Analysis of “Gauge Modes” in Linearized Relativity

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Abstract. By writing the complete set of 3+1 (ADM) equations for linearized waves, we are able to demonstrate the properties of the initial data and of the evolution of a wave problem set by Alcubierre and Schutz. We show that the gauge modes and constraint error modes arise in a straightforward way in the analysis, and are of a form which will be controlled in any well specified convergent computational discretization of the differential equations.
1. Introduction

The 3 + 1 (ADM [1]) form of the Einstein equations for vacuum may be written:

\[
R + K^2 - K_{cd}K^{cd} = 0 \tag{1}
\]

\[
\nabla_a K^a_b - \nabla_b K = 0 \tag{2}
\]

\[
\dot{\partial}_o g_{ab} = -2\alpha K_{ab} \tag{3}
\]

\[
\hat{\partial}_o K_{ab} = -\nabla_a \nabla_b \alpha + \alpha [R_{ab} - 2K_{al}K^l_b + KK_{ab}], \tag{4}
\]

where the derivative [2] \( \hat{\partial}_o \) is defined:

\[
\partial_o = \hat{\partial}_o + \mathcal{L}_\beta. \tag{5}
\]

Here \( \alpha \) is the lapse function, which relates the coordinate time between time instants to the proper time interval, \( \beta^i \) is a shift vector describing changes in the coordinization at one time and \( \mathcal{L}_\beta \) is the Lie derivative along \( \beta^i \). Because they are kinematical choices, we can, for some (perhaps small) interval, set \( \alpha \) and \( \beta^i \) arbitrarily. Here we take \( \alpha = 1, \beta^i = 0 \), following Alcubierre and Schutz, [3] who posed the problem we now define. We use the overdot and \( \partial_o \) interchangeably below to indicate the time derivative.

In Eqs. (1 - 4), all of the geometrical variables are 3-tensors. \( g_{ab} \) is the spatial 3 metric; \( K_{ab} \) is a spatial tensor (with \( \alpha = 1, \beta^i = 0 \), equal to \(-\frac{1}{2}g_{ab}\)); \( K \) is the 3-trace of \( K_{ab} \); \( R_{ab} \) is the 3-Ricci tensor obtained from \( g_{ab} \), and \( R \) is the trace of \( R_{ab} \). Linearizing the equation around flat space \( (g_{ab} = \delta_{ab} + h_{ab}; K_{ab} \ “small”) \) and introducing the traceless variables \( \dot{K}_{ab} = K_{ab} - \frac{1}{3}\delta_{ab}K \), and \( \dot{h}_{ab} = h_{ab} - \frac{1}{3}h_{cd}\delta^{cd}\delta_{ab} \), we have

\[
R = 0 \tag{6}
\]

\[
\dot{K}^{a}_b = \frac{2}{3}K_t^b \tag{7}
\]

\[
\partial_o \dot{h}_{ab} = -2\dot{K}_{ab} \tag{8}
\]

\[
\dot{h} = -2K \tag{9}
\]

\[
\partial_o \dot{K}_{ab} = R_{ab} - \frac{1}{3}\delta_{ab}R \tag{10}
\]

\[
\dot{K} = R. \tag{11}
\]
Here $R_{ab}$, the 3-Ricci tensor, is (for the linearized case):

$$R_{ab} = \frac{1}{2} \delta^{mn} (\ -h_{ab,mn} + h_{an,bm} + h_{bn,am} - h_{mn,ab})$$  \hspace{1cm} (12)

where the comma now denotes the partial derivative.

Formally, Einstein equations are a constrained hyperbolic system. Solution of the constraint equations (1 - 2) [or their linearized version (6 - 7)] will be preserved analytically in the evolution of the dynamical equations (3 - 4) [or their linearization]. By setting $\alpha = 1, \beta^i = 0$, we choose not to consider the admissible possibility of perturbations in $\alpha$ and $\beta^i$ of the same order as $h$. This is a coordinate choice (gauge choice). Further, there is the possibility to specify the gauge more completely at the perturbation level in a way that does not change $\alpha, \beta^i$. Alcubierre and Schutz [3], for instance, choose a transverse traceless gauge, $\delta^{mn} h_{mn} = 0$ and $\delta^{mn} \partial_m h_{ni} = 0$. Such a choice can be accomplished by a coordinate transformation at the original time; analytically this choice is preserved with $\alpha = 1, \beta^i = 0$ in the evolution. Gauges like this represent additional restrictions on the behavior of the solution; however they are not necessary to have a physically meaningful evolution. We shall see below that even in this linearized case, setting such gauges in computation is delicate.

