be \( 2B \), which is indeed the case [7]. The space \( J_{0,2} \) is two dimensional, with basis \( E_4(\tau)A^2 \) and \( B^2 \), \( E_4(\tau) \) being the normalized Eisenstein series of weight 4. So the elliptic genus is fixed by specifying \( \chi_0 \) and \( \chi \), leading to

\[ \chi_0 E_4 A^2 + \frac{\chi}{144} (B^2 - E_4 A^2) \]  

In the case that the manifold has strict \( SU(d) \) holonomy, which implies that \( \chi_0 = 2 \) the following predictions can be done

\[ \chi_1 = 8 - \frac{\chi}{6} \]  
\[ \chi_2 = 12 + \frac{2\chi}{3} \]

so that \( \chi \) should be a multiple of six, and there is a non trivial relation on the Hodge numbers

\[ 4(h^{1,1} + h^{3,1}) + 44 = 2h^{2,1} + h^{2,2} \]

as was recently noticed by Sethi, Vafa and Witten [6]. For a Calabi-Yau 3-fold, the elliptic genus is known to be [5]

\[ \frac{\chi}{2} \left( y^{-\frac{1}{2}} + y^{\frac{1}{2}} \right) \prod_{n=1}^{\infty} \frac{(1 - q^n y^2)(1 - q^n y^{-2})}{(1 - q^n y)(1 - q^n y^{-1})} \]

3 Product formulae.

In this section I look at the following generalization of the formulae in [4]

\[ Z = \sum_{\mu=-m+1}^{m} Z_{m,\mu}(T,U,V,\tau)h_\mu(\tau) \]
where the $h_\mu$ come from a function $\phi$, satisfying the transformation properties (1) and (2), and can be split like (6). For generality, I allow this function to have a pole of finite order $N$ for $\tau \to i\infty$, but nowhere else in the fundamental domain. So the function $\phi$ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{n \geq -N,r \in \mathbb{Z}+m} c(n, r)q^n y^r$$

converging for all $\tau$ with $\tau^2 > 0$ ($\tau^2 = \Im \tau$). The functions $Z_{m,\mu}$ are defined by

$$Z_{m,\mu}(T, U, V, \tau) = \sum_{m_1,m_2,n_1,n_2 \in \mathbb{Z} \mu} \sum_{b \in \mathbb{Z} + \mathbb{Z}} (-1)^{b-\mu} q^{\frac{1}{2}p_L^2} q^{\frac{1}{2}p_R^2}$$

$$\frac{1}{2}p_R^2 = \frac{1}{4Y}|m_1U + m_2 + n_1T + n_2(TU - mV^2) + bV|^2$$

$$\frac{1}{2}(p_L^2 - p_R^2) = \frac{b^2}{4m} - m_1n_1 + m_2n_2$$

$$Y = T_2U_2 - mV^2$$

The function $Z$ is manifestly invariant under the following transformations

$$U \to U + 2\lambda mV + \lambda^2 mT, V \to V + \lambda T + \mu$$

with $\lambda, \mu \in \mathbb{Z}$ if $m \in \mathbb{Z}$, and $\lambda, \mu \in 2\mathbb{Z}$ if $m \in \mathbb{Z} + \frac{1}{2}$. (This has the same effect as the substitutions

$$m_2 \to m_2 - \mu^2 mn_2 + \mu b$$

$$n_1 \to n_1 + \lambda^2 mm_1 - 2\lambda mmn_2 + \lambda b$$

$$b \to b + 2\lambda mm_1 - 2\mu mn_2$$

and these leave the inproduct $\frac{b^2}{4m} - m_1n_1 + m_2n_2$ invariant. In the same way one proves the other invariances.) It is also invariant under the generalization of $SL(2, \mathbb{Z})_T \times SL(2, \mathbb{Z})_U$, generated by

$$T \to T + 1$$

$$T \to -\frac{1}{T}, U \to U - \frac{V^2}{T}, V \to \frac{V}{T}$$

$$U \to U + 1$$
Furthermore, it is invariant under exchange of $T$ and $U$, and under a parity transformation

