Abstract

We present an exact solution to the problem of the relativistic motion of 2 point masses in (1 + 1) dimensional dilaton gravity. The motion of the bodies is governed entirely by their mutual gravitational influence, and the spacetime metric is likewise fully determined by their stress-energy. A Newtonian limit exists, and there is a static gravitational potential. Our solution gives the exact Hamiltonian to infinite order in the gravitational coupling constant.
The problem of motion is a notoriously difficult one in gravitational theory. Although approximation techniques exist [1], in general there is no exact solution to the problem of the motion of \( N \) bodies each interacting under their mutual gravitational influence, except in the case \( N = 2 \) for Newtonian gravity, or in \((2 + 1)\) dimensions, where the absence of a static gravitational potential allows one to generalize the static 2-body metric to that of two bodies moving with any speed [2].

We present here an exact solution to problem of the relativistic motion of 2 point masses under gravity in \((1 + 1)\) dimensions. The dimensionality necessitates that the gravitational theory we choose is such that the dilaton decouples from the classical equations of motion [3]. Consequently the motion of the bodies is governed entirely by their mutual gravitational influence, and the spacetime metric is likewise fully determined by their stress-energy [4]. Unlike the \((2 + 1)\) dimensional case, a Newtonian limit exists, and there is a static gravitational potential. Our solution gives the exact Hamiltonian to infinite order in the gravitational coupling constant. We can thus view the whole structure of the theory from the weak field to the strong field limits.

We shall work in the canonical formalism for which the action (1) is written in the form[5]

\[
I = \int d^2x \left\{ \frac{1}{2\kappa} \sqrt{-g} \left\{ \Psi R + \frac{1}{2} g^{\mu\nu} \nabla_\mu \Psi \nabla_\nu \Psi \right\} 
- \sum_{a=1}^{N} m_a \int d\tau_a \left\{ -g_{\mu\nu}(x) \frac{dz_a^\mu}{d\tau_a} \frac{dz_a^\nu}{d\tau_a} \right\}^{1/2} \delta^2(x - z_a(\tau_a)) \right\}
\]

where \( \Psi \) is the dilaton field and \( g_{\mu\nu}, g \) are the metric and its determinant, \( R \) is the Ricci scalar and \( \tau_a \) is the proper time of \( a \)-th particle, with \( \kappa = 8\pi G/c^4 \). It is straightforward to show that the system of field equations reduces to

\[
R = \kappa T^\mu_\mu = \frac{d}{d\tau_a} \left\{ \frac{dz_a^\mu}{d\tau_a} \right\} - \Gamma^\mu_{\alpha\beta}(z_a) \frac{dz_a^\alpha}{d\tau_a} \frac{dz_a^\beta}{d\tau_a} = 0 . \quad (2)
\]

and

\[
\frac{1}{2} \nabla_\mu \Psi \nabla_\nu \Psi - g_{\mu\nu} \left( \frac{1}{4} \nabla^\lambda \Psi \nabla_\lambda \Psi - \nabla^2 \Psi \right) - \nabla_\mu \nabla_\nu \Psi = \kappa T_{\mu\nu} \quad (3)
\]

where the stress-energy due to the point masses is

\[
T_{\mu\nu} = \sum_{a=1}^{N} m_a \int d\tau_a \frac{1}{\sqrt{-g}} g_{\mu\rho} g_{\nu\sigma} \frac{dz_a^\rho}{d\tau_a} \frac{dz_a^\sigma}{d\tau_a} \delta^2(x - z_a(\tau_a)) , \quad (4)
\]

and is conserved. Note that (2) is a closed system of \( N + 1 \) equations for which one can solve for the single metric degree of freedom and the \( N \) degrees of freedom of the point masses. The evolution of the dilaton field is governed by the evolution of the point-masses via (3). It is easy to show that the left-hand side of this equation is divergenceless (consistent with the conservation of \( T_{\mu\nu} \)), yielding only one independent equation to determine the single degree of freedom of the dilaton.

