Spherical Harmonic Expansion of Gamma Ray Burst Distributions: Probing Large Scale Structure?

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ABSTRACT

We have laid down the formalism and techniques necessary for computing the multipole components in a spherical harmonic expansion for bursting sources given any specific power spectrum of density perturbations. Using this formalism we have explicitly computed and tabulated the expected first few multipole components for the GRB distribution using a Cold Dark Matter power spectrum. Unfortunately, our analysis leads us to expect that an anisotropy signal for this model of structure formation will be below the shot noise level for the foreseeable future. Earlier studies by others had claimed that it may be possible within a decade or so to probe structure formation using the angular distribution of GRBs. We find that while it may be possible to probe the dipole due to our motion with respect to the cosmic rest-frame, it does not seem feasible to probe the multipole components due to intrinsic fluctuations if the actual power spectrum is fixed to within an order of magnitude by the Cold Dark Matter model of structure formation.
1. Introduction

The isotropy and paucity of weak bursts suggest that Gamma-ray bursts (GRBs) are most likely cosmological. Analysis of the number vs. peak flux distribution (Cohen & Piran, 1995; Fenimore et al., 1993) show that the bursts originate from $z \approx 1$. This suggests that GRBs could be applied as a tool for studying cosmology and the large scale structure of the Universe. If GRBs are cosmological, they constitute a homogeneous population with a narrow luminosity function (the peak luminosity of GRBs varies by less than factor of 10 (Cohen & Piran, 1995; Horack & Emslie, 1994; Horack, Emslie & Meegan, 1994; Ulmer, Wijers & Fenimore, 1995; Ulmer & Wijers, 1995) that is located at relatively high redshifts (Piran, 1992; Mao, & Paczyński, 1992; Wickramasinghe et al., 1993; Petrosian, 1993; Cohen & Piran, 1995). The universe and our Galaxy are transparent to MeV range $\gamma$-rays (see e.g. Zdziarski & Svensson, 1989). Hence GRBs constitute a unique uniform population which does not suffer from any angular distortion due to absorption by the galaxy or by any other object. Additionally, GRBs are sampled almost uniformly over $4\pi$ steradians. The information on matter fluctuations in the Universe is divided to local information obtained from red-shift surveys that span distances up to $z \approx 0.05$ or less and to information from the CMBR fluctuations that originate at $z \approx 1000$. The recent finding of Kollat & Piran (1996) that GRBs are correlated with Abell clusters suggest that GRBs do follow the matter distribution and hence they could provide a unique information about the distribution of matter at $z \approx 1$ and on scales that, at present, cannot be explored in any other way.

Recently, Lahav, Piran & Treyer (1996) developed a scheme for estimating the expected spherical harmonics that describe the fluctuations in the intensity of the x-ray background. We modify this scheme for estimating the spherical harmonics in the number of bursting sources and apply the modified scheme to the GRB distribution. There are two sources for such fluctuations: fluctuations in the number of sources that arises from fluctuations in the
matter density or from other effects such as the Sachs-Wolfe effect or the Compton-Getting effect due to the motion of the earth and random fluctuations that arises from Poisson noise. We estimate the expected fluctuations in the GRBs’ angular distribution that arise from both sources and we compare these theoretical estimates with recent and future observations. The outline of this paper is the following: In section 2. we develop the general formalism for estimating the spherical harmonics in a general population of bursting sources whose density fluctuates proportionally to the fluctuation in the matter density. Using this formalism we estimate in section 3. the number of GRBs needed in order that the real angular deviations of the GRB population will be larger than the Poisson noise. In section 4. we estimate the fluctuations due to the Sachs-Wolfe effect and the Compton-Getting effect. In section 5. we compare our results with previous works and with observational estimates.

2. The Formalism

We consider a general population of bursting sources that originate at cosmological distances and follow the matter distribution. The formalism is general but since we apply it to gamma-ray bursts (GRBs) we call the sources for simplicity GRBs in the rest of the discussion. We expand the number counts of GRBs in a given direction \( \hat{r} \) over the sky, \( \sigma(\hat{r}) \) in spherical harmonic expansion:

\[
\sigma(\hat{r}) = \sum_{lm} a_{lm} Y_{lm}(\hat{r})
\]  

(1)

where

\[
a_{lm} = \sum_{\text{sources}} Y_{lm}^{*}(\hat{r}_1)
\]  

(2)

