Higgs Branch, HyperKähler quotient and duality in SUSY N=2 Yang–Mills theories

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Abstract

Low–energy limits of N=2 supersymmetric field theories in the Higgs branch are described in terms of a non–linear 4–dimensional \(\sigma\)–model on a hyperKähler target space, classically obtained as a hyperKähler quotient of the original flat hypermultiplet space by the gauge group. We review in a pedagogical way this construction, and illustrate it in various examples, with special attention given to the singularities emerging in the low–energy theory. In particular, we thoroughly study the Higgs branch singularity of Seiberg–Witten \(SU(2)\) theory with \(N_f\) flavors, interpreted by Witten as a small instanton singularity in the moduli space of one instanton on \(\mathbb{R}^4\). By explicitly evaluating the metric, we show that this Higgs branch coincides with the Higgs branch of a \(U(1)\) N=2 SUSY theory with the number of flavors predicted by the singularity structure of Seiberg–Witten’s theory in the Coulomb phase. We find another example of Higgs phase duality, namely between the Higgs phases of \(U(N_c)\) \(N_f\) flavors and \(U(N_f – N_c)\) \(N_f\) flavors theories, by using a geometric interpretation due to Biquard et al. This duality may be relevant for understanding Seiberg’s conjectured duality \(N_c \leftrightarrow N_f – N_c\) in N=1 SUSY \(SU(N_c)\) gauge theories.

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1 Introduction

In the last couple of years there have been considerable progress towards an understanding of quantum field theories in the nonperturbative regime, mainly by focusing on theories with a large amount of symmetries that can give strong constraints on quantum effects. Extended supersymmetric theories indeed are prototypes of quantum field theories where quantum effects are under tight control, all the more as the number of supersymmetries becomes higher. A celebrated example is the $N=2$ $SU(2)$ with $N_f$ flavors model of Seiberg and Witten [1, 2], where special holomorphy properties and asymptotic behavior in the vector multiplet sector are powerful enough to allow for the determination of the full low energy theory on the Coulomb branch. On the other hand, it is known that the hypermultiplet manifold corresponding to the Higgs branch receives no radiative corrections [3, 2]. Indeed, one may assimilate the dynamical scale $\Lambda$ to a vector multiplet, and use the decoupling of vector multiplets from neutral hypermultiplets to argue that this should hold at nonperturbative level for any $N = 2$ super Yang–Mills theory [4].

By contrast, in models derived as low–energy limits of superstring theories, one in general expects gravitational ($\alpha'$) corrections or even perturbative or nonperturbative quantum effects when the dilaton which determines the string coupling is part of a hypermultiplet. In these cases a study of the hypermultiplet sector would allow new tests of string dualities, which so far rely mainly on the vector multiplet Coulomb branch structure. In particular, an understanding and a classification of singularities in the rigid limit, possibly in terms of the quantum numbers of the massless particles arising at the singular points would give new insights on string dynamics.

In this work, we investigate how non–trivial hypermultiplet manifolds can emerge by reduction of renormalizable $N=2$ super Yang–Mills theories to their low–energy limit, and how restoration of gauge symmetries and appearance of massless particles at certain points manifest themselves as singularities on these manifolds. In Section 2 we shall review in some detail the geometrical hyperKähler quotient construction of the low–energy theory, and apply it in Section 3 to models yielding a (quaternionic) one–dimensional moduli space with orbifold singularities. Section 4 will be concerned with the more interesting case of Seiberg–Witten $SU(2)$ model with $N_f$ flavors. Its moduli space can be interpreted as the moduli space of $SO(2N_f)$ 1–instantons on $\mathbb{R}^4$, and exhibits an isolated singularity at the point where the instanton shrinks to zero size, that we shall study in detail. We shall also be
able to further check the Seiberg–Witten conjectured singularity structure on the Coulomb phase, by explicitly proving the equivalence between the Higgs branches of the $SU(2)$ and $U(1)$ theories for suitable matter contents. Finally, we shall find in Section 5 another example of this kind of Higgs phase duality in the case of $U(N_c)$ theories with $N_f$ flavors. Thanks to a geometric interpretation of the Higgs branch in terms of the cotangent bundle of the complex Grassmannian $G_{N_c,N_f}$ due to ref. [5], we shall be able to prove the invariance of the Higgs branch under $N_c \leftrightarrow N_f - N_c$. This may be relevant for understanding Seiberg’s conjectured duality $N_c \leftrightarrow N_f - N_c$ in N=1 SUSY $SU(N_c)$ gauge theories, although at first sight this duality does not seem to generalize to more general gauge groups.

2 HyperKähler manifolds and hyperKähler quotients: a reminder

As is well known, rigid N=2 SUSY theories are constructed out of two types of multiplets: the vector multiplet comprises (together with two Weyl fermions) a gauge field and a complex scalar that takes its value in a special Kähler manifold, while the hypermultiplet comprises two complex scalars taking their values in a hyperKähler manifold [6] (together with two Weyl fermions). In this paper, we shall mainly be concerned with the hypermultiplet sector. The following subsections recall the basic facts about hyperKähler manifolds, triholomorphic isometries and quotient constructions, hopefully complementing in a pedagogical way the introductions already existing in the literature [7, 8, 9, 10, 11].

2.1 HyperKähler manifolds and hypermultiplets

A hyperKähler manifold is a $n$–dimensional riemannian manifold $(M, g)$ with three covariantly constant complex structures $I^x, x = 1, 2, 3$, verifying the quaternion algebra. We therefore have the following defining properties:

$$
I^x I^x = -1 \quad \quad \text{(almost complex structure)}
$$

$$
g(I^x X, I^x Y) = g(X, Y) \quad \quad \text{(hermiticity)}
$$

$$
\mathcal{N}^x (X, Y) = 0 \iff \nabla I^x = 0 \quad \quad \text{(integrability)}
$$

$$
I^x I^y = -\delta^{xy} 1 - \epsilon^{xyz} I^z \quad \quad \text{(quaternion algebra)}
$$
where $X, Y$ are vector fields and $N^x(X, Y) := [I^x X, I^x Y] - [X, Y] - I^x[X, I^x Y] - I^x[I^x X, Y]$ is the Nijenhuis integrability tensor $^4$.

The three hermitian endomorphisms $J^x := I^x/2i$ generate a unitary $n$-dimensional representation of $SU(2)_{\mathbb{H}}$ on the tangent space, which, since $J^x J^z = 3/4$, splits into irreducible components of spin $1/2$ (real dimension 4). As a consequence, the real dimension $n$ of $M$ has to be a multiple of four: $N_H = 4n$ (from now one, we shall by default refer to quaternionic dimension). Another way to see it is to note that for any vector $X$, the four vectors $X, I^1 X, I^2 X, I^3 X$ are orthogonal.

From these three complex structures one can construct three non degenerate antisymmetric 2–forms (the Kähler forms associated to the complex structure $I^x$)

$$\omega^x(X, Y) := g(I^x X, Y) \quad (1)$$

and since $d\omega^x(X, Y, Z) = A(g(\nabla_X I^x Y, Z))$, where $A$ is the antisymmetrization operator, one sees that the 2–forms $\omega^x$ are closed: $M$ is three times a symplectic manifold. Moreover, by privileging the third direction in $SU(2)_{\mathbb{H}}$, it can be checked that $\omega^h := \omega^2 + i\omega^3$ is a holomorphic closed form with respect to $I^1$, so that $(M, I^1, \omega^h)$ is actually a holomorphic–symplectic manifold.

HyperKähler manifolds can also be characterized by their riemannian holonomy group: the parallel transport preserves the symplectic form, so that the holonomy group must be contained in $Sp(N_H) \subset SO(4N_H)$. In particular, a hyperKähler manifold is Ricci–flat.

Practically, a way to prove that a manifold is hyperKähler is to exhibit three closed forms $\omega^x$ and a $SU(2)_{\mathbb{H}}$ action on the tangent space which preserves the metric and such that $\omega^x, x = 1, 2, 3$ transforms as a triplet.

### 2.2 Triholomorphic isometries and moment map

The coupling of hypermultiplets to vector multiplets is obtained by gauging a compact Lie group $G$ of triholomorphic isometries of the hyperKähler manifold [14]. Its Lie algebra $\mathcal{G}$ is generated by triholomorphic Killing vectors, ie

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$^4$We use the following standard notations [12, 13]: $X, Y, \ldots$ are vector fields, $[X, Y]$ their Lie bracket, $\mathcal{L}_X$ the Lie derivative along the vector $X$, $\nabla_X$ the Levi–Civita covariant derivative along $X$, $\phi$ the derivative of the scalar function $\phi$ along $X$, $\nabla \phi$ the gradient of $\phi$, $\langle \omega, X \rangle$ the contraction of the 1–form $\omega$ with the vector $X$, $d$ the exterior derivative on forms, $i_X$ the contraction operator with the vector $X$, and $TM$ the tangent bundle to the manifold $M$. 