We shall work in the canonical formalism for which the action (1) is written in the form[5]

\[
I = \int d^2x \left\{ \sum_{a} p_a \dot{z}_a \delta(x - z_a(x^0)) + \pi \dot{\gamma} + \Pi \dot{\Psi} + N_0 R^0 + N_1 R^1 \right\}
\]

where \( \gamma = g_{11}, N_0 = (-\theta^{00})^{-1/2}, N_1 = g_{10}, \pi \) and \( \Pi \) are conjugate momenta to \( \gamma \) and \( \Psi \) respectively, and

\[
R^0 = -\kappa \sqrt{\gamma} \gamma^{\pi^2} + 2\kappa \sqrt{\gamma} \Pi + \frac{1}{4\kappa \sqrt{\gamma}} (\Psi')^2 - \frac{1}{\kappa} \left( \frac{\Psi'}{\sqrt{\gamma}} \right)' - \sum_a \sqrt{\frac{p_a^2}{\gamma} + m_a^2} \delta(x - z_a(x^0)) \]

\[
R^1 = \frac{\gamma'}{\gamma} - \frac{\gamma}{\gamma} \Pi' \Psi + 2 \pi' + \sum_a \frac{p_a}{\gamma} \delta(x - z_a(x^0)) \]

(6)

(7)

with the symbols (\( ' \)) and (\( ' \)) denoting \( \partial_0 \) and \( \partial_1 \), respectively.

1
The quantities \( N_0 \) and \( N_1 \) are Lagrange multipliers which enforce the constraints \( R^0 = 0 = R^1 \). Since the only linear terms in these constraints are \( (\Psi'/\sqrt{\gamma'})^{\prime} \) and \( \pi' \), we may solve for these quantities in terms of the dynamical and gauge (i.e. co-ordinate) degrees of freedom. These latter degrees of freedom may be identified by writing the generator (which arises from the variation of the action at the boundaries) in terms of the former quantities, and then finding which quantities serve to fix the frame of the physical space-time coordinates in a manner analogous to the \((3 + 1)\)-dimensional case \([6, 7, 8]\).

Carrying out this procedure, we find that we can consistently choose the coordinate conditions \( \gamma = 1 \) and \( \Pi = 0 \). Eliminating the constraints, the action (5) then reduces to

\[
I = \int d^2x \left\{ \sum_a p_a \dot{z}_a \delta(x - z_a) - \mathcal{H} \right\} ,
\]

where the reduced Hamiltonian for the system of particles is \( H = \int dx \mathcal{H} = -\frac{\pi}{\kappa} \int dx \Delta \Psi \), where \( \Delta \equiv \partial^2/\partial x^2 \).

Here \( \Psi = \Psi(x, z_a, p_a) \) and is determined by solving the constraint equations which are now

\[
\triangle \Psi - \frac{1}{4} (\Psi')^2 + \kappa^2 \pi^2 + \kappa \sum_a \sqrt{p_a^2 + m_a^2} \delta(x - z_a) = 0 ,
\]

\[
2\pi' + \sum_a p_a \delta(x - z_a) = 0 .
\]

An exact solution to these equations in the 2-body case may now be obtained as follows. Consider first the case \( z_2 < z_1 \), for which we may divide spacetime into three regions: \( z_1 < x \) (\((+\) region)), \( z_2 < x < z_1 \) (\((0)\) region) and \( x < z_2 \) (\((-\)) region). Writing \( \Psi = -4\log|\phi| \) and \( \pi = \chi' \) we find in each region that \( \phi \) is the sum of growing and decaying exponentials in \( x \) and that \( \chi \) is a sum of terms linear in \( x \). Matching these solutions at the boundaries \( x = z_1 \) and \( x = z_2 \) of each region allows a determination of the coefficients of the exponentials in the + and - regions in terms of those in the 0 region.

Because the magnitudes of both \( \phi \) and \( \chi \) increase with increasing \( |x| \), it is necessary to impose a boundary condition which guarantees that the surface terms obtained in passing from (1) to (5) vanish. This condition has been shown to be \( \Psi^2 - 4\kappa^2 \chi^2 = 0 \), which must hold in the + and - regions \([5]\). Incorporating this into the matching conditions allows one to fully determine the coefficients of the exponentials in the 0 region in terms of the momenta and positions of the point masses.

