On a gauge invariant quantum formulation for non-gauge classical theory

I.L. Buchbinder
Department of Theoretical Physics,
Tomsk State Pedagogical University,
Tomsk 634041, Russia

V.D. Pershin
Department of Theoretical Physics,
Tomsk State University,
Tomsk 634050, Russia

G.B. Toder
Department of Physics and Chemistry,
Omsk State Academy of Railway Communications,
Omsk 644046, Russia

Abstract

We propose a method of constructing a gauge invariant canonical formulation for non-gauge classical theory which depends on a set of parameters. Requirement of closure for algebra of operators generating quantum gauge transformations leads to restrictions on parameters of the theory. This approach is then applied for illustration to bosonic string theory coupled to background tachyonic field. It is shown that within the proposed canonical formulation the known mass-shell condition for tachyon is produced.
The procedure of canonical quantization provides a natural and consistent approach to the construction of quantum models in modern theoretical physics. BFV method [1] is the most general realization of this procedure ensuring unitarity at the quantum level and consistency of theory symmetries and dynamics. Now BFV method has been studied in details [2, 3] yet formulation of new models in quantum field theory requires investigation of unexplored aspects of canonical formulation.

In this paper we discuss one of these aspects arising from bosonic string theory coupled to background fields [4]. A crucial point of string models is requirement of conformal invariance at the quantum level. It leads to restrictions on spacetime dimension in the case of free string theory [5, 6] and to effective equations of motion for massless background fields in the case of string theories coupled to background [7]. In terms of covariant functional methods this condition appears as independence of quantum effective action on the conformal factor of two-dimensional metrics or as vanishing of renormalized operator of the energy-momentum trace.

According to the prescription [5] generally accepted in functional approaches to string theory dynamical variables should be treated in different ways. Namely, functional integration is carried out only over string coordinates \(X^\mu(\tau, \sigma)\) while components of two-dimensional metrics \(g_{ab}(\tau, \sigma)\) are considered as external fields. Then one demands the result of such an integration to be independent on the conformal factor and the integrand over \(g_{ab}(\tau, \sigma)\) reduces to finite dimensional integration over parameters specifying string world sheet topologies. This prescription differs from the standard field theory rules when functional integral is calculated over every variable independently.

This approach can be as well applied to string theory interacting with massive background fields which is not classically conformal invariant. One demands that operator of the energy-momentum vanish no matter whether the corresponding classical action is conformal invariant or not. As was shown in [9] it gives rise to effective equations of motion for massive background fields. It means that a non-gauge classical theory depending on a set of parameters is used for constructing of a quantum theory that is gauge invariant under some special values of the parameters. Such a situation occurs in string theory if interaction with massless dilaton, tachyon or any other massive field is turned on.

The problem is how to describe this procedure in terms of canonical quantization. Due to the general BFV method one should construct hamiltonian formulation of classical theory, find out all constraints and calculate algebra of their Poisson brackets. Then one defines fermionic functional \(\Omega\) generating algebra of gauge transformations and bosonic functional \(H\) containing information of theory dynamics. Quantum theory is consistent provided the operator \(\hat{\Omega}\) is nilpotent and conserves in time. The corresponding analysis for bosonic string coupled to massless background fields was carried out in [10, 11].

In the case of string theory interacting with massive background fields components of two-dimensional metrics should be treated as external fields, otherwise classical equations of motion would be inconsistent. As a consequence, classical gauge symmetries are absent and it is impossible to construct classical gauge functional \(\Omega\). In this paper we propose a prescription allowing for some models to

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1. In string theory this independent integration would lead to appearance at the quantum level of an extra degree of freedom connected with two-dimensional gravity [8].
construct quantum operator $\hat{\Omega}$ starting with a classical theory without first class constraints. Quantum theory is gauge invariant if there exist values of theory parameters providing nilpotency and conservation of operator $\hat{\Omega}$. Then to illustrate how the prescription works we apply it to the theory of closed bosonic string coupled to tachyonic field.

2. Consider a system described by a hamiltonian

$$H = H_0(a) + \lambda^\alpha T_\alpha(a)$$

where $H_0(a) = H_0(q,p,a)$, $T_\alpha(a) = T_\alpha(q,p,a)$ and $q$, $p$ are canonically conjugated dynamical variables; $a = a_i$ and $\lambda^\alpha$ are external parameters of the theory.