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vector fields $K$ such that
\[ \mathcal{L}_K g = 0, \quad \mathcal{L}_K I^x = 0, \quad (2) \]
the Lie product on $\mathcal{G}$ being simply the Lie bracket of vector fields. These two equations imply that $\mathcal{L}_K \omega^x = 0$. Since $\mathcal{L}_K = di_K + i_K d$ and the forms $\omega^x$ are closed, one obtains
\[ d(i_K \omega^x) = 0 \quad (3) \]
This relation can be locally integrated to yield three functions $\mathcal{P}^x(K)$ on $M$ linearly dependent on the Killing vectors $K$, i.e. three moment maps from $M$ to the dual $\mathcal{G}^*$ of the Lie algebra $\mathcal{G}$, such that
\[ i_K \omega^x = d\mathcal{P}^x(K) \quad \forall K \in \mathcal{G} \quad (4) \]
When the second cohomology group of $\mathcal{G}$ vanishes (as occurs for all semi-simple algebras), the constants of integration can actually be locally chosen so as to impose the equivariance condition:
\[ \{\mathcal{P}^x(K), \mathcal{P}^x(L)\}^x = \mathcal{P}^x(\{K, L\}^x) \quad (5) \]
where $\{\cdot, \cdot\}^x$ is the Poisson bracket constructed from the symplectic form $\omega^x$ (for details see for example [11]). The left-hand side being globally defined, the moment maps on the right are then also globally defined. However, if $\mathcal{G}$ has a nontrivial center, it may happen that the equivariance condition cannot be imposed, or if it can, some integration constants may still remain undetermined. Those will be interpreted in a SUSY setting as Fayet–Iliopoulos terms.

As it appears, the moment map is a very general construction on symplectic manifolds with an action preserving the symplectic form. In classical mechanics, it corresponds to the linear momentum for the case of translations, or angular momentum for the case of rotations.

**2.3 N=2 SUSY theory, SUSY vacuum and moment map**

Having recalled in some detail the basics of the hyperKähler geometry, we now come to its implementation in N=2 SUSY field theory. The general
construction was worked out in a geometrical formalism in ref.[14], and we shall only focus on the elements relevant for our study of hypermultiplet moduli space.

The construction starts from a scalar manifold $M = M_V \otimes M_H$ which is the product of a special Kähler manifold $(M_V, g_{ij}^*)$ describing the vector multiplets and a hyperKähler manifold $(M_H, g_{uv})$ describing the hypers. The geometry of $M_V$ is defined by a holomorphic section $(Y_I, F_I^{\ast})$ of a $Sp(N_V)$ bundle over $M_V$, through $g_{ij}^* = \partial_i Y_I \partial_j \bar{F}_I - \partial_i \bar{F}_I \partial_j Y_I$. The gauge group of dimension $n_V$ acts on the scalars of both sides by (tri)holomorphic isometries generated by the Killing vectors $K_I^u$ and $k_i^I, k_i^{I*}$, and on the gauge vectors through its embedding in the symplectic group $Sp(N_V)$ of electromagnetic duality. The N=2 SUSY field theory is then defined by a supersymmetric gauged $\sigma$–model on $M$, with in particular the scalar potential

$$V = e^{2 \left( g_{ij}^* k_i^I k_j^J + 4 h_{uv} K_i^u K_j^v \right)} Y^I \bar{Y}^J + g_{ij}^* f_i^I \bar{f}_j^J + \frac{3}{2} \sum_{x=1}^{3} \mathcal{P}_x^I \mathcal{P}_x^J \geq 0 \quad (6)$$

where $f_i^I = \partial_i Y^I$ is usually invertible. This potential usually gives mass to most of the particles, however it may happen that some directions on $M$ remain unlifted, corresponding classically to massless particles. Their dynamics is then given by a non–linear $\sigma$–model whose target space is the set $\mathcal{M}$ of classical vacua of the theory, that is the moduli space of the theory. We shall restrict our attention to the $N = 2$ vacua, given by the equations

$$k_i^I \bar{Y}^I = 0, \quad k_i^{I*} Y^I = 0 \quad (7)$$

$$\mathcal{P}_x^I = 0 \quad (8)$$

$$K_i^u Y^I = 0 \quad (9)$$

These equations can be obtained by requiring $V = 0$, or equivalently by demanding that the N=2 SUSY variations of the fermions (in a trivial gauge background) vanish. Several cases may occur:

- There may be no solutions at all, in which case the theory (classically) breaks N=2 supersymmetry spontaneously. One should then look for nonzero minima of $V$. We shall not pursue this line here, except to note that the vacua may preserve some (N=1) supersymmetry [15, 16].

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6In that case, one can choose the coordinates $(z^i, \bar{z}^{i*})$ on $M_V$ so that $Y_I = z^i$. However, this is not always possible.
• There may be isolated solutions, so that no massless particles remain in the low–energy theory.

• There may be branches where the fields on the hyperKähler manifold are fixed at some point while the vectors are free to take their values in a submanifold $\mathcal{M}$ of $M_V$. This is usually called a Coulomb branch. One expects from N=2 SUSY that the low–energy scalar manifold $\mathcal{M}$, given by $k_j^I \bar{Y}^I = k_i^I Y^I = 0$ modulo $G$ is a special Kähler manifold. Although this can be checked in trivial renormalizable cases, it does not seem to follow directly from these equations in the general geometric setting of Ref.[14] (it is even not clear why $\mathcal{M}$ should be a complex manifold).

• On the contrary, the fields on $M_V$ may be fixed while the fields on $M_H$ are free to take value in a submanifold. One usually refers to this branch as a Higgs branch. The low–energy manifold $\mathcal{M}$ is given by $K_j^I Y^I = 0$ and $P_i^I = 0$ modulo $G$. It is easy to see that the first condition preserves the three complex structures, since they restrict to the tangent space of the submanifold $K_j^I Y^I = 0$. That the second condition also yields a hyperKähler manifold is the substance of the hyperKähler quotient construction [7] on which we shall dwell in the next sections: the zero level set of three moment maps modulo the gauge group $G$ is still hyperKähler. Note that when $G$ has a non trivial center, some constants in the definition of the moment maps have remained undetermined under the equivariance conditions. This freedom in defining the zero level set modulo elements of the center corresponds to the well–known Fayet–Iliopoulos couplings allowed by N=2 SUSY. Thus, the moduli space still depends on these free parameters.

• Finally the fields may take their value in a submanifold of $M_V \times M_H$, in which case one speaks of a mixed branch. It is not clear whether the scalar manifold should still be a product of a special Kähler manifold with a hyperKähler manifold, especially if the gauge group $G$ acts on $M_V$ and $M_H$ simultaneously, so that the quotient by $G$ a priori couples the vectors with the hypers.

In the following, we shall focus on the Higgs branch cases, where the vector multiplet is frozen to zero. We shall make heavy use of the hyperKähler quotient construction, which we shall now review in detail.
2.4 Diverse Quotients Constructions

**Symplectic quotient** – This construction is of daily use in classical me-
chanics: it is what allows to fix the total linear momentum (and forget about
the center of mass position) in order to study an isolated system of interact-
ing particles, or what allows to fix the angular momentum (and forget about
the actual angular position) in order to study the motion of a particle in a
central force field. The point here is that by restricting the phase space \( M \)
(i.e. a symplectic manifold) to the zero level set \( M_0 \) of the moment map,
and taking the quotient by the symmetry group \( G \), one obtains a manifold
\( \mathcal{M} = M_0/G \) of real dimension \( \dim M - 2 \dim G \) which is again a symplectic
manifold [17]. The symplectic form \( \omega' \) on \( \mathcal{M} \) is the unique symplectic form
whose pull–back on \( M_0 \) coincides with the restriction of the original form \( \omega \)
on \( M \) to \( M_0 \). \( \omega'(X', Y') \) is simply defined by taking any two vectors \( X, Y \)
on \( M_0 \) that project to \( X', Y' \), and letting \( \omega'(X', Y') := \omega(X, Y) \). That this does
not depend on the particular lift follows from \( \omega(X, K) = \langle dP_K, X \rangle = 0 \) for
\( X \in TM_0 \).

Note that the construction carries over if one replaces the zero–level set
\( M_0 \) by the preimage \( \mathcal{P}^{-1}(k) \) under \( \mathcal{P} \) of some non–zero invariant element
\( k \) of \( \mathcal{G}^* \) (which corresponds to the residual freedom in the definition of the
moment map), or even by the preimage of the whole orbit of an element of
\( \mathcal{G}^* \) under the action of \( \mathcal{G} \) (although this case does not seem to occur in the
setting of SUSY theories).

**Riemannian quotient** – If instead of taking a symplectic manifold one
starts with a Riemannian manifold \((M, g)\) with a continuous isometry group
\( G \) (acting freely on \( M \)), one can construct a canonical metric \( g' \) on the quo-
tient manifold \( \mathcal{M} = M/G \), by requiring that the projection \( \pi : M \to M/G \)
be a Riemannian submersion. The metric \( g'(X', Y') \) on the quotient is ob-
tained by horizontally lifting \( X', Y' \) to \( X, Y \), i.e. choosing two vectors \( X, Y \)
orthogonal to the Killing vectors \( K \) and projecting to \( X', Y' \) and then letting
\( g'(X', Y') := g(X, Y) \). The projection from \( M_0 \) to \( \mathcal{M} \) is then a Riemannian
submersion. Note that the pull–back of the metric \( g' \) to \( M_0 \) is not the re-
striction of \( g \) to \( M_0 \), for this pulled–back symmetric form is degenerate along
the action of the group.

This metric is actually the metric found by considering the classical low–
energy limit of a (non SUSY) gauged \( \sigma \)–model on \( M \), obtained by integrating
out the massive gauge bosons. Those couple to the scalars through a gauged
metric
\[ g_A(X, X) = g(X + eA^I K_I, X + eA^J K_J) \] (10)

At the classical level, gaussian integration of the gauge bosons can be carried out (without taking their kinetic terms into account) by simply minimizing \( g_A \) with respect to the corresponding gauge fields \( A^I \). As can easily be seen, this effectively projects the vector field \( X \) on the subspace orthogonal to the Killing vectors \( A^I \):

\[ \langle g_A(X, X) \rangle_A = g(X', X') \] (11)

where \( X' \) is the projection of \( X \) on the horizontal subspace, thus yielding the same metric as the quotient construction.