To compare the observed distribution with a theoretical one We first estimate the number of observed bursts in a given direction \( N(\hat{r}) \). To do so we must made several
assumption on the nature of the distribution of GRBs. We first assume that there is a linear biasing between the density of GRB sources and the mass fluctuation

$$\delta_x(r_c, \hat{r}) = b_\gamma \delta(r_c, \hat{r}), \quad (3)$$

where $b_\gamma$ is the bias factor between the density perturbations and the GRB sources, $\delta(r) \equiv \delta \rho / \rho(r)$ is the density perturbation and $r_c$ is the comoving distance. We also assume that the rate of bursts per comoving volume per unit proper time evolves with redshift as $(1 + z)^p$. At present $p$ is unknown (see e.g. Cohen & Piran, 1995). Finally, we must consider the detection procedure of the specific detector. BATSE, for example, is sensitive to $C$ the number of photons that arrive on a collecting area $A$, within a given period of time, $\Delta T$, in a given energy interval centered around $\bar{E}$:

$$C = \frac{L}{4\pi r_c^2 (1+z)^{1+\alpha}} \frac{A\Delta T}{\bar{E}}, \quad (4)$$

where $\alpha$ is the slope of the spectrum.

In general there exists a sensitivity function, $\Phi(C)$, that describes the detectability of a burst with a count rate $C$. Generally, $\Phi(C) = 1$ for $C \gg C_{\text{min}}$ and $\Phi(C) = 0$ for $C \ll C_{\text{min}}$, where $C_{\text{min}}$ is the critical sensitivity of the detector. We will usually assume for simplicity that the detector behaves like a Heaviside function and $\Phi(C) = 1$ for $C \geq C_{\text{min}}$ and $\Phi(C) = 0$ for $C < C_{\text{min}}$. In this case there will be a maximal comoving distance $r_{c,\text{max}}$ from which a burst is detected for which $C(r_{c,\text{max}}) = C_{\text{min}}$.

Using this assumptions we find that:

$$\mathcal{N}(\hat{r}) = \int_0^\infty dL \Phi(L) \int r_c^2 dr_c \ n(1+z)^{(p-1)}[1 + b_\gamma \delta(r_c, \hat{r})] \Phi[C(L, r_c)], \quad (5)$$

where $r_c$ is the comoving distance, $\Phi(L)$ is the GRB luminosity function, $n$ is an overall normalization such that for an observation period $T_{\text{obs}}$, $n(1+z)^p/T_{\text{obs}}$ equals the effective density (per unit comoving volume per unit observer time) of bursts. Eq. 5 differs from
the corresponding equation for the x-ray background (Lahav, Piran & Treyer, 1996) in two ways. First the x-ray background equation, in which we estimate the fluctuations in the flux, is weighted by the flux from a given source. There is no such weight here since we are interested in numbers of sources. Second, there is additional factor of \((1 + z)^{−1}\) in this equation since we are interested in bursting sources, while for the x-ray background we were considering steady state sources. The predicted harmonic coefficients are:

\[
a_{lm} = \int d\hat{r}N(\hat{r})Y_{lm}^*(\hat{r}) =
\int d\hat{r} \int_0^\infty dL \Phi(L) \int r_c^2 dr_c \ n(1 + z)^{(p-1)}[1 + b_1 \delta(r_c, \hat{r})] \Phi[r_{c,max}(L) - r_c] \ Y_{lm}^*(\hat{r}),
\]

(6)

If \(N\) is the total number of observed bursts, we can use the monopole \((l = 0)\), with \(Y_{00} = (4\pi)^{-1/2}\), to fix the normalization factor \(n\) in the following way:

\[
N = (4\pi)^{1/2}a_{00}.
\]

(7)

For the special case of standard candles, \(\Phi(L) = \delta(L - L_0)\), with no source evolution \((p = 0)\) we have

\[
N = \frac{4\pi r_{c,max}^3}{3} \left[ 1 + \frac{3H_0^2 r_{c,max}^2}{20c^2} - \frac{3H_0 r_{c,max}}{4c} \right] n,
\]

(8)

where \(H_0\) is the Hubble constant and \(r_{c,max}\) is the maximal comoving distance from which a GRB with luminosity \(L_0\) can be detected. Thus, for an Einstein-de Sitter universe we have:

\[
n = \frac{3N}{32\pi} \left( \frac{H_0}{c} \right)^3 \left[ 1 - \frac{1}{\sqrt{1+z_{max}}} \right]^3 \left[ 0.1 + \frac{0.6}{(1+z_{max})} + \frac{0.3}{\sqrt{1+z_{max}}} \right],
\]

(9)

where \(z_{max}(L) = z(r_{c,max}(L))\) is the maximal redshift from which a GRB with luminosity \(L\) can be observed.