2. “Gauge Modes”

Alcubierre and Schutz wrote the linearized evolution equations (8 - 11) in second order form without separating out the trace:

$$(-\partial_0^2 + \nabla^2)h_{ij} = \delta^{mn} (\partial_n \partial_i h_{jm} + \partial_n \partial_j h_{im} - \partial_i \partial_j h_{mn}).$$  \hspace{1cm} (13)

With this form, Alcubierre and Schutz[3] discovered secular, non-propagating, modes in their numerical implementation. In their attempt to evolve transverse traceless waves they find offending extra modes that violate the TT-gauge condition. Some of these modes appear to violate the constraints. By investigation of the properties of Eq. (13), Alcubierre and Schutz were able to show analytically the existence of modes similar to those seen numerically.
3. Preliminaries

From Eq. (12) for the linearized Ricci tensor, it is easy to obtain the linearized 3-scalar $R$:

$$ R = -\nabla^2 h + h^\text{mn},mn $$

and the linearized Hamiltonian constraint (6) sets this to zero.

A basic decomposition which we employ is:

$$ h_{mn} = h^{\text{TT}mn} + h^{\text{L}mn} + \frac{1}{3} \delta_{mn} h^* , $$

(15)

Here $h^{\text{TT}mn}$ is transverse:

$$ h^{\text{TT}mn},n = 0, $$

(16)

and traceless:

$$ h^{\text{TT}n} n = 0. $$

(17)

The quantity $h^{\text{L}mn}$ is longitudinal, and can be written:

$$ h^{\text{L}mn} = v_{m,n} + v_{n,m} $$

(18)

for some vector $v_m$. Notice that $h^{\text{L}mn}$ is not tracefree:

$$ h^{\text{L}m} m = 2v_m, m $$

(19)

(The construction (18 - 19) for $h^{\text{L}mn}$ is similar to that given by York[6] except that we do not subtract the trace from $h^{\text{L}mn}$). Additionally we posit an independent contribution to the trace, $h^*$. At this stage we distinguish $h_{mn}$, which need not be transverse, from $h^{\text{TT}mn}$, which is transverse.

Firstly, notice that $h^{\text{L}mn}$ annihilates the scalar curvature. This is not surprising because such longitudinal components arise from classic “gauge terms” as in Eq. (18).

Secondly, notice that if we set (only) nonvanishing $h^{\text{TT}mn}$ data, $R = 0$ becomes

$$ \nabla^2 h^* = 0. $$

(20)

In this study we are concerned with computational relativity, for “physically realistic” disturbances. In particular, we will expect fall-off “at infinity” (this can be
accomplished by a mixed outer boundary condition on the computational domain), and we exclude singularities in the interior of the domain. The solution to Eq. (20) then is \( h = 0 \). Hence, we expect the trace, \( h \), to arise only in the context of longitudinal components, \( h_{mn}^L \), as in Eq. (19).

4. Linearized Data Setting and Evolution

We continue the Alcubierre and Schutz analysis of the linearized case, but take more careful note of the constraint equations in the analysis. We will also give a complete data-setting analysis.

An immediate result from the Hamiltonian constraint \( R = 0 \) and from the second order form of the linearized evolution equations (8 - 11) is:

\[
\partial^2_t h = 0 \quad (!)
\]

Hence the trace of the metric perturbation is solved by

\[
h = a(x^i) + b(x^i)t,
\]

if one enforces the Hamiltonian constraint to this order. Notice that since \( K = -\frac{1}{2}\dot{h} \), we have \( K = -\frac{1}{2}b(x^i) \). Notice also that because \( h \) can only arise from longitudinal modes, we have found Alcubierre and Schutz’s growing modes:

\[
h_{mn}^L = A_{mn}^L(x^i) + tB_{mn}^L(x^i),
\]

However, also note that \( B_{mn}^L \) is related to \( K_{mn} \) and is restricted by the momentum constraint.