In more usual field theoretical terms, gauge bosons couple to matter through gauge currents \( J_\mu^I \), which are nothing but the pull–back to ambient space of the 1–form \( J_K \) on target space defined by

\[ J_K := g(K, \cdot) \] (12)

Gaussian integration of the gauge bosons of mass \( m \) yields a term \( J_\mu^I J^\mu_I \) in the effective lagrangian, which combines with the original metric to give the effective projected metric \( g' \).

Note that if the gauge group does not act freely on the manifold, it may happen that at some points part of the gauge symmetry gets restored, \( i.e. \) the Killing vectors become linearly dependent. Consequently some gauge bosons remain massless, while the local dimension of the quotient \( M \), given by the dimension of the horizontal subspace of the tangent space of \( M_0 \) at the given point, increases. This corresponds to extra scalars becoming massless, and to a singularity in the differentiable structure of the quotient at the corresponding point. It is similar to the singularity that occurs at the apex of a cone, where the tangent space is exceptionally 3–dimensional while 2D elsewhere.

**Kähler quotient** — Since a Kähler manifold is a special case of symplectic manifold, the symplectic quotient construction applies and yields a symplectic form \( \omega' \) on the quotient \( M_0/G \) if the action of \( G \) is symplectic (\( i.e. \mathcal{L}_K \omega = 0 \)). If moreover \( K \) is a Killing vector of \( M \) (\( i.e. \mathcal{L}_K g = 0, \text{ so that } \mathcal{L}_K I = 0 \)), it is also a Killing vector on \( M_0 \) with the restricted metric \( g \), so that \( M_0/G \) receives a metric \( g' \) through the Riemannian quotient construction. One can also check that the complex structure \( I \) restricts to the horizontal subspace
of $TM_0$, and thus descends to a complex structure $I'$ on the quotient. The compatibility of $g', I', \omega'$ ensures that $\mathcal{M}$ is indeed a Kähler manifold.

That $\mathcal{M}$ is a complex manifold can more easily be seen by noting that it coincides with the quotient of $M$ by the action of the complexified group $G^\mathbb{C}$ generated by the holomorphic vector fields $K \pm iIK$. Indeed, the orbit of a generic point $x$ of $M$ under the imaginary part $e^{iG^\mathbb{C}}$ of the complexified gauge group generically intersects the zero–level submanifold in exactly one point $x_0 = g_x x$ (since the flow corresponding to the vectors $IK$ is orthogonal to $M_0$). On the other hand two points $x, y$ in $M$ equivalent under the action of $G^\mathbb{C}$ are mapped under this procedure into two points $x_0, y_0$ on $M_0$ equivalent under the action of $G$ itself. The holomorphic symplectic structure on $\mathcal{M} = M_0/G$ is then the same as the quotient holomorphic symplectic structure on $M/G^\mathbb{C}$. This is summarized in the following diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{g_x} & M_0 \\
\downarrow & & \downarrow \\
M/G^\mathbb{C} & \xrightarrow{\text{Hol. Sympl}} & M_0/G
\end{array}
\]

Actually, since $G^\mathbb{C}$ is non compact, the quotient is in general ill-defined at some points, and we should restrict $M$ to the set of stable points, i.e. those which have a point in $M_0$ in the closure of their orbit under $G^\mathbb{C}$ (This is pedagogically explained in ref.[18]).

One can use this mapping to pull the Kähler metric from the Kähler quotient back to the complex manifold $M$, and a formula for the resulting Kähler potential has been given in [5], exploiting an idea of [7]:

\[
K'(x) = K(g_x x) + \frac{1}{4\pi} \ln |\chi(g_x)|^2
\]  

(13)

The second term is only present in presence of Fayet–Iliopoulos terms, i.e. when one considers, instead of the zero–level set $M_0$ of the moment map, the preimage of a $G$-invariant element $k$ of $G^\mathbb{C}$. This element can be seen as the differential of a character $\chi : G \to U(1)$ at the unity of $G$: $d\chi = -2\pi ik$. In equation (13), $\chi$ is naturally extended to the complexified group $G^\mathbb{C} \to \mathbb{C}^\ast$.

**HyperKähler quotient** — This construction can now be generalized to the case of hyperKähler manifolds [7, 8]. Here we define $M_0$ as the intersecting zero level set of the three moment maps: $M_0 = \{z \in M/P^x = 0, x = 1, 2, 3\}$, and $\mathcal{M} = M_0/G$. $\mathcal{M}$ is thus of real dimension $\dim M - 4 \dim G$, and as it
turns out still hyperKähler. To see why it is a special case of the previous construction, note that $M_h = \{ z \in M : P^1 = P^2 = 0 \}$ is a complex submanifold of $(\hat{M}, I^1)$, inheriting its Kähler structure from $\hat{M}$, and stable under the action of $\mathbb{C}G$. The previous construction then yields a Kähler structure on its Kähler quotient $\mathcal{M}$. The same can be done by privileging the two other complex structures in turn, and the three Kähler structures on $\mathcal{M}$, as it turns out, make it into a hyperKähler manifold. The three complex structures on $\mathcal{M}$ are simply the restrictions of the original ones to the horizontal subspace of the tangent space of $M_0$, as this subspace is stable under $I^x, x = 1, 2, 3$. In the cases where there remains unbroken gauge symmetry, the local dimension of the quotient increases by a multiple of four (since adding a vector $X$ to the horizontal subspace automatically brings in $I^x X, x = 1, 2, 3$, and these four vectors are linearly independent as well as independent from the original ones), and correspondingly extra hypermultiplets become massless, one for each unbroken gauge symmetry. This is the N=2 version of the Higgs effect, the gauge vector multiplet becoming massive by “eating” a matter hypermultiplet. As a consequence, the index $N_V - N_H$ is a constant over the Higgs branch.

As a holomorphic–symplectic manifold, the hyperKähler quotient also coincides for stable points with the holomorphic quotient of the complex submanifold $M_h$ by the complexified group $\mathbb{C}G$. One may ask whether it can be obtained directly form $\hat{M}$ by quotient by an hypothetical “quaternionized” group $G^\mathbb{H}$. The problem here is that there is no good Lie algebra structure on the tensor product $G \otimes \mathbb{H}$.

### 2.5 Renormalizable N=2 SUSY field theories

In the following, we shall be interested in low–energy effective theories corresponding to microscopic renormalizable gauge theories, i.e. corresponding to a non linear $\sigma$–model with flat target space. The way to obtain these flat manifolds is to take a $\mathbb{C}$–vector space $V$ of complex dimension $N_H$, acted upon by a linear unitary representation of a Lie group $G$ (which has to be a subgroup of $Sp(N_H)$). $G$ acts on its dual $V^*$ by the contragradient representation, and we choose $M = V \oplus V^*$. Letting $Q^i, \tilde{Q}^i, i = 1 \ldots N_H$ be the

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A vector multiplet can also become massive when a central charge appears in the SUSY algebra, but no example of this involving hypermultiplets has been found.
coordinates on $M$, we have the hyperKähler structure

$$\omega^x = (dQ^+_i \quad d\tilde{Q}_i) \sigma^x \wedge \left( \frac{dQ_i}{dQ^+_i} \right) \quad g = (dQ^+_i \quad d\tilde{Q}_i) \otimes \left( \frac{dQ_i}{dQ^+_i} \right)$$

where the $\sigma^x$ are the Pauli matrices. The $SU(2)_H$ action on the cotangent bundle is such that $(dQ^+_i \quad d\tilde{Q}_i^+)$ transforms as a doublet. This is not to be mistaken with an extra $SU(2)_R$ isometry that acts on the flat hyperKähler space itself, under which the coordinates $(Q^i \quad \tilde{Q}^+_i)$ themselves transform as doublets \(^8\) (while the vector multiplet scalars would be singlets). Although this isometry is not triholomorphic, it still generates a symmetry in the full gauged theory, if one asks that the gauginos transform as doublets while the hyperinos are singlets.

The Killing vectors associated to $G$ read

$$K_I = \left( \partial_{Q^+_i} \quad \partial_{\tilde{Q}_i} \right) T^i_{jI} \left( \frac{Q^j}{Q^+_j} \right) - \left( Q^+_i \quad \tilde{Q}_i \right) T^i_{jI} \left( \partial_{Q^+_j} \right)$$

where $T^i_{jI}$ is the antihermitian representation of the generator associated to $K_I$, while the equivariant moment maps are

$$P^x_{K_I} = (Q^+_i \quad \tilde{Q}_i) \sigma^x T^i_{jI} \left( \frac{Q^j}{Q^+_j} \right)$$

up to Fayet–Iliopoulos terms for the generators in the center of $G$.

In N=1 superfield formalism, the moment map $P^3$ simply corresponds to the D–term coming from the canonical kinetic terms, while the (anti)holomorphic moment maps $P^1 \pm iP^2$ correspond to the F–terms of the vector multiplet induced by the superpotential

$$W = \tilde{Q}_i T^i_{jI} \Phi^I Q^j$$

where $\Phi^I$ stands for the chiral superfield of the vector multiplets. For short, we shall call D–flatness (resp. F–flatness) the conditions $P^3 = 0$ (resp. $P^1 + iP^2 = 0$).