The fluctuations in the background are expressed by higher harmonics \(l > 0\):

\[
a_{lm} = n \int dL \Phi(L) \int d\hat{r} \int r_c^2 dr_c \ \Phi[C(L, r_c)] b_q \delta(r_c, \hat{r}) \ Y_{lm}^*(\hat{r})(1 + z)^{p-1}.
\]

(10)
Expressing the fluctuations in the matter density in terms of the Fourier components $\delta_k$ (see Eq. 4) and using the orthogonality of Spherical Harmonics $\int d\omega Y_{lm}(\hat{\mathbf{r}})Y_{l'm'}^*(\hat{\mathbf{r}}) = \delta^m_{m'}$ we find:

$$a_{lm} = \frac{1}{2\pi^2} \int d^3k b_\gamma \delta_k(z) Y_{lm}(\mathbf{k}) \int dL \Phi(L) \int r_c^2 dr_c \Phi[C(L, r_c)] j_l(kr_c)(1 + z)^{p-1}. \quad (11)$$

We are interested in sources at $z \approx 1$ and at large wavelengths. The relevant fluctuations that contribute to this integral are still in the linear regime. This means that:

$$\delta_k(z) = \delta_{k0}/(1 + z) \quad (12)$$

where $\delta_{k0}$ is the present linear amplitude of fluctuations normalized to fit the CMB observations. Eq. 12 is gauge dependent (in some gauges perturbation that are larger than the horizon do not grow). However, it can be checked numerically that the contribution of very large scale perturbations is small and hence this gauge choice is not important.

Taking the mean-square values and using Parseval’s theorem and

$$\langle \delta_k(z) \delta_{k'}^*(z) \rangle = (2\pi)^3 P(k, z) \delta^{(3)}(k - k') \quad (13)$$

we obtain

$$\langle|a_{lm}|^2\rangle = \frac{32}{\pi} b_\gamma^2 \left(\frac{c}{H_0}\right)^6 n^2 \int dk k^2 P(k) |\Psi_l(k)|^2. \quad (14)$$

The function $\Psi_l(k)$ is defined (for an Einstein-de Sitter universe) as:

$$\Psi_l(k) \equiv \int dL \Phi(L) \int_0^{z_{max}} dz (1 + z)^{p-7/2} \left[1 - \frac{1}{\sqrt{1 + z}}\right]^2 j_l(kr_c)\Phi[C(L, z)]. \quad (15)$$

### 3. Estimates for GRBs

We consider standard candles GRBs with no density evolution i.e. $p = 0$. For simplicity we use in this section $\Phi(C) = 1$ for $C \geq C_{min}$ and $\Phi(C) = 0$ otherwise. The power-spectrum
for CDM can be written down in the form (Padmanabhan, 1993):

\[ P(k) = \frac{A k^n}{(1 + B k + C k^{3/2} + D k^2)^2}, \]

(16)

For a scale-invariant or Harrison-Zel’dovich spectrum, \( n = 1 \). Further, by fitting to available data we get the best-fit values of the parameters as follows: \( A = (24 \text{Mpc})^4 \), \( B = 1.7 \text{Mpc} \), \( C = 9 \text{Mpc}^{3/2} \) and \( D = 1 \text{Mpc}^2 \). Let us scale out the dimensions of \( k \) in the following way,

\[ \tilde{k} \equiv k(100 \text{Mpc}) \]

(17)

By doing this rescaling the expression for \( \langle |a_{lm}|^2 \rangle \) can be written completely in terms of dimensionless numbers in the following form:

\[ \langle |a_{lm}|^2 \rangle = \frac{(3.0 \times 10^{-5}) N^2}{\left[ 1 - \frac{1}{\sqrt{1 + z_{\text{max}}}} \right]^6 [0.1 + \frac{0.6}{(1 + z_{\text{max}})} + \frac{0.3}{\sqrt{1 + z_{\text{max}}}}]^2} b^2 \int d\tilde{k} \frac{\tilde{k}^3 |\Psi_l(k)|^2}{D(\tilde{k})} \]

(18)

where

\[ D(\tilde{k}) = (1 + 0.017 \tilde{k} + 0.009 \tilde{k}^{3/2} + 0.0001 \tilde{k}^2)^2. \]

(19)

Thus, the quantity which needs to be estimated numerically is:

\[ \tilde{I}(l, z_{\text{max}}) \equiv \int d\tilde{k} \frac{\tilde{k}^3 |\Psi_l(k)|^2}{D(\tilde{k})} \]

(20)

Let us now obtain an order of magnitude estimate of the above effect and see whether this is going to be detectable in the near future (Tegmark et al., 1995; 1996). Let us re-write the earlier equation 18 in a form convenient for tabulation of values for various values of \( z_{\text{max}} \)

\[ \langle |a_{lm}|^2 \rangle = p(z_{\text{max}}) N^2 b^2 \tilde{I}(l, z_{\text{max}}) \]

(21)

Here \( p(z_{\text{max}}) \) is the numerical pre-factor which depends on \( z_{\text{max}} \), which we shall tabulate below.
TABLE 1

<table>
<thead>
<tr>
<th>$z_{\text{max}}$</th>
<th>0.2</th>
<th>0.5</th>
<th>1.0</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(z_{\text{max}})$</td>
<td>$9.0 \times 10^4$</td>
<td>1.4</td>
<td>$1.3 \times 10^{-1}$</td>
<td>$2.4 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

One can estimate the magnitude of $\bar{I}(l, z_{\text{max}})$ for various values of $l$.