$$T \leftrightarrow U$$

$$V \rightarrow -V$$

These transformations generate a group isomorphic to $Sp(4, \mathbb{Z})$ if $m = 1$, and to a paramodular subgroup of $Sp(4, \mathbb{Q})$ for $m > 1$. Since $\tau_2 Z$ is invariant under modular transformation of $\tau$, as will be shown later, the following integral is well defined and can be evaluated explicitly by the methods of 

$$I = \int_{\mathcal{F}} \frac{d^2 \tau}{\tau_2} (Z - c(0, 0))$$

The subtraction is to remove the logarithmic singularities due to the massless hypermultiplets, and is needed only if $m$ is integer. If it is not, I define $c(0, 0)$ to be zero. Poisson resummation on $m_1, m_2$ leads to

$$\sum_{m_1, m_2} q^{\frac{1}{2} m_1^2} q^{\frac{1}{2} m_2^2} = \sum_{k_1, k_2} \frac{Y}{U_2 \tau_2} q^{\frac{1}{4} m_1 b^2} \exp G$$

where

$$G = -\frac{\pi Y}{U_2^2 \tau_2} |A|^2 - 2\pi i T(n_1 k_2 + n_2 k_1) + \frac{\pi b}{U_2} (V \tilde{A} - \bar{V} A)$$

$$- \frac{\pi mn_2}{U_2} (V^2 \tilde{A} - \bar{V}^2 A) + \frac{2\pi imV^2}{U_2^2} (n_1 + n_2 \bar{U}) A$$

$$A = -k_1 + n_1 \tau + k_2 U + n_2 \tau U$$

$$\tilde{A} = -k_1 + n_1 \tau + k_2 \bar{U} + n_2 \tau \bar{U}$$

By applying another Poisson resummation on $b$, it is easy to find the following transformation properties of $Z_{\mu,m}$

$$Z_{\mu,m}(-1/\tau) = \sqrt{\tau/2mi} \sum_{\nu=-m}^{m} e^{\frac{\mu \nu}{2m}} Z_{\nu,m}(\tau)$$
which together with the known properties (12) of the $h_\mu$ prove the modular invariance of $\tau_2 Z$. Following [8, 9] a bit further I find

$$I_0 = \frac{Y}{U_2} \int \frac{d^2 \tau}{\tau_2^2} \phi(\tau, 0) = \frac{\pi}{3} E_2(\tau) \phi(\tau, 0)_{q^0}$$

(42)

$$I_d = \sum_{b \in \mathbb{Z}+m} 2\pi c(0, b) \left[ b^2 \frac{V_2^2}{U_2} - |b| V_2 + \frac{U_2}{6} \right] - c(0, 0) \ln(kY)$$

(43)

- $\ln \prod_{(l>0, b \in \mathbb{Z}+m), (l=0, 0 < b \in \mathbb{Z}+m)} |1 - e[lU + bV]|^{4c(0, b)}$

(This under the assumption that $0 \leq V_2 \leq U_2/|b|$ for all $b$ with $c(0, b) \neq 0$).

Here

$$k = \frac{8\pi}{3\sqrt{3}} e^{1-\gamma}$$

(44)

$$I_{nd} = -\ln \prod_{k>0, l \in \mathbb{Z}, b \in \mathbb{Z}+m} |1 - e[kT + lU + bV]|^{4c(kl, b)}$$

(45)

(This for $T_2$ large enough). Putting this all together, I obtain

$$I = -2 \ln(kY)^{\frac{1}{2}c(0, 0)} \left| e[pT + qU + rV] \prod_{(k, l, b) > 0} (1 - e[kT + lU + bV])^{c(kl, b)} \right|^2$$

(46)

where the coefficients $p, q, r$ are given by

$$p = \sum_{b \in \mathbb{Z}+m} \frac{b^2}{4m} c(0, b)$$

(47)

$$q = \sum_{b \in \mathbb{Z}+m} \frac{1}{24} c(0, b)$$

(48)

$$r = \sum_{b \in \mathbb{Z}+m} -\frac{|b|}{4} c(0, b)$$

(49)

and the summation condition means $k > 0$ or $k = 0, l > 0$ or $k = l = 0, b > 0$ (always with $k, l \in \mathbb{Z}$ and $b \in \mathbb{Z} + m$). Applying these formulae to $2B$, the elliptic genus of $K3$, I recover the result of Kawai [4]. Now consider the elliptic genus of a Calabi-Yau fourfold,