One may ask how the mass terms appear in this formalism: the only way to introduce them is to consider them as frozen N=2 vector multiplets gauging a flavor group. In order to be consistently frozen, their fermionic

\(^8\)This $SU(2)_R$ is actually nothing but the action of the unit quaternions on the quaternionic vector space $V \oplus V^*$. 

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components should be invariant under N=2 symmetry, i.e., one should have $k_i^I Y^I = k_i^I Y^I = 0$. These vector multiplets would then give extra contribution to the scalar potential, and would change the vacuum equations $K^I_i Y^I = 0$. They would also bring in extra moment map equations which however may be dropped by choosing an infinite metric on the vector multiplet manifold in the direction corresponding to the flavor vector multiplets.

In the following we will present explicit calculations of hyperKähler quotients of some examples of these flat manifolds, and study the singularities that emerge.

3 1–dimensional examples and ADE singularities

HyperKähler quotient manifolds of dimension 1 are the most easy to study, since it is often possible to explicitly parameterize $M_0$ modulo the gauge group. Moreover, 1–dimensional asymptotically locally euclidean (ALE) hyperKähler manifolds have been under active investigation as gravitational backgrounds for general relativity, and have been completely classified in terms of resolutions of the quotient of $\mathbb{C}^2$ by a discrete subgroup of $SU(2)_R$ [19].

3.1 U(1) Gauge theory and Eguchi–Hanson gravitational instantons

Let us consider a renormalizable theory with $N_f$ hypermultiplets $(Q_i, \tilde{Q}_i$ with charges $e_i^\alpha$ under $N_c$ U(1) vector multiplets $\Phi^\alpha$. The Higgs branch vacuum equations, as read on the superpotential $W = e_i^\alpha \tilde{Q}_i \Phi^\alpha Q_i$, are

$$e_i^\alpha \tilde{Q}_i Q_i = \xi^\alpha \in \mathbb{C}$$

$$e_i^\alpha (Q_i^+ Q_i - \tilde{Q}_i \tilde{Q}_i^+) = \nu^\alpha \in \mathbb{R}$$

where $\xi$ and $\nu$ denote the three Fayet–Iliopoulos parameters. In the case $N_f = N_c + 1$ and $e_i^\alpha$ of maximal rank, one expects a dimension 1 Higgs branch with all the $U(1)$’s broken by expectation values of the hypermultiplets. For vanishing Fayet–Iliopoulos terms, we can actually give an explicit parameterization of the manifold of solutions after some changes of basis. First, modulo interchange of hypermultiplets we can assume that the square
submatrix $e_{i}^{\alpha}$, $\alpha, i = 1..N_{c}$ is invertible, and act by $GL(N_{c}, \mathbb{R})$ combinations of $U(1)$’s to turn it into unity (whereas this does not preserve the scalar potential, it definitely preserves the flat directions). This leaves the values $e_{N_{f}}^{\alpha} = q^{\alpha}$. If any of these is zero, 1 vector and 1 hyper decouple. Otherwise we can rescale the $U(1)$’s to achieve

$$e_{i}^{\alpha} = \begin{pmatrix} 1/q_{1} & 1/q_{2} & \cdots & 1 \\ 1/q_{2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1/q_{N_{c}} \\ 1/q_{N_{c}} & \cdots & 1 & 1 \end{pmatrix}$$

(20)

and rescale the hypers to choose $q^{\alpha} = 1$. The Higgs branch is then parametrized by

$$\begin{pmatrix} Q_{i} \\ \tilde{Q}_{i} \end{pmatrix} = \sqrt{q_{i}} \begin{pmatrix} a \\ b \end{pmatrix}, \quad \begin{pmatrix} Q_{N_{f}} \\ \tilde{Q}_{N_{f}} \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix}, \quad (a, b) \in \mathbb{C}^{2}$$

(21)

The gauge current vanishes for this parameterization, so we actually have a slice of $M_{0}$ orthogonal to infinitesimal gauge transformations. However, there may still remain a discrete subgroup $\Gamma$ of the $U(1)$’s relating some $(a, b)$’s, by which we should quotient $\mathbb{C}^{2}$. Note that the precise subgroup depends crucially on the charges and is not invariant under linear redefinitions of $U(1)$’s. For the previous $q^{\alpha} = 1$ charge assignment, a $U(1)^{N_{c}}$ transformation $(e^{i\theta_{1}}, \ldots, e^{i\theta_{N_{c}}})$ on $(Q_{i}, \tilde{Q}_{i})$ can be reabsorbed in a change of $(a, b)$ for

$$e^{i\theta_{1}} = \ldots = e^{i\theta_{N_{c}}} = e^{-i\theta_{N_{c}}}$$

(22)

that is $\theta_{i} = \theta$, $(N_{c} + 1)\theta \equiv 0 \pmod{2\pi}$. The moduli space is then $\mathbb{C}^{2}/\mathbb{Z}_{N_{f}}$, where the discrete group acts as $(a e^{2i\pi/N_{f}}, b e^{-2i\pi/N_{f}}) \equiv (a, b)$.

On the other hand, for the case of two hypers of charges $(p, q)$ under one $U(1)$, the gauge transformation acts as

$$\begin{pmatrix} Q_{1} \\ Q_{1} \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \rightarrow \begin{pmatrix} a e^{ip\theta} & -b e^{ip\theta} \\ b e^{-ip\theta} & a e^{-ip\theta} \end{pmatrix}$$

(23)

so that it can be reabsorbed in a change of $(a, b)$ for $(p+q)\theta \equiv 0 \pmod{2\pi})$. The corresponding subgroup is $\mathbb{Z}_{r}$ where $r$ is the smallest integer so that $pr/(p+q)$ is integer, that is $r = \text{lcm}(p, p+q)/p$.

As for the metric on $\mathcal{M}$, it is anyway obtained by pulling back the metric (14) of the unconstrained $Q, \tilde{Q}$ fields:

$$ds^{2} = (1 + \sum |q_{i}|)(da_{1} + db_{1})$$

(24)
This is still the flat metric on $\mathbb{C}^2/\Gamma$, but due to the quotient the space is not flat, but rather has an orbifold singularity in curvature at the origin (its holonomy group is the discrete group $\Gamma$ rather than the trivial group).

This space (for $q_i = 1$) is actually the multi–Eguchi–Hanson gravitational instanton [20, 21], in the limit where the $N_f$ instantons collapse to one point. Switching on Fayet–Iliopoulos terms generically removes the points with unbroken gauge symmetry and therefore yields a smooth manifold, indistinguishable from the singular one at long distance. The orbifold singularity for vanishing Fayet–Iliopoulos terms is resolved into a family of $N_c$ intersecting two spheres, and one can retrieve the value of the $3N_f - 3$ real Fayet–Iliopoulos parameters by integrating the three closed hyperKähler forms on these spheres [22].

On the other hand, there are also $3N_f - 3$ real parameters in the multi–Eguchi–Hanson specifying the relative positions of the instantons, and one can check that those are exactly the Fayet–Iliopoulos parameters. Setting $n$ triplets of Fayet–Iliopoulos terms to zero is then equivalent to shrink $n$ 2–spheres to zero, or to make $n$ instantons collapse at the same point. The space then looks locally like $\mathbb{C}/\mathbb{Z}_n$.

### 3.2 $SU(N_c)$ with $N_f = N_c$ flavors gauge theory

Another easily workable example is the case of $N=2$ SQCD $SU(N_c)$ with $N_f = N_c$ flavors, where we also expect a 1–dimensional Higgs phase, among other phases with partially restored symmetry. The Lie algebra of $SU(N_c)$ has a trivial center, so no Fayet–Iliopoulos terms are available, and the F– and D–flatness equations take the form

\begin{align}
\tilde{Q} Q &\propto \mathbb{I}_{N_c} \\
Q^+ Q - \tilde{Q} Q^+ &\propto \mathbb{I}_{N_c}
\end{align}

where in the general case $(N_f, N_c)$ case $Q$ (resp $\tilde{Q}$) is a $N_c \times N_f$ matrix (resp. $N_f \times N_c$) (For this subsection we restrict to $N_f = N_c$). The right-hand side can be seen as the contribution of the Lagrange multiplier imposing that the vector multiplet is traceless.

An obvious and gauge–orthogonal solution is $Q = a \mathbb{I}_{N_c}, \tilde{Q} = b \mathbb{I}_{N_c}$, $(a, b) \in \mathbb{C}^2$, but here again $(a, b)$ and $(a e^{2i\pi/N_c}, b e^{-2i\pi/N_c})$ are gauge equivalent, so the moduli space is really $\mathcal{M} = \mathbb{C}^2/\mathbb{Z}_{N_c}$, again with a flat metric and

---

9In Ref.[23] it was proved that the Kähler classes of the hyperKähler forms are actually linear in the Fayet–Iliopoulos terms, so that the periods of these forms yield the Fayet–Iliopoulos terms for suitable normalizations.
an orbifold singularity at the origin, just like the $U(1)^{N_c}$ with $N_c + 1$ flavors case. This example shows that there is no hope to identify the restored gauge symmetry at a singularity by inspection of a Higgs branch only. However, for this model we have only looked at a part of the moduli space, namely the baryonic branch in the terms of ref.[4] (so called because the baryonic operators $\det Q, \det \tilde{Q}$ are non vanishing), and there are also a variety of non baryonic branches meeting the latter at the origin. The complete structure of those branches may be sufficient to characterize the singularity at the origin.

