Lattice Background Effective Action: a Proposal
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We propose a method based on the Schrödinger functional for computing on the lattice the gauge invariant effective action for external background fields. We check this method by studying the U(1) lattice gauge theory in presence of a constant magnetic background field.

1. INTRODUCTION

We propose a method, based on the Euclidean Schrödinger functional \cite{1}, to evaluate on the lattice the gauge invariant effective action. The Euclidean Schrödinger functional in Yang-Mills theories without matter fields is:

\[ Z[A^{(f)}, A^{(i)}] = \langle A^{(f)} | \exp(-HT) \mathcal{P} | A^{(i)} \rangle \]  \hspace{1cm} (1)

where \( \mathcal{P} \) is the operator that projects onto the physical states and \( H \) the pure gauge Yang-Mills Hamiltonian in the temporal gauge \cite{2}.

The Schrödinger functional on the lattice becomes \cite{3,4}:

\[ Z[U^{(f)}, U^{(i)}] = \int D\mathcal{U} \exp(-S) \]  \hspace{1cm} (2)

with \( S \) the Wilson action modified to take into account the boundaries at \( x_4 = 0 \), and \( x_4 = T \):

\[ U(x)|_{x_4=0} = U^{(i)}, \quad U(x)|_{x_4=T} = U^{(f)}. \]  \hspace{1cm} (3)

The Schrödinger functional is invariant under arbitrary lattice gauge transformations of the boundary links.

2. LATTICE EFFECTIVE ACTION

We want to investigate the lattice effective action for external background fields.

In our approach \cite{5} we use periodic boundary conditions also in the time direction:

\[ U(x)|_{x_4=0} = U(x)|_{x_4=T} = U^{\text{ext}}(0, \vec{x}), \]  \hspace{1cm} (4)

so that \( S \) is the standard Wilson action \( S_W \).

On the lattice (\( \mathcal{P} \) is the path-ordering operator):

\[ U^{\text{ext}}(x) = \mathcal{P} \exp \left\{ +i a g \int_0^1 dt A^{\text{ext}}_\mu(x + a \hat{t}) \right\}, \]  \hspace{1cm} (5)

with the continuum gauge field (\( \lambda_a/2 \) generators of the SU(N) Lie algebra):

\[ \vec{A}^{\text{ext}}(\vec{x}) = \vec{A}^{\text{ext}}(x) \lambda_a/2. \]  \hspace{1cm} (6)

The lattice effective action for the background field \( \vec{A}^{\text{ext}}(\vec{x}) \) is defined by means of the lattice Schrödinger functional Eq. (2)

\[ \Gamma[\vec{A}^{\text{ext}}] = -\frac{1}{T} \ln \left\{ \frac{Z[U^{\text{ext}}]}{Z(0)} \right\}, \]  \hspace{1cm} (7)

where \( T \) is the extension in Euclidean time and

\[ Z[U^{\text{ext}}] = Z[U^\text{ext}, U^{\text{ext}}] = \int D\mathcal{U} \exp(-S_W). \]  \hspace{1cm} (8)

\( Z(0) \) is the lattice Schrödinger functional without external background field (i.e. with \( U^{\mu}_{\text{ext}} = 1 \)). In the continuum, where \( T \to \infty, \Gamma[\vec{A}^{\text{ext}}] \) becomes the vacuum energy in presence of the background field \( \vec{A}^{\text{ext}}(\vec{x}) \).

Our effective action is by definition gauge invariant and can be used for a non-perturbative investigation of the properties of the quantum vacuum.