Similarly writing the second-order equation for \( \overset{\circ}{h}_{ab} \), we find:

\[
(-\partial^2_o + \nabla^2) \overset{\circ}{h}_{ij} = 2\partial_n \partial_i \overset{\circ}{h}^n_{\ j} - \frac{1}{3}\partial_i \partial_j h - \frac{1}{3}(2\partial_n \partial_m \overset{\circ}{h}^{mn} - \frac{1}{3}\nabla^2 h)\delta_{ij}.
\]

The last term (in parenthesis, proportional to \( \frac{1}{3}\delta_{ij} \)) in Eq. (24) is in fact part of the Ricci scalar, and again using the Hamiltonian constraint it equals \(-\frac{1}{3}\nabla^2 h\delta_{ij}\):

\[
(-\partial^2_o + \nabla^2) \overset{\circ}{h}_{ij} = 2\partial_n \partial_i \overset{\circ}{h}^n_{\ j} - \frac{1}{3}(\partial_i \partial_j + \delta_{ij} \nabla^2) h.
\]
Clearly, $TT$ data annihilates the righthand side of Eq. (25), so $h^{TT}_{mn}$ satisfies the source free wave equation. It is also straightforward to verify using Eq. (18) that $h^L_{mn}$ annihilates the sum of the spatial derivatives in Eq. (25), so that

$$\partial_o^2 h^L_{mn} = 0,$$  \hspace{1cm} (26)

consistent with the behavior we have already found for the trace. Thus this system is simply hyperbolic in the notation of reference [2]: disturbances travel at speed zero ($h^L_{mn}$), or unity ($h^{TT}_{mn}$). In order to proceed, we should also investigate the time derivative of the right side of (24). Let us proceed by taking note of one of the remarkable lessons learned from the hyperbolic analysis, for instance like that carried out in reference [2], that one should consider higher temporal derivatives of the evolution equations. For instance, an additional time derivative gives:

$$\left(-\partial_o^2 + \nabla^2\right) \tilde{K}_{ij} = \partial_i (\tilde{K}_{j,n}) + \partial_j (\tilde{K}_{n,i}) - \frac{1}{3} \partial_i \partial_j K - \frac{1}{3} \delta_{ij} \nabla^2 K.$$  \hspace{1cm} (27)

By the momentum constraint, (7) the first two terms on the right are equal; each equal to $\frac{2}{3} K_{,ij}$.

Thus:

$$\left(-\partial_o^2 + \nabla^2\right) \tilde{K}_{ij} = \partial_i \partial_j K - \frac{1}{3} \nabla^2 K \delta_{ij}$$

$$= \text{given function of spatial coordinates}$$

$$= -\frac{1}{2} \left( \partial_i \partial_j b - \frac{1}{3} \delta_{ij} \nabla^2 b \right).$$

Hence;

$$\tilde{K}_{ij} = \tilde{K}_{ij} \text{ (wave)} + \nabla^{-2} \left( \partial_i \partial_j K - \frac{1}{3} \nabla^2 K \delta_{ij} \right).$$  \hspace{1cm} (28)

Where $\nabla^{-2}$ is the Greens function with appropriate boundary conditions. Furthermore, one can consider the temporal evolution of $\delta^{il} \tilde{K}_{ij,l}$, the divergence of $\tilde{K}_{ij}$.