### 3.3 Kronheimer–Nakajima construction

So far we only have seen $\mathbb{Z}_n$ type of orbifold singularities emerge. As already mentioned, ALE 1–dimensional hyperKähler manifolds are completely classified, and are desingularizations of quotients of $\mathbb{C}^2 = \mathbb{H}$ by a discrete subgroup $\Gamma$ of $SU(2)_R$, the latter being in one–to–one correspondence with simply laced ADE Dynkin diagrams. It should therefore be possible to obtain any kind of ADE orbifold singularity by looking at low–energy limits of suitable $N=2$ SUSY field theories.

This construction has actually be found by Kronheimer [19], reviewed for physicists in [24, 10], and then extended by Nakajima [25] in the formalism of *quiver varieties*, more natural for a SUSY field theory interpretation. This formalism has recently found its way in field theories through the study of D-branes [26], so we shall briefly review the construction.

For any oriented diagram, *i.e.* a collection of points and arrows joining (some of) them, we associate to each point a gauge group $U(N_i)$ with the corresponding vector multiplets, and to each arrow $i \rightarrow j$ hypermultiplets in the representation $(N_i, \bar{N}_j)$ of $U(N_i) \times U(N_j)$ $^{10}$. This defines a $N=2$ field theory with gauge group $\prod U(N_i)$ (one of the $U(1)$ being decoupled) and a Higgs branch which is a hyperKähler manifold of dimension computable in terms of the $N_i$’s and the connection matrix of the diagram (non singular for generic values of the Fayet–Iliopoulos terms). In particular, 1–dimensional manifolds occur when one chooses a simply–laced extended Dynkin diagram and associates to each point the Coxeter number of the corresponding representation (the corresponding diagrams with the Coxeter numbers can be

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$^{10}$This field content is typical of the states found from open strings in the background of D-branes, which carry Chan–Paton gauge indices labeling the branes on which each end of the string lives.
found in ref.[10],fig.1 and 2). The resulting manifold is then a ALE gravitational instanton, asymptotically equivalent to $\mathbb{C}^2/\Gamma$.

For instance, a $D_4$ singularity would be obtained by studying the Higgs branch of a $SU(2) \times U(1)^4$ theory with 4 doublets of hypers each one of charge one under a different $U(1)$. Unfortunately, we could not find any parameterization of the Higgs branch that would exhibit the $D_4$ discrete action. A $E_8$ singularity could be obtained at the price of a rather heavy gauge group:

$$U(1) \times (U(2) \times U(3))^2 \times U(4) \times U(5) \times U(6)$$

4 Seiberg–Witten theory and small instanton singularity

Models with $N_H \geq 2$ are usually rather difficult to solve unless they possess some special flavor symmetries. The case of $N=2$ SUSY $SU(2)_G$ gauge theory with $N_f$ flavors is particularly favorable, since the pseudoreality of the $SU(2)_G$ representation 2 implies that the symmetry group is enhanced from $SU(N_f) \times SU(2)_G \times SU(2)_R$ to $SO(2N_f) \times SU(2)_G \times SU(2)_R$. The Higgs branch is then specified by only one real parameter up to symmetries. Moreover, a nonzero expectation value for a single doublet of $SU(2)$ already breaks the gauge symmetry completely, so gauge symmetry enhancement can only happen at the origin in field space, and a single isolated singularity is expected. The Higgs branch of this model has actually been briefly worked out in [2], and subsequently in more detail in the context of small instantons in heterotic $SO(32)$ string theory [27]. Here we shall concentrate on the study of the singularity occurring at the origin, after recalling Seiberg and Witten’s description of the moduli space.

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11We use a subscript $G$ to distinguish the $SU(2)$ gauge group from the other $SU(2)$’s that will occur in the following. In the context of hypermultiplets it is more useful to think of this $SU(2)_G$ as a $Sp(1)_G$.

12Although for $N_f = 1$ there is no Higgs branch, as we shall see in the following.
4.1 \( SO(2N_f) \times SO(4) \) symmetry and solutions of the vacuum equations

Using the same notations as in subsection 3.2 and embedding \( SU(N_f) \subset SO(2N_f) \) through \( A + iB \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \), the hypermultiplets can be recast in a pseudoreal \((2N_f,2,2)\) representation \( q^a_{I \alpha} \) of \( SO(2N_f) \times SU(2)_G \times SU(2)_R \):

\[
q^a_{I_1} = \left( \begin{array}{c} Q^i_a + \epsilon^{ab} \tilde{Q}^i_b \\ iQ^i_a - i\epsilon^{ab} \tilde{Q}^i_b \end{array} \right) \quad q^a_{I_2} = \left( \begin{array}{c} \epsilon^{ab} Q^i_b + \tilde{Q}^i_a \\ -i\epsilon^{ab} Q^i_b - i\tilde{Q}^i_a \end{array} \right)
\]

where the indices \( I, \alpha, a \) label the defining representations of \( SO(2N_f), SU(2)_R, SU(2)_G \). The pseudoreality condition reads \(^{13}\):

\[
(q^a_{I \alpha})^+ = \epsilon^{\alpha \beta} \epsilon^{a b} q^b_{I \beta}
\]

In this formalism, the metric and moment maps translate into

\[
ds^2 = \epsilon^{\alpha \beta} \epsilon^{a b} dq^a_{I \alpha} \otimes dq^b_{I \beta} \quad P^{(ab)} = \epsilon^{\alpha \beta} \epsilon^{a b} q^a_{I \alpha} q^b_{I \beta}
\]

where the adjoint representation of \( SU(2)_G \) is written as a symmetric tensor \( P^{(ab)} \). Vanishing of the three moment maps requires

\[
q^a_{I \alpha} q^b_{I \beta} \propto \epsilon^{\alpha \beta} \epsilon^{a b}
\]

This can be more easily exploited if one uses the decomposition \( SO(4) = SU(2)_G \otimes SU(2)_R/\mathbb{Z}_2 \) \(^{14}\) under which the pseudoreal \((2,2)\) representation corresponds to a real vector of \( SO(4) \) through

\[
q^\mu_I := (\sigma^\mu)_{\alpha \beta} \epsilon^{\alpha \beta} q^\alpha_{I \beta}
\]

\(^{13}\)Should we have taken matter in the adjoint representation of \( SU(2)_G \), there would have been no such reality condition to be imposed on the \( SO(2N_f) \times SU(2)_G \times SU(2)_R \) \((2N_f,3,2)\) representation to cut the number of degrees of freedom by half, so that a \( SO(2N_f) \) enhancement could not occur. On the other hand, it would still take place if one chooses a gauge group \( Sp(N_c)_G \) rather than \( SU(2)_G \) with \( N_f \) hypers in the fundamental representation; indeed, as noted in Ref.[27], one could use the antisymmetric tensors of \( Sp(N_c) \) and \( SU(2)_R \) to impose a reality condition on the \((2N_f,N_c,2)\) representation of \( SO(2N_f) \times Sp(N_c)_G \times SU(2)_R \).

\(^{14}\)This generalizes to the \( Sp(N_c) \) case by using the embedding \( Sp(N_c)_G \times SU(2)_R \subset SO(2N_c) \) under which \( 2N_c \) decomposes precisely as \((N_c,2)\). The flatness conditions \( q^a_{I \alpha} q^b_{I \beta} \propto \epsilon_{\alpha \beta} \) would not however have a simple interpretation in terms of \( SO(2N_c) \).
where $\sigma^\mu$ are the generalized 4 Pauli matrices (see for instance Ref.[28]). In this formalism the flatness conditions and metric read

$$q^\mu_\ell q^\nu_\ell \propto \delta^{\mu\nu}$$

$$ds^2 = dq^\mu_\ell dq^\nu_\ell$$

A point on the flat directions is thus given by 4 orthogonal real vectors of $\mathbb{R}^{2N_f}$, each of the same undetermined length $\rho$. $SO(2N_f)$ acts irreducibly\(^{15}\) on the bases of 4 vectors of a given length, so we can bring the four vectors along the first four directions of $\mathbb{R}^{2N_f}$ and recover the general solution by a $SO(2N_f)$ rotation. We can consequently parameterize the Higgs branch as

$$q^\mu_\ell = \Omega_{2N_f} \begin{pmatrix} \rho \\ \rho \\ \rho \\ \rho \end{pmatrix}, \quad \Omega_{2N_f} \in SO(2N_f)$$

$$\rho \in \mathbb{R}^+$$

However, a subgroup $SO(2N_f - 4)$ of $SO(2N_f)$ lets the solution unaffected, and furthermore one should identify configurations differing by the action of the gauge group $SU(2)_G \subset SO(4)$. One therefore obtains as a moduli space

$$\mathcal{M} = \mathbb{R}^+ \times \frac{SO(2N_f)}{SO(2N_f - 4) \times SU(2)_G}$$

where $SO(2N_f - 4) \times SU(2)_G$ acts in $SO(2N_f)$ as

$$\Omega_{2N_f} \overset{=}{\rightarrow} \Omega_{2N_f} \cdot \begin{pmatrix} \mathbb{I}_4 \\ \Omega_{2N_f - 4} \end{pmatrix}, \quad \Omega_{2N_f - 4} \in SO(2N_f - 4)$$

with $\Omega_{2N_f - 4} \in SO(2N_f - 4)$ and $\Omega_4 \in SO(4)$ the embedding of $SU(2)_G$ in $SO(4)$.