We suppose that $T_\alpha(a)$ are some functions of the form

$$T_\alpha(a) = T^{(0)}_\alpha(a) + T^{(1)}_\alpha(a)$$

and closed algebra in terms of Poisson brackets is formed by $T^{(0)}_\alpha(a)$, not by $T_\alpha(a)$:

$$\{T^{(0)}_\alpha(a), T^{(0)}_\beta(a)\} = T^{(0)}_\gamma(a)U^{\gamma}_{\alpha\beta}(a)$$

$$\{H_0(a), T^{(0)}_\alpha(a)\} = T^{(0)}_\gamma(a)V^{\gamma}_{\alpha}(a)$$

Such a situation may occur, for example, if $T^{(0)}_\alpha(a)$ correspond to a free gauge invariant theory and $T^{(1)}_\alpha(a)$ describe a perturbation spoiling gauge invariance.

At the quantum level both the algebras of $T^{(0)}_\alpha(a)$ and $T_\alpha(a)$ are not closed in general case

$$[T_\alpha(a), T_\beta(a)] = i\hbar(T_\gamma(a)U^{\gamma}_{\alpha\beta}(a) + A_{\alpha\beta}(a)),$$

$$[H_0(a), T_\alpha(a)] = i\hbar(T_\gamma(a)V^{\gamma}_{\alpha}(a) + A_\alpha(a)),$$

and operators $A_{\alpha\beta}$, $A_\alpha$ do not vanish in the limit $\hbar \to 0$ due to absence of classical gauge invariance.

We define quantum operators $\Omega$ and $H$ as follows:

$$\Omega = c^\alpha T_\alpha(a) - \frac{1}{2} U^{\gamma}_{\alpha\beta}(a) : \mathcal{P}_\gamma c^\alpha c^\beta :$$

$$H = H_0(a) + V^{\gamma}_{\alpha}(a) : \mathcal{P}_\gamma c^\alpha :$$

where $: :$ stands for some ordering of ghost fields. Square of such an operator $\Omega$ and its time derivative take the form

$$\Omega^2 = \frac{1}{2} i\hbar (A_{\alpha\beta}(a) + G_{\alpha\beta}(a)) : c^\alpha c^\beta :$$

$$\frac{d\Omega}{dt} = \frac{\partial\Omega}{\partial t} + [H, \Omega] =$$

$$= \left( \frac{\partial T_\alpha(a)}{\partial t} - A_\alpha(a) - G_\alpha(a) \right) c^\alpha - \frac{1}{2} \frac{\partial U^{\gamma}_{\alpha\beta}(a)}{\partial t} : \mathcal{P}_\gamma c^\alpha c^\beta :$$

where $G_{\alpha\beta}(a), G_\alpha(a)$ are possible quantum contributions of ghosts.
It is natural to suppose that every operator of the theory can be performed as a linear combination of an irreducible set of independent operators $O_M(q,p)$:

$$A_{\alpha\beta}(a) + G_{\alpha\beta}(a) = E^M_{\alpha\beta}(a)O_M(q,p)$$

$$\frac{\partial T_\alpha(a)}{\partial t} - A_\alpha(a) - G_\alpha(a) = E^M_\alpha(a)O_M(q,p)$$

(7)

$E^M_{\alpha\beta}(a)$, $E^M_\alpha(a)$ being some $c-$valued functions of the parameters $a$.

In general case $\Omega^2 \neq 0$ and $d\Omega/dt \neq 0$. However, if equations

$$E^M_{\alpha\beta}(a) = 0, \quad E^M_\alpha(a) = 0, \quad \frac{\partial U^\gamma_{\alpha\beta}(a)}{\partial t} = 0,$$

(8)

have some solutions $a_i = a_i^{(0)}$ then operator $\Omega$ is nilpotent and conserves for these specific values of parameters and hence the quantum theory is gauge invariant. Thus, there exists a possibility to construct quantum theory with given gauge invariance that is absent at the classical level.