3. U(1) IN A CONSTANT BACKGROUND FIELD

As a first step we check the consistency of our proposal by analyzing the well known U(1) lattice gauge theory.
We consider background fields that give rise to constant field strength. In this case $\Gamma[A_{\text{ext}}]$ is proportional to the spatial volume $V$ and the relevant quantity is the density of the effective action:

$$\varepsilon[A_{\text{ext}}] = -\frac{1}{\Omega} \ln \left[ \frac{Z[U_{\text{ext}}]}{Z(0)} \right], \quad \Omega = V \cdot T. \quad (9)$$

We study the U(1) l.g.t. in a constant background magnetic field directed along the $x_3$ direction. In the Landau gauge:

$$A_{\text{ext}}^\mu(x) = \delta_{\mu,2} x_1 B. \quad (10)$$

On the lattice:

$$U_2^{\text{ext}}(x) = \exp [i a g B x_1] , \quad U_1^{\text{ext}}(x) = U_3^{\text{ext}}(x) = U_4^{\text{ext}}(x) = 1. \quad (11)$$

Since we adopt periodic boundary conditions the external magnetic field gets quantized:

$$a^2 g B = \frac{2\pi}{L_1} n^{\text{ext}} \quad (12)$$

with $n^{\text{ext}}$ integer and $L_1$ the lattice extension in the $x_1$ direction (in lattice units).

We perform numerical simulations of U(1) l.g.t. with the standard Wilson action. The links belonging to the time slice $x_4 = 0$ are frozen to the configuration (11). We also impose that the constraint (11) applies to links at the spatial boundaries (in the continuum this condition amounts to the usual requirement that the fluctuations over the background field vanish at the infinity).

We first analyze the behaviour of the magnetic field. To this end we look at the field strength tensor measured at a given time slice:

$$F_{\mu\nu}(x_4) = \sqrt{\beta} \left\langle \frac{1}{V} \sum_{\vec{x}} \sin \theta_{\mu\nu}(\vec{x}, x_4) \right\rangle. \quad (13)$$

Only the component $F_{12}$ of the field strength tensor is present in our data. Moreover, in agreement with previous studies [6], we find that in the confined region $\beta < 1$ the external magnetic field is shielded after a small penetration, while in the Coulomb region $\beta > 1$ the field penetrates indicating that the gauge system supports a long range magnetic field.

Let us turn now to the evaluation of the density of the effective action Eq. (9). In this case we are faced with the problem of computing a partition function. To overcome this problem we consider the derivative of $\varepsilon[A_{\text{ext}}]$ with respect to $\beta$:

$$\varepsilon'[A_{\text{ext}}] = \frac{\partial \varepsilon[A_{\text{ext}}]}{\partial \beta} = \left\langle \frac{1}{\Omega} \sum_{x,\mu>\nu} \cos \theta_{\mu\nu}(x) \right\rangle_0 - \left\langle \frac{1}{\Omega} \sum_{x,\mu>\nu} \cos \theta_{\mu\nu}(x) \right\rangle_{A_{\text{ext}}} \quad (14)$$

The density of the effective action can be recovered by integrating $\varepsilon'$ over $\beta$. Note that the contributions to $\varepsilon'[A_{\text{ext}}]$ due to the frozen time slice at $x_4 = 0$ and to the fixed boundary conditions at the lattice spatial boundaries must be subtracted, i.e. only the dynamical links must be taken into account in evaluating $\varepsilon'[A_{\text{ext}}]$. We denote by $\Omega = L_1 L_2 L_3 L_4$ the total number of lattice sites (i.e. the lattice volume). $\Omega_{\text{int}}$ are the lattice sites whose links are fixed according to Eq. (4):

$$\Omega_{\text{ext}} = L_1 L_2 L_3 + (L_4 - 1) \times (L_1 L_2 L_3 - (L_1 - 2)(L_2 - 2)(L_3 - 2)). \quad (15)$$

Hence $\Omega_{\text{int}} = \Omega - \Omega_{\text{ext}}$ is the volume occupied by the dynamical lattice sites. In Figure 1 we display the derivative of the energy density due to “internal” links versus $\beta$ for the $64 \times 12^3$ lattice and $n^{\text{ext}} = 2$.

Figure 1 shows that in the weak coupling region ($\beta \gg 1$) $\varepsilon'[A_{\text{ext}}]$ tends to the derivative of the external...
action (action due to the external links):
\[ \epsilon_\text{ext} = \frac{1}{\beta} \frac{\partial}{\partial \beta} \Omega^\text{ext} = 1 - \cos \left( \frac{2\pi}{L_1 n_\text{ext}} \right). \] (16)

This means that for large \( \beta \) the effective action agrees with the classical action:
\[ \lim_{\beta \to \infty} \epsilon[\vec{A}^\text{ext}] = \beta \left[ 1 - \cos \left( \frac{2\pi}{L_1 n_\text{ext}} \right) \right] = \epsilon_\text{ext}[\vec{A}^\text{ext}], \] (17)
so that in the continuum limit \( a \to 0 \) and \( B \) fixed we get the classical energy density \( B^2/2 \).