To evaluate the metric on this space\(^{16}\), we first pull the flat metric (33) back on $\mathbb{R}^+ \times SO(2N_f)$:

$$ds^2 = tr \left( d(\rho \Omega_{2N_f}) \begin{pmatrix} \mathbb{I}_4 \\ \mathbb{I}_4 \end{pmatrix} \right) d(\rho \Omega_{2N_f})^t$$

\(^{15}\)This is not quite true for $2N_f = 4$ where the orientation of 4 vectors in $\mathbb{R}^4$ distinguishes two connected components (the so called baryonic and nonbaryonic branches) that can be mapped to each other by a $O(2N_f)$ ($Q_i^a \leftrightarrow -e^{ab} Q_b^a$) or $O(4)$ parity transformation. The action of $SO(2N_f)$ is then irreducible on each branch.

\(^{16}\)Note that there is not a unique $SO(2N_f)$ invariant metric on the coset $SO(2N_f)/SO(2N_f - 4) \times SU(2)_G$.
\[
= d\rho^2 + \rho^2 \text{tr} \left( \Omega^I_{2N_f} d\Omega_{2N_f} \left( \begin{array}{c} 1 \\ \end{array} \right) d\Omega^I_{2N_f} \Omega_{2N_f} \right)
\] (38)

Here we note that the scale fluctuations \(d\rho\) are orthogonal to the fluctuations of \(\Omega^I_{2N_f}\), and that the fluctuations along \(SO(2N_f-4)\) are effectively projected out. Then we should retain only the fluctuations of \(q\) orthogonal to the gauge group \(SU(2)\). Scale fluctuations are already orthogonal to \(SU(2)\), so one simply has to project the fluctuations \(d\Omega_{2N_f}\) on the subspace orthogonal to the subalgebra \(SU(2)\) for the scalar product \(\text{tr} \left( d\Omega^I_{2N_f} \left( \begin{array}{c} 1 \\ \end{array} \right) d\Omega^I_{2N_f} \Omega_{2N_f} \right)\):

\[
ds^2 = d\rho^2 + \rho^2 \text{tr} \left( \Omega^I_{2N_f} d\Omega_{2N_f} \right) \left( \begin{array}{c} 1 \\ \end{array} \right) d\Omega^I_{2N_f} \Omega_{2N_f} \right)\) (39)

This a special case of a warped product \(M \otimes_{f^2} M'\) of two Riemannian manifolds \((M,ds^2)\) and \((M',ds'^2)\), that is a Riemannian manifold \(M \times M'\) with metric \(d\tilde{s}^2 = ds^2 + f^2(x)ds'^2\) where \(f(x)\) is a function of the coordinates on \(M\) only. These manifolds have actually been under investigation in the context of Einstein spaces [29] and have yielded numerous examples of non-homogeneous Einstein manifolds, though non compact. The case where \(\dim M = 1\) has in particularly been completely worked out, and it is known that Ricci-flat warped products are obtained only for \(f(x) = x\) and \(M'\) an Einstein manifold of definite Einstein constant. This is precisely the case here.

### 4.2 Conical singularity

Singularities on such manifolds may arise when \(f(x)\) vanishes, and this actually occurs in the case at hand when \(\rho\) vanishes. It was noticed by Witten [27] that the Higgs branch we are considering actually describes the moduli space of one \(SO(2N_f)\) instanton in \(\mathbb{R}^4\), as one learns from the general ADHM construction [30]. The four directions \(q^\mu\) describe the embedding of the \(SU(2)\) describing the instanton in \(SO(2N_f)\), while \(\rho\) specifies its size. The singularity at the origin thus corresponds to the zero size limit of the instanton, and signals the appearance of a “nonperturbative” \(SU(2)\) gauge symmetry enhancement.

The warped product structure makes it fairly easy to study the singularity that occurs at the origin, since it does not come from either of the components but only of the function \(f(x)\) that couples them. The Riemann tensor of a
warped product can be easily expressed in terms of the Riemann tensors of \( M \) and \( M' \) (expressions can be found in [29], but we give in appendix another version of them with more conventional notations), and it is found that the only non vanishing component is obtained when all the vectors are on the homogeneous side:

\[
\tilde{R}(X', Y')Z' = R'(X', Y')Z' + (g'(X', Z')Y' - g'(Y', Z')X')
\]  

(40)

that is, the only effect of the warping is to add a constant negative curvature term to the Riemann tensor on the homogeneous side. As a consequence, for closed paths at fixed value \( \rho \) arbitrarily close to zero, the holonomy can be calculated in terms of \( M' \) only, using the previous expression as an effective Riemann tensor, and is independent of \( \rho \). Except in the case where special cancellation between the two terms occurs, the holonomy remains non trivial when the path shrinks to zero, implying a singularity in the Riemann tensor at this point, even though the components of the Riemann tensor do not show any divergence. Cancellation can only occur when \( M' \) is of constant positive curvature, \( i.e. \) locally a sphere. This is actually what happens for \( 2N_f = 4 \), since

\[
\frac{SO(4)}{SU(2)_G} \equiv \frac{SU(2)_R}{\mathbb{Z}_2} \equiv S_3/\mathbb{Z}_2
\]  

(41)

so that the holonomy group around the origin is a discrete group \( \mathbb{Z}_2 \), corresponding to an orbifold singularity. This agrees with the result \( \mathcal{M} = \mathbb{C}^2 / \mathbb{Z}_2 \) of the previous section. Note that in general the singularity is much worse than an orbifold singularity, since the local holonomy group is not even discrete.

### 4.3 Global symmetries on the Higgs branch

From the previous formulation of the Higgs branch, it is easy to determine the global symmetry breaking pattern at a given point in the moduli space (as was already done in the original paper by Seiberg and Witten). The four vectors of \( SO(2N_f) \) break the flavor symmetry down to \( SO(2N_f - 4) \), however \( SO(4) \) rotations in the space of the four vectors can actually be compensated by orthogonal linear combinations of the same vectors (this is only true because they are orthonormal), \( i.e. \) by \( SU(2)_G \times SU(2)_R \) rotations. The remaining global symmetry group is thus

\[
SO(2N_f - 4) \times SU(2)_G' \times SU(2)_R'
\]  

(42)
where \( SU(2)_G \) is the diagonal group of \( SU(2)_G \) times a \( SU(2) \) subgroup of \( SO(4) \subset SO(2N_f) \), while \( SU(2)_{R'} \) is the diagonal group of \( SU(2)_R \) times the other \( SU(2) \) subgroup of \( SO(4) \). This symmetry group acts on the tangent space of \( \mathcal{M} \), which splits into real irreducible representations

\[
(2N_f - 4, 2, 2) \oplus (1, 1, 3) \oplus (1, 1, 1) \tag{43}
\]

as obtained by decomposing the adjoint representation of \( SO(2N_f) \) under the unbroken group and forgetting the adjoint representations of \( SO(2N_f - 4) \times SU(2)_G \) by which we quotient. This corresponds to the representations of the massless particles for the given point in moduli space. The representations of the fermions can also be worked out, taking into account their different quantum numbers under \( SU(2)_R \), yielding

\[
(2N_f - 4, 2, 2) \oplus (1, 1, 2) \oplus (1, 1, 2) \tag{44}
\]

The unusual quantum numbers of the massless particles under \( SU(2)_{R'} \) should cause no surprise: the action of \( SU(2)_{R'} \) is distinct from that of the \( SU(2)_H \) generated by the three complex structures, which is an isometry of the tangent space but not a symmetry of the theory in general.

### 4.4 \( \mathbb{N}=2 \) duality and Higgs branches

As is now well known, the \( U(1) \) Coulomb phase of the Seiberg–Witten model \( SU(2)_G \) with \( N_f \) flavors presents singularities where hypermultiplets charged under \( U(1) \) become massless. From these points Higgs branches emerge corresponding to giving vacuum expectation values to this multiplet, and these branches should actually be the same as the Higgs branches of the microscopic \( SU(2)_G \) theory, since the Higgs branch does not receive any quantum corrections. This was checked at the level of global symmetry breaking in \cite{2}, but we can slightly strengthen their result by explicitly evaluating the metric on the Higgs branch emanating from a \( U(1) \) theory with \( N_e \) massless hypers \( Q^i, \tilde{Q}^i \).

As for the previous case, the Higgs branch is parameterized by a real parameter up to flavor and gauge rotation. Indeed, by a \( SU(N_e) \) rotation one may choose \( Q^i \) along the first flavor; the F–flatness condition \( Q^i \tilde{Q}^i = 0 \) implies that \( \tilde{Q}^i \) has no component along this direction, so we can use \( SU(N_e - 1) \) to bring it along the second flavor:

\[
\begin{pmatrix}
Q^i \\
\tilde{Q}^{i+}
\end{pmatrix} =
\begin{pmatrix}
\rho & 0 & 0 & \cdots & 0 \\
0 & \bar{\rho}^+ & 0 & \cdots & 0
\end{pmatrix}
\tag{45}
\]
The D–flatness implies that \( \rho = \tilde{\rho} \) up to a \( U(1) \) gauge rotation, and finally one may choose \( \rho \in \mathbb{R}^+ \) by a \( SU(2) \subset SU(N_e) \) rotation along the two first flavors. We can therefore parameterize the moduli space by

\[
\mathcal{M} = \mathbb{R}^+ \times \frac{SU(N_e)}{SU(N_e - 2) \times U(1)}
\]

where the subgroup acts in \( SU(N_e) \) through \(^{17}\)

\[
\Omega_{N_e} \mapsto \Omega_{N_e} \cdot \begin{pmatrix} \mathbb{I}_2 & \Omega_{N_e - 2} \\ \Omega_{N_e - 2} & \mathbb{I}_2 \end{pmatrix} \cdot \begin{pmatrix} e^{i\theta} & e^{-i\theta/(N_e - 2)} \\ e^{-i\theta/(N_e - 2)} & e^{i\theta} \end{pmatrix},
\]

The riemannian structure turns out to be also a warped product:

\[
ds^2 = d\rho^2 + \rho^2 \text{tr} \left( \Omega_{N_e}^+ d\Omega_{N_e} \right) \left( \mathbb{I}_2 \right) \left( d\Omega_{N_e}^+ \Omega_{N_e} \right) \perp
\]

where one now projects orthogonally to the \( U(1) \) subalgebra.