TABLE 2

<table>
<thead>
<tr>
<th>$z_{\text{max}}$</th>
<th>0.2</th>
<th>0.5</th>
<th>1.0</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{I}(1, z_{\text{max}})$</td>
<td>$3.09 \times 10^{-9}$</td>
<td>$8.20 \times 10^{-9}$</td>
<td>$1.064 \times 10^{-8}$</td>
<td>$1.036 \times 10^{-8}$</td>
</tr>
<tr>
<td>$\bar{I}(2, z_{\text{max}})$</td>
<td>$3.90 \times 10^{-9}$</td>
<td>$1.061 \times 10^{-8}$</td>
<td>$1.439 \times 10^{-8}$</td>
<td>$1.487 \times 10^{-8}$</td>
</tr>
<tr>
<td>$\bar{I}(3, z_{\text{max}})$</td>
<td>$4.72 \times 10^{-9}$</td>
<td>$1.310 \times 10^{-8}$</td>
<td>$1.826 \times 10^{-8}$</td>
<td>$1.949 \times 10^{-8}$</td>
</tr>
<tr>
<td>$\bar{I}(4, z_{\text{max}})$</td>
<td>$5.51 \times 10^{-9}$</td>
<td>$1.559 \times 10^{-8}$</td>
<td>$2.214 \times 10^{-8}$</td>
<td>$2.411 \times 10^{-8}$</td>
</tr>
<tr>
<td>$\bar{I}(5, z_{\text{max}})$</td>
<td>$6.29 \times 10^{-9}$</td>
<td>$1.804 \times 10^{-8}$</td>
<td>$2.598 \times 10^{-8}$</td>
<td>$2.868 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

We, of course have to compare the magnitude of the signal given in the previous equation to the shot noise. The magnitude of the shot noise in our case is given by:

$$\langle |a_{lm}|^2 \rangle_{\text{sn}} = \frac{1}{4\pi} \int dV_c \frac{n}{(1+z)} = \frac{N}{4\pi}$$  \hspace{1cm} (22)

where $N$ is the number of bursts observed. Let us now estimate what is the number of bursts one needs to get to observe a signal above noise. This means we want,

$$p(z_{\text{max}}) N^2 b_\gamma^2 \bar{I}(l, z_{\text{max}}) > \frac{N}{4\pi}$$  \hspace{1cm} (23)

We can obtain an estimate for the number of bursts required to get a signal above noise for any desired assumption about the location of the most distant sources of GRBs.
In particular let us obtain the required number of bursts if we assume the most distant sources of GRBs is at $z_{max} = 0.2$. We start at Eq. 21 and use the value of $p(z_{max})$ given in Table 1 to get the relation,

$$\langle |a_{lm}|^2 \rangle = N^2 (9.0 \times 10^1) b_\gamma^2 \bar{I}(l, 0.2)$$

(24)

One can numerically compute the magnitude of $\bar{I}(l, 0.2)$ for various values of $l$. We did this for $l = 1$ to $l = 5$. The magnitude does not change significantly and is given by, $\bar{I} \simeq 4 \times 10^{-9}$. Thus the relevant inequality now becomes,

$$N^2 (9.0 \times 10^1) b_\gamma^2 \bar{I}(l, 0.2) > \frac{N}{4\pi}$$

(25)

Thus if the most distant GRBs are at $z_{max} = 0.2$ then the number of bursts required to get a signal over noise must be given by $N > 2 \times 10^5$. In general, if the most distant GRBs are at $z_{max}$ then the number of bursts required to get a signal over noise must be given by $N > N_c(l, z_{max}) b_\gamma^{-2}$, where

$$N_c(l, z_{max}) = \left[ 4\pi p(z_{max}) \bar{I}(l, z_{max}) \right]^{-1}$$

(26)

Below we tabulate the Number of bursts, $N_c(l, z_{max})$ required to observe the multipole component $l$ for a given maximum redshift of sources $z_{max}$:

TABLE 3
These results are perhaps discouraging because it means that the expected signal is not detectable in the foreseeable future. However, given the fact that there are ongoing efforts to measure the multipole components of the GRB distributions (Tegmark et al., 1995; 1996), this is still a significant result. It means that if a signal is detected it is not due to the effect discussed here by us and an alternative explanation must then be sought for the signal.