4. U(1) FINITE SIZE SCALING

To further convince ourselves of the consistency of our evaluation of the lattice effective action, we also determined the critical parameters of U(1) l.g.t. by applying the standard finite size scaling analysis [7]. To this purpose we simulated U(1) l.g.t. on lattices with sizes \( L_1 = 64 \), \( L_2 = L_3 = L_4 \equiv L = 6, 8, 10, 12, 14 \) and \( n_\text{ext} = 2 \). In Figure 2 we display the derivative of the energy density near the critical region for different lattice sizes \( (n_\text{ext} = 2) \). The effective linear dimension

\[ L_{\text{eff}} = (\Omega_\text{int}(L_1, L_2, L_3, L_4))^{1/4}. \] (18)

We apply the f.s.s. to our generalized susceptibility \( \epsilon '[\vec{A}^\text{ext}] \). Near the critical region (see Fig. 2):
\[ \epsilon '[\vec{A}^\text{ext}] = \frac{a_1(L_{\text{eff}})}{a_2(L_{\text{eff}})[\beta - \beta^*(L_{\text{eff}})]^2 + 1} \] (19)

According to f.s.s. the peak of the derivative of the U(1) energy density \( (a_1(L_{\text{eff}}) \) in Eq. (19) should behave as
\[ a_1(L_{\text{eff}}) = c L_{\text{eff}}^{-\gamma/\nu}. \] (20)

In Fig. 3 the peak of \( \epsilon '[\vec{A}^\text{ext}] \) vs. lattice size with superimposed fit Eq. (20) is displayed. Analogously the pseudocritical couplings at various lattice sizes are fitted according to (see Fig. 4):
\[ \beta^*(L_{\text{eff}}) = \beta_c + k L_{\text{eff}}^{-1/\nu}. \] (21)

Note that the values of the critical exponents are quite consistent with the hyperscaling relation:
\[ \gamma = 2 - \nu d \implies \frac{1}{\nu} = d \frac{1}{2} + \frac{1}{2 \nu}. \] (22)

Indeed, using the value of \( \gamma/\nu = 0.84(14) \) obtained from the fit to the value of the peak of \( \epsilon '[\vec{A}^\text{ext}] \) and the
hyperscaling relation we get $1/\nu = 2.42(7)$, which is compatible with $1/\nu = 3.48(65)$ obtained from the fit Eq. (21) to the pseudocritical couplings.

We also checked that the universality law in the critical region

$$L_{\text{eff}}^{-\gamma/\nu} \varepsilon_{\text{int}} \left[ A_{\text{ext}}^{\nu} \right] = \tilde{\phi} \left[ L_{\text{eff}}^{1/\nu} (\beta - \beta_c) \right]$$

holds quite well for the lattices with $L = 10, 12, 14$ corresponding to $L_{\text{eff}} = 13.75, 16.16, 18.46$ respectively (see Fig. 5).

5. CONCLUSIONS

We have presented a method that allows to investigate the effective action for external background fields in gauge systems by means of Monte Carlo simulations. We have successfully tested this method for the U(1) pure lattice gauge theory in an external magnetic field. In particular we found that the external magnetic field is screened for strong couplings while penetrates for color weak couplings. We have also verified that in the continuum limit the effective action agrees with the classical U(1) action. Moreover our estimations of the critical parameters and of the infinite volume critical coupling are in perfect agreement with the values extracted from the specific heat on lattices with closed topology or with fixed boundary conditions, where there is evidence of a continuous phase transition [8–10].

We used this definition of lattice effective action to analyze the SU(2) lattice gauge theory in presence of an external magnetic abelian field, finding evidence for the so-called Nielsen-Olesen unstable modes [11].

REFERENCES

11. P. Cea and L. Cosmai, these Proceedings.