By reorganizing Eq. (28) to have only the second time derivative on the left, and taking the divergence, we obtain

$$\partial_o^2 \tilde{K}_{ij} = 0,$$  \hspace{1cm} (30)
which in principle allows
\[ \overset{\circ}{K}_{ij} = c_i(x^j) + d_i(x^j)t. \] (31)

If we inspect Eq. (28) again, however, we find that solutions consist of solutions to the homogeneous wave equation (which, by Eq. (30) must be divergenceless) plus a “particular” solution which has no time dependence, because the right hand side is time independent. Hence \( d_i(x^j) \equiv 0 \) in Eq. (30), and we may write
\[ \overset{\circ}{K}_{ij} = K_{ij} + \nabla^2(\partial_i \partial_j K - \frac{1}{3} \delta_{ij} \nabla^2 K). \] (32)

Consider an arbitrary longitudinal \( K_{ij}^L = w_{i,j} + w_{j,i} \). A general 3-tensor of this form does not satisfy the momentum constraint \( K_{i,j} = \delta_i^j K_{j} \):
\[ w_{i,j} + w_{j,i} - 2w_{i,j} = w_{i,j} - w_{i,j} \] (33)
Hence in order to satisfy the momentum constraint we must have
\[ (w_{i,j} - w_{j,i})_k \delta^{jk} = 0. \] (34)
The simplest solution is to take \( w_l = \psi, l \), the gradient of a scalar. Although more general solutions may be possible, but likely excluded by reasonable boundary conditions. With \( w_l = \psi, l \) we have
\[ 2\nabla^2 \psi = K. \] (35)
The corresponding \( K_{ij}^L \) is
\[ \overset{\circ}{K}_{ij}^L = 2\psi,_{ij} - \frac{2}{3} \delta_{ij} \nabla^2 \psi = 2\psi,_{ij} - \frac{1}{3} \delta_{ij} K. \] (36)

5. Transverse Data

As an aside, we state how one sets \( TT \) data. Obviously, if the data are set with a given (fixed) wave vector direction, one can algebraically make the tensors \( TT \). Also, if one wishes to Fourier decompose the data, then each Fourier component can be made transverse. In general, one can use the following procedure. The idea is to pose an arbitrary traceless field \( C_{mn} \), and compute its divergence.

Then, find the longitudinal component \( L_{mn} = w_{m,n} + w_{n,m} \) by solving
\[ C_{m,n} = \nabla_n (\nabla_m w^n + \nabla^n w_m), \] (37)
i.e.:
\[
\nabla_n \nabla^2 w_m + \nabla_m (\nabla \cdot w) = C_{m,n} n.
\]

(38)

The operator on the left is easily derived from the minimization of \( \nabla (m w) \nabla (m w) \) and so is strongly elliptic, guaranteeing the existence of unique solutions. If, for instance one postulates compact support for \( C_{mn} \), one can solve this equation using a scalar potential field \( \phi : w_m = \phi_m \)

\[
2 \nabla^2 \phi_m = C_n m n,
\]

(39)

so that

\[
w_m = \phi_m = \nabla^{-2} \left( \frac{1}{2} C_{m,n} \right) \sim \frac{1}{r} + O\left( \frac{1}{r^2} \right).
\]

(40)

The resulting longitudinal component \( L_{mn} \) thus has behavior \( \sim O(r^{-2}) \), and the transverse data now is: \( C_{mn} - L_{mn} \), and no longer has compact support.

6. Complete Data Setting

Set \( h_{mn}^{TT}, h_{mn}^L = v_{m,n} + v_{n,m} \) with appropriate locality (e.g. compactness; at least “fall-off at infinity). Then the trace \( h \) is specified; cf. eqs. (19 - 20).

Set \( K_{mn}^{TT}, K \) with appropriate locality. Then \( K_{mn}^L \) is set by Eq. (36) in terms of \( \psi \) where \( 2 \nabla^2 \psi = K \) and \( K_{mn}^L \) satisfies the momentum constraint. The Hamiltonian constraint is maintained by any \( K_{mn}^L \), in particular one of form (36).

7. Expected Numerical Behavior - Unconstrained Case

From Eq. (32) \( \bar{h}_{ij} \) satisfies

\[
\bar{h}_{ij}^H = \bar{h}_{ij}^L (t_o) - 2t \nabla^2 (\partial_i \partial_j K) - \frac{1}{3} \delta_{ij} \nabla^2 K) + h_{ij}^{TT} \text{ (wave).}
\]

(41)

Note that \( \bar{h}_{ij} \) (wave) is transverse because its time derivative \( -2K_{ij}^{TT} \) is transverse.