4. Sachs-Wolfe and Compton-Getting

We turn now to fluctuations in the redshift that influence the observed count rate and induce fluctuations in the number density given by Eq. (5). So far we have introduced \( n = n_0 + \delta n \) where \( n_0 \) is the background density and \( \delta n \) are the fluctuations and we were interested in those fluctuations (we assumed that \( \delta n = b_\gamma n_0 \delta \rho \) where \( b_\gamma \) is a biasing factor and \( \delta \rho \) the fluctuations in the matter density. Now we consider fluctuations in the count rate \( C \) and in the redshift factor \( (1 + z) \) that arise due to Doppler motions of the sources and the observer \( (V_0 - v_{||}) \) (where \( V_0 \) is our motion and \( v_{||} \) is the motion of the source), or due to fluctuations in the gravitational potential \( \delta \phi \). For simplicity we discuss in this section a standard candle population. The generalization to a luminosity function is trivial.

The variation in the factor of \( (1 + z) \) causes fluctuations in the time dilation between the sources and the observer and hence fluctuations in the observed rate of events from
different regions of space-time. These lead to the following fluctuations in the angular count rate:

\[ \delta N_1 = \int r^2 dr c n(1+z)^{p-1}[-(V_0 - v_{||})/c - \delta \phi/3c^2]. \]  

(27)

The first term \(V_0\) is a constant vector that gives rise, just like in the CMBR, to the Dipole and a second order quadrupole term:

\[ \frac{\Delta N_{d1}}{N} = \frac{V_{obs}}{c} \cos(\theta) + \left( \frac{V_{obs}}{c} \right)^2 \left[ 1 - \cos^2(\theta)/2 \right]. \]  

(28)

Note that for sources that are at \(z \approx 1\) the intrinsic fluctuations discussed earlier (Eq. (21) and Tables 1 and 2) give a dipole at the \(10^{-4}\) level. The Doppler dipole discussed above in Eq. (28) is at a few times \(10^{-3}\) level. Thus the intrinsic dipole is comparable to but about an order of magnitude below the Doppler dipole. However, by examining the same equations and tables as above we see that the intrinsic quadrupole is clearly much greater than the Doppler quadrupole. In general the higher Doppler multipole components are suppressed by increasing powers of \((V_{obs}/c)\) whereas the higher multipole moments of the intrinsic fluctuations are of the same order as the lower ones (see Table 2).

The second and third terms, \(v_{||}\) and \(\delta \phi\), reflect the fluctuations of the velocity field and the gravitational field. Like in the discussion of the density fluctuations we turn to Fourier space. Using:

\[ \delta \phi_k = \frac{3}{2} \frac{H_0^2 a^2}{k^2} \delta_k(z) = \frac{3}{2} \frac{H_0^2}{k^2} (1+z) \frac{\delta_k(z)}{k^2} \]  

(29)

for \(\delta \phi\) and

\[ v_{||} = Ha \frac{\delta_k(z)}{k} = H_0 \sqrt{1 + z} \frac{\delta_k(z)}{k} \]  

(30)

for the line-of-sight velocity \(v_{||}\), we obtain:

\[ \frac{\delta \phi/c^2}{2c^2} = \frac{3H_0^2}{2c^2} (1+z) \frac{1}{2\pi^2} \sum_{lm} (i^l)^* Y_{lm}^*(\hat{r}) \int d^3k \frac{\delta_k(z)}{k^2} Y_{lm}(\hat{k}) j_i(kr) \]  

(31)

and (Fisher, Scharf & Lahav, 1994)

\[ \frac{v_{||}(\hat{r})}{c} = \frac{H_0}{2\pi^2 c} \sum_{lm} (i^l)^* Y_{lm}^*(\hat{r}) \int d^3k \frac{\delta_k(z)}{k} Y_{lm}(\hat{k}) j_i(kr). \]  

(32)
Substitution of Eqs. 31 and 32 into 27 yields:

$$\delta N_1(\hat{r}) = \frac{1}{2\pi^2} \int r_c^2 dr_c n(1+z)^{p-1} \sum_{lm} (i^l)^* Y_{lm}^*(\hat{r}) \left[ \frac{H_0^2}{2} \int d^3k \frac{\delta_k}{k^2 c^2} Y_{lm}(\hat{k}) j_l(kr) + \frac{H_0}{\sqrt{1+z}} \int d^3k \frac{\delta_k(z)}{kc} Y_{lm}(\hat{k}) j'_l(kr) \right].$$

To calculate the corresponding $a_{lm}$ we expand $\delta N(\hat{r})$, expressed in Eq. 33:

$$a_{lm,1} = \frac{(i^l)^*}{2\pi^2} n \int d^3k Y_{lm}(\hat{k}) \delta_{k0} \left[ \frac{H_0^2}{2k^2 c^2} \Psi_{l,sw} + \frac{H_0}{kc} \Psi_{l,v} \right],$$

where

$$\Psi_{l,sw}(k) \equiv 4\left(\frac{c}{H_0}\right)^3 \int_0^{z_{max}} dz (1+z)^{p-5/2} \left[ 1 - \frac{1}{\sqrt{1+z}} \right]^2 j_l(kr_c),$$

and

$$\Psi_{l,v}(k) \equiv 4\left(\frac{c}{H_0}\right)^3 \int_0^{z_{max}} dz (1+z)^{p-4} \left[ 1 - \frac{1}{\sqrt{1+z}} \right]^2 j'_l(kr_c).$$