Solving the constraints and the equations for \( N_0 \) and \( N_1 \) yields

\[
N_0 = A e^{-\frac{1}{4} \Psi} = A \phi^2 = \begin{cases} A \phi_+^2 & \text{(+)} \text{region} \\ A \phi_0^2 & \text{(0)} \text{region} \\ A \phi_-^2 & \text{(-)} \text{region} \end{cases}
\]

and

\[
N_{1(+)} = \epsilon (A \phi_+^2 - 1) \quad N_{1(-)} = -\epsilon (A \phi_-^2 - 1)
\]

\[
N_{1(0)} = \epsilon A \frac{L_1 L_2}{L_0^2} \left\{ \frac{L_2}{M_1} e^{\frac{\pi}{4} L_0 (x - z_1)} - \frac{L_1}{M_2} e^{\frac{-\pi}{4} L_0 (x - z_2)} \right\} + \frac{\kappa \epsilon}{2} A \frac{L_1 L_2}{L_0} x + C_0
\]

where \( A = L_0/(L_1 + L_2 - \frac{\pi^2}{4}(z_1 - z_2)L_1 L_2) \), \( C_0 = A(M_1 - M_2 - \frac{\pi}{4}(z_1 + z_2)L_1 L_2)/L_0 \), \( \epsilon^2 = 1 \), and

\[
\phi_+ = \left( \frac{L_1}{M_1} \right)^{1/2} e^{\frac{\pi}{4} L_0 (x - z_1)} \quad \phi_- = \left( \frac{L_2}{M_2} \right)^{1/2} e^{-\frac{\pi}{4} L_0 (x - z_2)}
\]

\[
\phi_0 = \left( \frac{L_1 L_2}{L_0} \right)^{1/2} \left[ \left( \frac{L_2}{M_1} \right)^{1/2} e^{\frac{\pi}{4} L_0 (x - z_1)} + \left( \frac{L_1}{M_2} \right)^{1/2} e^{-\frac{\pi}{4} L_0 (x - z_2)} \right]
\]

as the solutions for the relevant field variables. For convenience we have defined the quantities

\[
M_a \equiv \sqrt{p_a^2 + m_a^2 + \epsilon \eta_{ab} p_b} \quad L_\pm = 4X \pm \epsilon(p_1 + p_2) \quad L_1 \equiv L_0 - M_2 \quad L_2 \equiv L_0 - M_1
\]
where $\eta_{11} = \eta_{22} = 1, \eta_{12} = \eta_{21} = 0$ and $L_0 = 4X - \epsilon(p_1 - p_2)$. These quantities are related by the equation
\[
L_1 L_2 = M_1 M_2 e^{\frac{\kappa}{2} L_0(z_1 - z_2)}
\]  
which determines $X$.

The parameter $\epsilon$ is a constant of integration associated with the metric degree of freedom. We have two types of solutions corresponding to $\epsilon = \pm 1$. Under time reversal, these solutions transform into each other, ensuring invariance of the whole theory under this symmetry. It is straightforward to show from this solution that the Ricci scalar vanishes everywhere except at the locations $(z_1(t), z_2(t))$ of the point masses.

A calculation shows that the Hamiltonian of the system is $H = 4X$. Consequently (18) gives an exact solution for the Hamiltonian $H$ in terms of the co-ordinate and metric degrees of freedom. It is straightforward to show that the total momenta $p_1 + p_2$ is conserved, and so we may choose a frame of reference with $p_1 = -p_2 = p$. Hamilton’s equations then yield
\[
\begin{align*}
\dot{\rho} &= -\frac{\kappa}{4} A L_1 L_2 \\
\dot{z}_1 &= \epsilon - \epsilon A \frac{L_1}{\sqrt{p^2 + m_1^2}} \\
\dot{z}_2 &= -\epsilon + \epsilon A \frac{L_2}{\sqrt{p^2 + m_2^2}}
\end{align*}
\]
as the dynamical equations for the 2-body system coupled to gravity.