$$\phi = \chi_0 E_4 A^2 + \frac{\chi}{144} (B^2 - E_4 A^2)$$

(50)
Amazingly, the $\chi$-dependent part equals the coefficients of Gritsenko and Nikulin’s second product formula [3], which is known to be associated to the generalized Kac-Moody algebra which is an automorphic form correction to the Kac-Moody algebra defined by the symmetrized generalized Cartan matrix

$$G_2 = \begin{pmatrix} 4 & -4 & -12 & -4 \\ -4 & 4 & -4 & -12 \\ -12 & -4 & 4 & -4 \\ -4 & -12 & -4 & 4 \end{pmatrix}$$  \hspace{1cm} (51)

Unfortunately, there is no such formula for the $\chi_0$-dependent part. So for a Calabi-Yau fourfold I find

$$I = -\chi_0 \ln((kY)^6|\Pi_6(\Omega)|^2) - \frac{\chi}{3} \ln((kY)^2|F_2(\Omega)|^2)$$  \hspace{1cm} (52)

where $F_2$ is Gritsenko and Nikulin’s product and $\Pi_6$ is

$$e^{[2V]} \prod_{(k,l,b) > 0} (1 - e^{[kT + lU + bV]})^c^{(kl,b)}$$  \hspace{1cm} (53)

of weight 6, with coefficients $c$ coming from $2E_4 A^2$. The following section describes the product formula for a Calabi-Yau 3-fold.

4 Calabi-Yau 3-folds.

In this section I apply my formulae to equation (21), without the factor $\chi/2$. Expanding this in $q$ gives

$$(y^{-\frac{1}{2}} + y^{\frac{1}{2}}) + O(q)$$  \hspace{1cm} (54)

so that $c(0, -\frac{1}{2}) = c(0, \frac{1}{2}) = 1$, and the corresponding product formula reads

$$F_0(T, U, V) = p^{\frac{1}{12}} q^{\frac{1}{12}} y^{-\frac{1}{2}} \prod_{(k,l,b) > 0} (1 - p^k q^l y^b)^c^{(kl,b)}$$  \hspace{1cm} (55)

of weight zero, where now $p = e[T]$, $q = e[U]$, $y = e[V]$. In the limit $V \to 0$, this product behaves like

$$V \eta^2(p) \eta^2(q)$$  \hspace{1cm} (56)
as can be expected for $\chi = 2$. This product can be expanded in terms of $p$ (since it is valid for $T_2$ large enough). It turns out to be useful to consider $F_0(T, U, 2V)$. Thus

$$F_0(T, U, 2V) = \sum_{k \in \mathbb{Z} \geq 0 + \frac{1}{12}} \phi_k(q, y)p^k$$

(57)

This is a variant of what is known as a Fourier-Jacobi expansion. The transformation properties of $F_0(T, U, V)$ imply that the coefficients $\phi_m$ should be Jacobi forms of weight 0 and index $6k$, with a possible multiplier system. From the product formula it is possible to read of the lowest order coefficient

$$\phi_{\frac{1}{12}}(q, y) = q^{\frac{1}{12}}(y^{-\frac{1}{2}} - y^{\frac{1}{2}}) \prod_{n>0} (1 - q^n y)(1 - q^n y^{-1}) = \theta_{11}(q, y)\eta^{-1}(q)$$

(58)

by the product formula for theta-functions. This is indeed a Jacobi cusp form of weight 0 and index $\frac{1}{2}$ with multiplier system [2], which can serve as a consistency check. It can be written as a sum as follows

$$\left(\sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{(2n+1)^2}{8}} y^{\frac{(2n+1)}{4}}\right) \left(\sum_{n \geq 0} p(n) q^{n-\frac{1}{24}}\right)$$

(59)

where $p(n)$ is the partition function. Now unlike the case of $F_2(\Omega)$ from [3], it doesn’t seem to be possible to write the entire product as a lifting of its first Fourier-Jacobi coefficient. It does seem to be likely that this function is also related to some generalized Kac-Moody algebra. This is under investigation. The final result for the Calabi-Yau 3-fold calculation is

$$\mathcal{I} = -\chi \ln |F_0(\Omega)|^2$$

(60)

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References