The fluctuations in the gravitational field and in the velocity field also influence the detectability of bursts that are marginally detectable and this leads to another effect. The BATSE detector is sensitive to $C$ the number of photons in a given energy interval that arrive to the detector in a given period of time. The count rate $C$ varies due to variation in $z$ which arise from the Doppler effect and the gravitational redshift. Using Eq. 4 we find:

$$\frac{\partial C}{\partial z} = -(1+\alpha)C \left(1 + \frac{1}{\sqrt{1+z}} \right).$$

Substitution of Eq. 37 to Eq. 5 and linearizing we obtain:

$$\delta N_2 = \int_0^\infty n(1+z)^{p-1} \frac{\partial \Phi}{\partial C} (1+\alpha)C[-(V_0 - v_{||})/c - \delta \phi/3c^2] r_c^2 dr_c.$$
We can express the volume element as a Jacobian times $dC$. Thus, we arrive at the following trivial integral for $\delta N_2$:

$$
\delta N_2 = \left( \frac{c}{H_0} \right) \int_0^\infty \frac{n}{(1+z)} \delta(C - C_{\text{min}}) \left[ \frac{(V_0 - v_\parallel)}{c} + \frac{\delta \phi/3c^2}{r_c^2} \right] \frac{1}{\sqrt{1+z}} dC = 
$$

$$
\left( \frac{c}{H_0} \right) \frac{n}{(1+z_{\text{max}})^{3/2}} \left[ \frac{(V_0 - v_\parallel)(z_{\text{max}})}{c} + \frac{\delta \phi(z_{\text{max}})/3c^2}{r_c^2(z_{\text{max}})} \right].
$$

(39)

The quantities: $v_\parallel(z_{\text{max}})$, $\delta \phi(z_{\text{max}})$ and $r_c^2(z_{\text{max}})$ are all calculated on the surface $z_{\text{max}}$ which corresponds to $C_{\text{min}}$ with our standard candle luminosity.

The contribution from $V_0$ is again a dipole and a second order quadrupole term:

$$
\Delta N_{d2} = \frac{(V_{\text{obs}}/c) \cos(\theta) + (V_{\text{obs}}/c)^2 [1 - \cos^2(\theta)/2]}{\sqrt{1 + z_{\text{max}} - 1} [0.1(1 + z_{\text{max}}) + 0.3\sqrt{1 + z_{\text{max}} + 0.6}]}.
$$

(40)

We discuss the magnitude of these terms as well as those given in Eq. 28 in the next section.

The fluctuation terms can be dealt in the same manner that we have used earlier. First, we substitute Eqs. 31 and 32 into Eq. 38. Then, after multiplying by $Y_{lm}(\hat{r})$ and integrating over $d\Omega$ we get:

$$
a_{lm,2} = \frac{(i)^*}{2\pi^2} n \int d^3k \ Y_{lm}(\hat{k}) \delta_{k0} \left[ \frac{H_0^2}{2k^2c^2} \Psi_{l,sw,S} + \frac{H_0}{kc} \Psi_{l,v,S} \right],
$$

(41)

where:

$$
\Psi_{l,sw,S}(k) \equiv 4 \left( \frac{c}{H_0} \right)^3 (1 + z_{\text{max}})^{p-3/2} \left[ 1 - \frac{1}{\sqrt{1 + z_{\text{max}}}} \right]^2 j_l(kr_c),
$$

(42)

and

$$
\Psi_{l,v,S}(k) \equiv 4 \left( \frac{c}{H_0} \right)^3 (1 + z_{\text{max}})^{p-3} \left[ 1 - \frac{1}{\sqrt{1 + z_{\text{max}}}} \right]^2 j_l'(kr_c).
$$

(43)

We can combine now all fluctuating terms (Eqs. 11, 34 and 41) to a single equation for $a_{lm}$ which resembles Eq. 11 with $\Psi_l$ replaced by $\Psi_l + \Psi_{l,sw} + \Psi_{l,sw,S} + \Psi_{l,v} + \Psi_{l,v,S}$:

$$
a_{lm} = \frac{(i)^*}{2\pi^2} n \int d^3k \ Y_{lm}(\hat{k}) \delta_{k0} \left[ b_\gamma \Psi_l + \frac{H_0^2}{2k^2c^2} (\Psi_{l,sw} + \Psi_{l,sw,S}) + \frac{H_0}{kc} (\Psi_{l,v} + \Psi_{l,v,S}) \right].
$$