Repeating this analysis for $z_1 < z_2$ yields the general equation for the Hamiltonian
\[
\begin{align*}
(r &> z_1 - z_2) \\
&= (\sqrt{p_1^2 + m_1^2} - \epsilon \tilde{p}_2 - H)(\sqrt{p_2^2 + m_2^2} + \epsilon \tilde{p}_1 - H) \\
&= (\sqrt{p_1^2 + m_1^2} - \epsilon \tilde{p}_2)(\sqrt{p_2^2 + m_2^2} + \epsilon \tilde{p}_1) e^{\frac{2}{\kappa} (H - \epsilon (\tilde{p}_1 + \tilde{p}_2))} |r|
\end{align*}
\]  
where $r = z_1 - z_2$ and $\tilde{p}_0 = p_0 \text{sgn}(z_1 - z_2)$.

Equation (22) describes the surface in $(r, p, H)$ space of all allowed phase-space trajectories. Since $H$ is a constant of the motion, (a fact easily verified by differentiation of (22) with respect to $t$) a given trajectory in the $(r, p)$ plane is uniquely determined by setting $H = H_0$ in (22). For $H_0$ sufficiently small, the trajectory can be regarded as a relativistic perturbation of the Newtonian case. However once $H_0$ is sufficiently large there exist a qualitatively new set of trajectories which cannot be understood as relativistically ‘correcting’ the Newtonian ones. The transition from the first set to the second set is smooth.

In the equal-mass case, the Hamiltonian is
\[
H = -8 \frac{W \left( \left( \frac{-\kappa}{8} \sqrt{p^2 + m^2} - \epsilon p r \right) \exp \left[ \frac{\kappa}{8} \left( \frac{1}{r} \right) \sqrt{p^2 + m^2} - \epsilon p r \right] \right) \right) + \sqrt{p^2 + m^2} + \epsilon p \text{sgn}(r)}{\kappa |r|} 
\]
where $W(x)$ is the Lambert W-function defined via
\[
y \cdot e^y = x \quad \implies \quad y = W(x)
\]
and has two real branches [9] which join smoothly onto each other. The principal branch (for which $W(x) \geq -1$) in (23) reduces to the Newtonian limit for small $\kappa$; the second set of trajectories mentioned above are described by the other branch.

Some characteristic phase-space plots for the equal mass case are given in figs. 1–2. In each of these we have included the corresponding trajectory from the Newtonian theory for comparison. We see that as $H_0$ increases, the trajectory becomes more S-shaped, with the particles reaching their maximum separation for some positive value of $p$, where the velocity $\dot{r} = 0$. This occurs because $p$ depends upon the momenta of the particles and of the metric. Under time-reversal, the trajectory for a given value of $H_0$ is obtained by reflection in the $r = 0$ axis. Fig. 3 shows three phase space plots for the unequal mass case, with the mass of particle 2 being 1/10, equal to and twice the mass of particle 1.
An interesting limiting case of (22) may be obtained by setting $z_1 = z, z_2 = 0, m_1 = \mu, m_2 = m, p_1 = p$ and $p_2 = 0$. Taking the lowest order terms of $\mu$, we obtain

$$H = m + \sqrt{p^2 + \mu^2 e^{\frac{4\pi}{m} |z|}} - \epsilon p \text{sgn}(z) \left( e^{\frac{4\pi}{m} |z|} - 1 \right).$$

which is just the Hamiltonian of a test particle of mass $\mu$ under the influence of a static source at the origin of mass $m$, in which the metric is $N_0 = e^{\frac{4\pi}{m} |z|}, N_1 = \epsilon \text{sgn}(z) \left( e^{\frac{4\pi}{m} |z|} - 1 \right)$ [5].

We hope to extend our analysis of this problem to the many-body problem and to more general theories of dilaton gravity.

Acknowledgements

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References

[1] see the article by T. Damour in Three Hundred Years of Gravitation eds. S.W. Hawking and W. Israel (Cambridge University Press, 1987)


Figure Captions

Fig. 1  A comparison of relativistic (s-shaped) and non-relativistic (oval) phase-space trajectories for $H = 2.2m$. All axes are in units of $m$.

Fig. 2  A comparison of relativistic (s-shaped) and non-relativistic (oval) phase-space trajectories for $H = 4m$. All axes are in units of $m$.

Fig. 3  Phase-space trajectories for $H = 4m_1$, with $m_2 = 2m_1$ (innermost curve), $m_2 = m_1$ and $m_2 = .1m_1$ (outermost curve). All axes are in units of $m_1$. 