Since we have specified the intermediate variable \( \psi \) by \( 2 \nabla^2 \psi = K \), we may also write:

\[
\bar{h}_{ij} = \bar{h}_{ij}^L - 2t (2 \partial_i \partial_j \psi - \frac{1}{3} \delta_{ij} K) + h_{ij}^{TT}.
\]

(42)
To proceed, let us suppose that we wish, as Alcubierre and Schutz did, to set purely \( TT \) data.

In the Alcubierre-Schutz case, plane waves were set, so \( TT \) data can be algebraically enforced, and we are solving only

\[
\Box h_{nm}^{TT} = 0. \tag{43}
\]

However, errors in setting \( TT \) data, can lead to a nonzero longitudinal part. Since we solve to zero the longitudinal component via an elliptic equation in a second order scheme we expect the longitudinal error to be second under small:

\[
||\Delta h_{mn}^L|| \sim \left( \frac{\Delta x}{\lambda} \right)^2 ||h_{mn}^{TT}|| \tag{44}
\]

where \( \Delta x \) is the discretization scale, and \( \lambda \) is the typical scale of the \( TT \) part. Then, since longitudinal terms grow as \( t \), one expects longitudinal contamination equal to the transverse signal at

\[
\frac{t}{\lambda} \sim \left( \frac{\Delta x}{\lambda} \right)^{-2}. \tag{45}
\]

For typical discretizations, \( (\frac{\Delta x}{\lambda}) \lesssim 10^{-2} \) at worst, which suggests evolutions for times

\[
t \sim 10^4 \lambda \tag{46}
\]

before the longitudinal signals are comparable to the transverse signal, and another factor of order \( O \left( ||h_{mn}^{TT}||^{-1} \right) \) before the longitudinal mode violates the linearization criterion \( ||h_{mn}^L|| \ll 1 \). Unfortunately, a poorly controlled computational scheme could put error (noise) in the longitudinal mode at short scales (so that \( \frac{\Delta x}{\lambda} \sim 1 \)). In that case one would expect to see the longitudinal mode grow to the amplitude of the transverse signal after only a few time steps; in a poorly designed differencing approach these secular zone-to zone oscillations would crash the program shortly thereafter. Notice that errors in the trace \( h \) will grow similarly to the \( h_{mn}^L \) modes and with potentially worse effect, because they violate the Hamiltonian constraint. Without a close inspection of the Alcubierre-Schutz code, we cannot comment on why they found such rapid growth of non-\( TT \) modes.

In a recent paper\[5\], Gundlach and Pullin point out a mechanism for instabilities arising from the violation of the constraints in a free evolution. They
used perturbation analysis in double null coordinates on a Reissner-Nordström background, and found that a free evolution led to exponentially growing gauge violating modes. Their results can be taken to the flat space limit. In contrast to the Gundlach and Pullin result, our flat background analysis finds modes that grow at most linearly with time. Choptuik[4] suggests that the presence of the $r = 0$ singularity (persisting even as one takes the $M \to 0$ and $q \to 0$ limit) in [5] results in exponentially growing modes rather than the linearly growing modes that we see in our analysis, which explicitly assumed regularity.

8. Constrained Evolution

In view of Eq. (20), violations of the Hamiltonian constraint lead to spurious $h^*$. This can be solved for $h^*$:

$$h^* = \nabla^{-2} R.$$  (47)

This $h^*$ can then be subtracted from the solution at any particular instant, reasserting the Hamiltonian constraint.

Gauge drift, which would arise from an error in setting exactly zero longitudinal data, can be similarly suppressed. Section 5 shows how to remove longitudinal data from arbitrarily set data. This step can be carried out at any particular instant, reasserting the $TT$ requirement.

9. Summary

We have shown that gauge and constraint error modes arise in the analysis of the 3+1 form of the Einstein equations for linearized waves. These modes are shown to grow linearly in time and have a form that can be controlled in any well specified convergent computational discretization of the evolution equations. By imposing the constraints on the free-evolution these modes may be supressed.

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References