(44)
Now, we continue and calculate $\langle a_{lm} a_{lm}^* \rangle$ taking the mean-square values and using Parseval’s theorem and Eq. 13:

$$\langle |a_{lm}|^2 \rangle = \frac{32}{\pi} \left( \frac{c}{H_0} \right)^6 n^2 \int dk k^2 P(k) \left| b_n \Psi_l + \frac{H_0^2}{2k^2c^2} (\Psi_{l,sw} + \Psi_{l,sw,S}) + \frac{H_0}{kc} (\Psi_{l,v} + \Psi_{l,v,S}) \right|^2.$$ \hspace{1cm} (45)

¿From the above expressions we can calculate the most general r.m.s. amplitude $\langle a_{lm} a_{lm}^* \rangle$. It is worthwhile, however, to estimate the order of magnitude of the relevant terms. To do so we recall that if $\lambda$ is the length scale being probed then (Padmanabhan, 1993):

$$\frac{\delta \phi}{c^2} \simeq \left( \frac{\delta \rho}{\rho} \right) \lambda \left( \frac{H \lambda}{c} \right)^2,$$ \hspace{1cm} (46)

and

$$\left( V_0 - v_{||} \right)/c \simeq \left( \frac{\delta \rho}{\rho} \right) \lambda \left( \frac{H \lambda}{c} \right).$$ \hspace{1cm} (47)

These estimates suggest that the Sachs-Wolfe terms in Eq. 44 are smaller than the intrinsic fluctuation term by a factor of $(H \lambda/c)^2$. The velocity terms are smaller than the intrinsic fluctuation by a factor of $H \lambda/c$.

Comparison of the surface terms in the Sachs-Wolfe and the velocity terms to the volume integral show that these terms are comparable for small $z_{max}$ and become smaller as $z_{max}$ becomes larger. This is intuitively explained by the fact that at lower $z_{max}$ there are relatively more faint bursts than at higher $z_{max}$.

¿From our expressions it is clear that there is a systematic expansion in terms of the parameter $(H_0/kc)$. These results show that the Sachs-Wolfe and the velocity terms can be ignored if one is interested in the lowest order effect and an order of magnitude calculation as we have been interested here. The difference between GRBs and CMBR is that the first arise from sources at relatively low red-shift, for which the intrinsic fluctuations have grown
and are much more important than the fluctuations induced by a varying gravitational field or from the random peculiar velocity of the sources.

5. Discussion

We have laid down the formalism and techniques for evaluating the expected multipole components of the GRB count distribution. The largest term, which is a dipole of order $10^{-2}$, almost two orders of magnitude larger than all other terms arises due to the Compton-Getting effect resulting from our motion relative to the GRB distribution. This term is not new. It was first calculated by Maoz (1994) who estimated to be of order $\approx 10^{-2}$. Later Scharf et al. (1995) repeated the calculations and have considered also the fluence weighted dipole (which we don’t consider here).

Smaller multipole components arise from intrinsic fluctuations of the source distribution that are related to the power spectrum of density perturbations. We have explicitly computed the first few multipole components for a Cold Dark Matter power spectrum. These are in fact the very multipole components that are least likely to be affected by the positional inaccuracy in locating GRBs. We feel that the expression of the fluctuations in the GRB distribution in terms of a multipole moments is particularly useful in view of the general usage of these moments to express fluctuations in the CMBR (Bond & Efstathiou, 1987) and other background fields (Peebles, 1973; Lahav, 1994; Fisher, 1993; Heavens & Taylor, 1994; Lahav, Piran & Treyer, 1996). There have now been serious and exhaustive efforts (Tegmark et al., 1995; 1996) to examine the deviation from isotropy of GRBs using an expansion in terms of spherical harmonics. Further, this observational program is likely to continue into the future and so a calculation of the expected signal in terms of various multipole components is a useful calculation to do.
Our work would certainly be incomplete if we failed to mention the earlier work of Lamb & Quashnock (1993) who were the first to address the issue of whether it is possible to probe Large Scale Structure using the distribution of Gamma Ray Bursts, and did not compare our results to their analysis and conclusions. LQ use the angular two-point correlation function to examine the expected deviations from isotropy, whereas we have used an expansion in terms of spherical harmonics. While undoubtedly both formalisms have their usefulness we have already mentioned that we do feel that an expansion in terms of spherical harmonics is better suited to address the issue of deviations from isotropy. However, having said all that in favor of doing a spherical harmonic expansion of the GRB distribution it still remains a true statement that such conclusions as the number of bursts required to see an anisotropy signal over the noise should be basis independent.