We can now easily check the duality conjectures on the Higgs branches. For \( N_f=2 \), there are two singularities on the Coulomb branch with two hypermultiplets of same charge becoming massless at each point. One \( \mathbb{R}^+ \times SU(N_e = 2)/\mathbb{Z}_2 \) Higgs branch emanates from each point, corresponding to the two Higgs branches \( \mathbb{R}^+ \times SO(4) / SU(2)_G \) of the microscopic \( SU(2)_G \) theory. Moreover, the two metrics (39) and (48) coincide.

For \( N_f=3 \), Seiberg and Witten predict two singularities on the Coulomb branch, one with only 1 charged hyper becoming massless (thereby giving no Higgs branch), and the other with 4 massless hypers in the spinor of \( SO(6) \). Using the isomorphism \( SO(6) = SU(4)/\mathbb{Z}_2 \), where the \( \mathbb{Z}_2 \) is the square of the center \( \mathbb{Z}_4 \) of \( SU(4) \) \(^{18}\), one can check that the \( SO(2) \times SO(4) \) subgroup of \( SO(6) \) translates into the \( U(1) \times SU(2) \times SU(2) \) subgroup of \( SU(4) \) so that we have at the level of differentiable manifolds

\[
\mathbb{R}^+ \times \frac{SO(6)}{SO(2) \times SU(2)_G} = \mathbb{R}^+ \times \frac{SU(4)}{SU(2) \times U(1)}
\]

The metrics can now be seen to coincide, with the peculiarity that the role of the orthogonal projection on the gauge group on one side is played by the insertion of \( \left( \mathbb{I} \right) \) on the other side.

\(^{17}\)For \( N_e = 2 \) the \( U(1) \) is reduced to a \( \mathbb{Z}_2 \) in the center of \( SU(2) \). For \( N_e < 2 \) there is obviously no Higgs branch.

\(^{18}\)This \( \mathbb{Z}_2 \) is actually contained in \( SO(2) \times SU(2)_G \), so that is disappears in the quotient.
We therefore further check that the global structure as well as the metric of the Higgs branch is compatible with the singularity structure conjectured by Seiberg and Witten, and also with the non-renormalization theorem on the Higgs branch [4] (since those branches coincide not only asymptotically, but also in the nonperturbative regime).

5 Multicolor Higgs branches

When the number of colors is increased, the phase structure of the theory gets much more involved, in particular since vev’s of quarks do not necessarily completely break the gauge group, so that the gauge symmetry becomes partially restored on submanifolds of the Higgs branch. The structure of vacua for $SU(N_c)\ N=2$ theories with $N_f$ flavors has already been carefully analyzed in [4], with the purpose of understanding Seiberg’s conjectured $N_c\leftrightarrow N_f-N_c$ duality. In particular, it was noted that the baryonic Higgs branch of a $SU(N_c)$ theory with $N_f$ flavors could also be interpreted as the Higgs branch of a $SU(N_f-N_c)\times U(1)^{2N_c-N_f}$ theory with the same number of flavors and $2N_c-N_f$ color singlets charged under the $U(1)$'s. Here we shall find another manifestation of this kind of duality, and prove the exact equivalence (at the level of hyperKähler manifolds) of the Higgs branches of $U(N_c)$ and $U(N_f-N_c)$ theories with $N_f$ flavors. As a first hint in this direction, note that the dimension of the Higgs branch is $N_H = N_f N_c - (N_c^2 - 1) = N_c (N_f - N_c)$, obviously invariant under $N_c\leftrightarrow N_f-N_c$. We shall then comment on the extension of this result to theories with other gauge groups.

5.1 $U(N_c)\ N=2$ theory with $N_f$ flavors

The hyperKähler quotient corresponding to a theory $U(N_c)$ with $N_f$ flavors in the fundamental representation has actually already been worked out in the mathematical literature [5], though with very different motivations than ours. As it turns out, the quotient can be interpreted geometrically as a cotangent bundle of a complex Grassmannian $G_{N_c,N_f}$. Now, there is a fairly trivial equivalence of the Grassmannians $G_{N_c,N_f}$ and $G_{N_f-N_c,N_f}$, and we shall prove that this equivalence carries over to the hyperKähler structure of their cotangent bundles (that is, their holomorphic–symplectic structure together with their metric).
Using the same notations as in section 3.2, the equations describing the flat directions in presence of a Fayet–Iliopoulos term along the third direction in $SU(2)_H$ read

$$\tilde{Q}Q = 0$$  \hspace{1cm} \text{(50)}

$$Q^+Q - \tilde{Q}\tilde{Q}^+ = 2k \mathbb{I}_{N_c}$$  \hspace{1cm} \text{(51)}

where $k$ is a fixed real number that can be chosen non-negative. $Q$ (resp. $\tilde{Q}$) can be seen as a linear endomorphism $\mathbb{C}^{N_c} \rightarrow \mathbb{C}^{N_f}$ (resp. $\mathbb{C}^{N_f} \rightarrow \mathbb{C}^{N_c}$).

At the level of holomorphic–symplectic manifolds, the hyperKähler quotient coincides with the quotient of the stable points verifying the first equation by the action of the complexified gauge group, here $GL(N_c, \mathbb{C})$. The stable points under $GL(N_c, \mathbb{C})$ are those for which the matrix $Q$ has maximal rank $N_c$; the invariants under this action are the the $N_c$-dimensional vector subspace $P = \text{Im}Q \subset \mathbb{C}^{N_f}$ and the $N_f \times N_f$ meson matrix $M = Q\tilde{Q}$. $M$ and $P$ are however constrained by the F-flatness condition which implies that $P \subset \ker M$. Since $\text{Im} M \subset P$, $M$ is really an endomorphism $\mathbb{C}^{N_f}/P \rightarrow P$. The doublet $(P \subset \mathbb{C}^{N_f}, M : \mathbb{C}^{N_f}/P \rightarrow P)$ has a geometrical interpretation: it defines a point in the cotangent bundle $T^*G_{N_c,N_f}$ of the complex Grassmannian $G_{N_c,N_f}$. Indeed the complex Grassmannian $G_{N_c,N_f}$ is the set of $N_c$-dimensional subspaces $P$ in $\mathbb{C}^{N_f}$, while tangent vectors correspond to displacements of $P$, that is linear mappings from $P$ to its supplementary, $\mathbb{C}_f^{N_f}/P$. $M$ is thus a cotangent vector at $P \in G_{N_c,N_f}$. Cotangent bundles of complex manifolds have a canonical holomorphic–symplectic structure, and it can be checked that it is the same structure as the one obtained by hyperKähler quotient:

$$\mathcal{M}^{Hok.\ Sympl} \equiv T^*G_{N_c,N_f}$$  \hspace{1cm} \text{(52)}

The Grassmannians $G_{N_c,N_f}$ and $G_{N_f-N_c,N_f}$ being isomorphic as complex manifolds, this is the first hint to the already discussed duality of the corresponding Higgs branches.

However, we want to show that the isomorphism holds at the level of hyperKähler manifolds, so we should compare the Kähler metric on the two quotients. This metric was precisely computed by [5], as a generalization of Calabi’s formula [31] for the hyperKähler on the cotangent bundle $T\mathbb{C}P^n$.

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\textsuperscript{19}Whereas in the general case it is not clear how hyperKähler quotients with rotated Fayet–Iliopoulos terms are related, there is no such problem here, since a $SU(2)_H$ rotation of the Fayet–Iliopoulos terms can be compensated by a $SU(2)_R$ rotation of the solutions.
of the complex projective space, and we shall sketch here their derivation. Applying equation (13), one determines the hermitian endomorphism \( g_x \in e^{i\theta} \) that takes a point \((Q, \tilde{Q})\) of \( M_0 \) to \( M_0 \), such that \( g^{-1}Q^+Qg^{-1} = g\tilde{Q}Q^+g = 2k1 \) and uses \( \chi(g) = \det(g)^{2\pi k} \) to describe the Fayet–Iliopoulos term. As a result,

\[
K'(Q, \tilde{Q}) = \frac{k}{2} \ln \det(Q^+Q) + \text{tr}(\frac{1}{2}\gamma\gamma^+ - \frac{k}{2}\ln(\gamma\gamma^+))
\]

(53)

where \( \gamma = \sqrt{Q^+Q} \) verifies a biquadratic equation that yields

\[
\gamma\gamma^+ = k(1 + \sqrt{1 + \frac{1}{k^2}MM^+})
\]

(54)

The first term in (53) is simply the pull back of the Kähler metric on \( G_{N_{c},N_f} \) defined by the scalar product \( ds^2 = 2ktr(XX^+) \) where \( X : P \to \mathbb{C}^{N_f}/P \) is a tangent vector to the Grassmannian. Indeed, taking \( Q = \left( \begin{array}{c} I_{N_c} \\ Q \end{array} \right) \) one finds a Kähler potential \( \frac{k}{2} \ln \det(I_{N_c} + QQ^+) \) which yields upon derivation a generalization of the Fubini–Study metric on \( \mathbb{C}P^N \). The second term in (53) can be interpreted in terms of the curvature tensor of \( G_{N_{c},N_f} \) (cf [5]). In the limit of vanishing Fayet–Iliopoulos term, the Kähler potential (53) reduces to

\[
K'(Q, \tilde{Q}) = \frac{1}{2} \text{tr}\sqrt{MM^+}
\]

(55)

so that the metric degenerates along the base manifold \( G_{N_{c},N_f} \).