We now turn to comparing our results about the number of bursts required to see a signal above noise to those obtained by LQ. The central observational quantity in the formalism that LQ use is the mean of the angular two-point correlation function, \( \bar{w}(\theta) \). The statistical uncertainty in \( \bar{w}(\theta) \) is denoted by \( \delta \bar{w}(\theta) \). In order to detect a signal it is therefore necessary to require \( \delta \bar{w}(\theta)/\bar{w}(\theta) < 1 \). The above inequality can be transformed into an inequality which determines the minimum number of bursts required to detect a signal due to the actual anisotropy of the distribution as opposed to the expected statistical fluctuations. This bound is written down by LQ as

\[
N_b \geq \frac{2\pi^2 \sqrt{8} D^3}{C I(k_o) P(k_o)},
\]

where \( N_b \) is the number of bursts, \( D \) is the distance to the edge of the source distribution, \( k_o \) is the largest wavenumber that can be probed, \( P(k) \) is the power spectrum of density perturbations, \( C \) is a constant introduced by LQ and \( I(k_o) \) is the following integral introduced by LQ,

\[
I(k_o) = \int_{k_o}^\infty \frac{dk}{k} (kD)^2 W(k) .
\]
Here $W(k)$ is a window function introduced by LQ. While LQ are somewhat ambiguous about the exact numerical value of the window function used to arrive at their results, they do argue that in the regime $D^{-1} < k < (D\theta)^{-1}$ the window function can be computed analytically tending for large values of $kD$ to $\frac{5.64}{kD}$. Further, they argue that most of the contribution to the integral, $I(k_o)$ comes from wavenumbers $k \sim k_o$. Using these inputs we obtain an order of magnitude estimate for $I(k_o)$ to be

$$I(k_o) \simeq 5.64k_oD.$$  \hspace{1cm} (50)

We also need to input the power spectrum to extract the bound on the number of bursts for a detectable signal. To make a fair comparison let us do this along the lines suggested by Figure 1 of LQ. Thus we take, $k_o = 0.1hMpc^{-1}$ and $P(k_o) \simeq 5 \times 10^2h^3Mpc^3$. LQ also argue that $C \simeq 1.5$ in order for their convolution to match those from Limber equation for small values of $\theta$. Further they choose $D = 1h^{-1}Gpc$ for illustrative purposes. Note that this corresponds to $z_{max} \simeq 0.3$. Putting in all these values we get the bound on the number of bursts, $N_b$ for a detectable signal to be,

$$N_b \geq 2 \times 10^5.$$ \hspace{1cm} (51)

This result agrees in order of magnitude with our results for comparable $z_{max}$. Thus we see that within the context of our analysis both the results of the LQ formalism and the spherical harmonic expansion agree. However, there is one claim made by LQ with which we disagree. Namely, they claim that if an anisotropy signal is not detected with $\sim 3000$ bursts then it would cast a doubt on the cosmological origin of GRBs. Such a statement we think is unwarranted. Certainly if one stays within the context of the Cold Dark Matter models whose spectrum we have used in our analysis and the maximum redshift from which GRBs originate $z_{max} \geq 0.2$ there would be no reason to expect an anisotropy signal above that of shot noise with only $\sim 3000$ bursts.
Let us now summarize our results in the light of the observational situation. We first note that BATSE has accumulated $\sim 1000$ bursts in 3 years of operation. If it continues operating for a period of 10 years it will accumulate $\sim 3000$ bursts. In light of our results displayed in Table 3 this means that if GRBs originate at redshifts $z_{\text{max}} \geq 0.2$, and the Cold Dark Matter power spectrum is right to within an order of magnitude, then we are unlikely to see an anisotropy signal in GRB distributions above the shot noise within the next 10 years or so. Given that there are now serious and exhaustive efforts to measure and analyze the spherical harmonic components of the GRB distribution (Tegmark et al., 1995; 1996), we think an accurate estimate of the signal expected from Large Scale Structure was necessary.

We have laid down in this paper the formalism and techniques necessary for computing the various multipole components in a spherical harmonic expansion for bursting sources given any specific power spectrum of density perturbations. We have further explicitly computed and tabulated the first few multipole components for GRBs distribution using a Cold Dark Matter power spectrum. Unfortunately, our analysis leads us to expect that an anisotropy signal for this model of structure formation will be below the shot noise level for the foreseeable future.

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Appendix: Fourier Preliminaries

Fourier decomposition of the density perturbations

\[ \delta(r) \equiv \frac{\delta \rho}{\rho}(r) = \frac{1}{(2\pi)^3} \int d^3k \, \delta_k \, e^{-i k \cdot r} \]  

(1)

and the inverse:

\[ \delta_k = \int d^3r \, \delta(r) \, e^{i k \cdot r} . \]  

(2)

Rayleigh expansion:

\[ e^{i k \cdot r} = 4\pi \sum_{lm} i^l j_l(kr) Y_{lm}(\hat{r}) Y_{lm}^*(\hat{k}) . \]  

(3)

Combining Eqs. (1) and (3):

\[ \delta(r) = \frac{1}{2\pi^2} \sum_{lm} (i^l)^* Y_{lm}(\hat{r}) \int d^3k \, \delta_k Y_{lm}(\hat{k}) \, j_l(kr) . \]  

(4)
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