The isomorphism between the grassmannians \( G_{N_{c},N_f} \) and \( G_{N_f-N_{c},N_f} \) is obtained by sending a \( N_c \)-dimensional plane \( P \) in \( C^{N_f} \) to its hermitian orthogonal space \( Q \) such that \( C^{N_f} = P \oplus Q \). This amounts to take \( Q = \left( \begin{array}{c} I_{N_c} \\ Q \end{array} \right) \) to \( Q = \left( \begin{array}{c} Q^+ \\ I_{N_f-N_c} \end{array} \right) \). The cotangent form \( M : C^{N_f}/P \equiv Q \to P \) is then sent to its hermitian conjugate \( M^+ : P \equiv C^{N_f}/Q \to Q \). The Kähler potential (53) is obviously invariant under this operation, which proves that the moduli spaces of the \( U(N_c) \) and \( U(N_f-N_c) \) theories with \( N_f \) flavors are actually the same at the level of hyperKähler manifolds. This may be relevant for an understanding of Seiberg’s duality [32] in \( N = 1 \) SUSY \( SU(N_c) \) theory.

5.2 From \( U(N_c) \) to \( SU(N_c) \)

We can now fairly easily adapt the previous construction to the case of a gauge group \( SU(N_c) \), where there is no Fayet–Iliopoulos term anymore. The
flat directions are now given by

\[ \tilde{Q}Q \propto I_{N_c} \quad (56) \]
\[ Q^+Q - \tilde{Q}\tilde{Q}^+ \propto 2k I_{N_c} \quad (57) \]

so that the solutions are those of the \( U(N_c) \) equations when one does not impose any value to the Fayet–Iliopoulos terms \(^{20}\). The complexified gauge group is now \( SL(N_c, \mathbb{C}) \), so that not only is the subspace \( P = \text{Im}Q \) preserved, but also the antisymmetric \( N_c \)-form \( \varpi \) induced from \( \mathbb{C}^{N_c} \) to \( P \) through \( Q \):

\[ \varpi(x_1, \ldots, x_{N_c}) := \det(Q^{-1}x_1, \ldots, Q^{-1}x_{N_c}) \quad \text{for } x_i \in \mathbb{C}^{N_f} \quad (58) \]

Adding in the \( \tilde{Q} \) degrees of freedom and imposing the F–flatness conditions, we find that the moduli space is actually the cotangent bundle \( T^*G_{N_c,N_f} \) of the “complex grassmannian with a volume form” \( G_{N_c,N_f}^V \), that is the set of \( N_c \)-dimensional subspaces \( P \) of \( \mathbb{C}^{N_f} \) with any antisymmetric \( N_c \)-form on \( P \).

The metric on this space is simply obtained from the previous construction by choosing the Fayet–Iliopoulos term \( k \) so that \( g_x \in G^C \), that is \( \det g_x = 1 \). One then simply finds

\[ K'(Q, \tilde{Q}) = \frac{1}{2} \text{tr} \sqrt{k^2 + M M^+} \quad (59) \]

with \( k \) determined by \( \det(k I_{N_f} + \sqrt{k^2 I_{N_f} + M M^+}) = \det(QQ^+) \).

However, the previous duality between \( G_{N_c,N_f} \) and \( G_{N_f-N_c,N_f} \) has no chance to carry over to this \( SU(N_c) \) case, since there is no way a volume form on \( P \) could induce a volume form on its orthogonal.

### 5.3 From \( SU(N_c) \) to \( SO(N_c) \) and \( Sp(N_c) \)

In the cases \( SO(N_c) \) (resp. \( Sp(N_c) \)), the gauge group is further reduced, so that there are more invariants. In particular, from the \( Q \) sector we obtain on the invariant subspace \( P \) a bilinear symmetric (resp. antisymmetric) form, so that we expect the moduli space to be the cotangent space of the complex Grassmannian “with a symmetric (resp. antisymmetric) form”. Once again, given such a form on \( P \) there is no way to construct a form on the orthogonal, and one should not expect the duality \( N_c \leftrightarrow N_f - N_c \) to apply.

\(^{20}\)That is, the moduli space of \( SU(N_c) \) is a fibered over the moduli space of \( U(N_c) \), the fiber corresponding to the three Fayet–Iliopoulos terms together with a \( U(1) \) angle associated to the central \( U(1) \) of \( U(N_c) \).
6 Conclusion

In this paper, we gave a self-contained introduction to hyperKähler geometry and studied in detail some examples of Higgs branches occurring in N=2 SUSY Yang–Mills theories, complementing the existing literature on the subject. In particular, we showed how the hyperKähler structure on the Higgs branch naturally emerges through a hyperKähler quotient construction, and devoted special attention to the singularities appearing at points with enhanced symmetry. While 1-dimensional branches only displayed ADE orbifold singularities, we explored a higher dimensional example of nontrivial singularity by pursuing the work of Refs.[2, 27] on SU(2) and U(1) Higgs branches. We proved the precise equivalence of the two Higgs branches for special matter content, thereby supporting both the conjectured singularity structure of the SU(2) theory [2] and the nonrenormalization theorem on Higgs branches [4]. Finally, we elaborated on the work of Ref.[5] to prove the invariance of the Higgs branch of a U(Nc) Nf flavors theory under Nc ↔ Nf − Nc. The connection with Seiberg’s conjectured duality [32] in N=1 is not clear at present, all the more as this Higgs branch duality does not seem to extend to other gauge groups.

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Appendix: Riemann tensor of a Warped Product

We consider the warped product \( \tilde{M} \) of two Riemannian manifolds \((M, g)\) and \((M', g')\), of dimension \(n\) and \(n'\), defined by
\[
\tilde{d}s^2 = ds^2 + e^{2\phi(x)} ds'^2
\]
(60)
Each vector vector field \( \tilde{X} \) on \( \tilde{M} \) can be split along the tangent spaces of \( M \) and \( M' \) as \( \tilde{X} = X + X' \).

The Levi–Civita connection on \( \tilde{M} \) corresponding to the above metric is given by
\[
\nabla_{\tilde{X}} Y = \nabla_X Y
\]
(61)
\[ \tilde{\nabla}_X Y' = (X.\phi)Y' \]
\[ \tilde{\nabla}_X Y = (Y.\phi)X' \]
\[ \tilde{\nabla}_X Y'' = \nabla_{X'} Y' - \tilde{g}(X', Y') \tilde{\nabla}_X \phi \]

for which one evaluates the Riemann tensor
\[ R(X, Y) Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z: \]
\[ \tilde{R}(X,Y) Z = R(X,Y) Z \]
\[ \tilde{R}(X,Y) Z' = 0 \]
\[ \tilde{R}(X', Y') Z = 0 \]
\[ \tilde{R}(X', Y') Z' = R'(X', Y') Z' + ||\tilde{\nabla}_X \phi||^2 (\tilde{g}(X', Z') Y' - \tilde{g}(Y', Z') X') \]
\[ \tilde{R}(X,Y) Z = \big((X.\phi)(Z.\phi) + \langle \tilde{\nabla}_X d\phi, Z \rangle \big) Y' \]
\[ \tilde{R}(X,Y) Z' = -\tilde{g}(Y', Z') \big((X.\phi) \tilde{\nabla}_X \phi + \tilde{\nabla}_X (\tilde{\nabla}_X \phi)\big) \]

the Ricci tensor \( S(X,Y) := \text{tr}(X \to R(X,Y) Z) \):
\[ \tilde{S}(X,Y) = S(X,Y) - n' \big((X.\phi)(Y.\phi) + \langle \tilde{\nabla}_X d\phi, Y \rangle \big) \]
\[ \tilde{S}(X', Y) = 0 \]
\[ \tilde{S}(X', Y') = S'(X', Y') - \tilde{g}(X', Y') \big( \tilde{\Delta}_\phi + n' ||\tilde{\nabla}_X \phi||^2 \big) \]

and the scalar curvature
\[ \tilde{s} = s + e^{-2\phi} s' - n' \big( n' ||\tilde{\nabla}_X \phi||^2 + \tilde{\Delta}_\phi \big) \]

In the special case where \( M \) is 1–dimensional and \( d\tilde{s}^2 = d\rho^2 + \rho^2 ds'^2 \), all the components of the Riemann tensor vanish except for
\[ \tilde{R}(X', Y') Z' = R'(X', Y') Z' + (g'(X', Z') Y' - g'(Y', Z') X') \]

As a check, taking the \( n' \)–sphere with its canonical metric for \( M' \), the two terms in the last expression cancel in accordance with \( \tilde{M} = \mathbb{R}^{n'+1} \). We note that rescaling the metric on the sphere by a factor distinct from unity turns \( \tilde{M} \) into a revolution cone with a given deficit angle at the apex, and consequently the cancellation does not take place anymore.
References