It is important that eqs.(8) are not anomaly cancellation conditions because an anomaly represents breaking of classical symmetries at the quantum level. In the theory under consideration classical symmetries are absent and eqs.(8) express conditions of quantum symmetries existence.

In specific models eqs.(8) may have no solutions at all or, conversely, be fulfilled identically. The latter possibility was described in [12].

3. As an example where the described procedure really works and leads to eqs.(8) with non-trivial solutions for parameters $a$ we consider closed bosonic string theory coupled with massive tachyonic fields. Structure of this theory has been studied in details within covariant functional methods and we are going to show that our procedure reproduces the correct equation for tachyonic field.

The theory is described by the classical action

$$S = -\frac{1}{2\pi\alpha'} \int d^2\sigma \sqrt{-g}\{\frac{1}{2}g^{ab}\partial_aX^\mu\partial_bX^\nu\eta_{\mu\nu} + Q(X)\},$$

(9)

$\sigma^a = (\tau,\sigma)$ are coordinates on string world sheet, $\eta_{\mu\nu}$ is Minkowski metrics of $D-$dimensional spacetime and $Q(X)$ is background tachyonic field.

If components of two-dimensional metrics $g_{ab}$ were considered as independent dynamical variables the corresponding classical equations of motion would be fulfilled only for vanishing tachyonic fields:

$$g_{ab}\frac{\delta S}{\delta g_{ab}} = -\frac{1}{2\pi\alpha'}\sqrt{-g}Q(X) = 0.$$

(10)

Similar situation arises for string theories coupled to another massive fields [9] or to massless dilaton [10].

Hence, to construct a theory with non-trivial tachyon we have to treat components of $g_{ab}$ as external fields. This treatment quite corresponds to covariant methods where functional integral is calculated only over $X^\mu$ variables.

After the standard parametrization of metrics

$$g_{ab} = e^\gamma \begin{pmatrix} \lambda^2 & \lambda^3 \\ \lambda^1 & 1 \end{pmatrix}$$

(11)
the hamiltonian takes the form

\[ H = \int d\sigma \ (p_\mu \dot{\omega}^\mu - L) = \int d\sigma \ (\lambda_0 T_0 + \lambda_1 T_1), \]  

(12)

where

\[ T_0 = T_0^{(0)} + \frac{1}{2\pi \alpha'} e^\gamma Q(X), \quad T_0^{(0)} = \frac{1}{2} \left( 2\pi \alpha' P^2 + \frac{1}{2\pi \alpha'} X^\gamma \right), \]

\[ T_1 = T_1^{(0)} = P_\mu X^\mu, \]  

(13)

\( P_\mu \) are momenta canonically conjugated to \( X^\mu, \) \( X^\mu = \partial X^\mu / \partial \sigma; \) \( T_0^{(0)} \) and \( T_1^{(0)} \) represent constraints of free string theory and form closed algebra in terms of Poisson brackets. \( \lambda_0 \) and \( \lambda_1 \) play the role of external fields and so \( T_0 \) and \( T_1 \) cannot be considered as constraints of classical theory. In free string theory conditions \( T_0^{(0)} = 0, \)

\( T_1^{(0)} = 0 \) result from conservation of canonical momenta conjugated to \( \lambda_0 \) and \( \lambda_1. \)

According to our prescription in string theory with tachyon \( \lambda_0 \) and \( \lambda_1 \) can not be considered as dynamical variables, there are no corresponding momenta and conditions of their conservation do not appear.

The role of parameters \( \alpha \) in the theory under consideration is played by tachyonic field \( Q(X) \) and conformal factor of two-dimensional metrics \( \gamma(\tau, \sigma). \) The theory (12) is of the type (1) with \( H_0 = 0, \) structural constants of classical algebra being independent on time. It means that the condition of conservation for the operator \( \Omega \) (6) will be met if operators (13) do not depend on time explicitly, that is

\[ \dot{\gamma}(\tau, \sigma) = \frac{\partial \gamma(\tau, \sigma)}{\partial \tau} = 0. \]  

(14)

To derive the first condition (8) one has to construct quantum algebra of operators \( T_0, T_1. \) We will use Fourier components of the operators

\[ L_n = \int_0^{2\pi} d\sigma e^{-i n \sigma} \frac{1}{2} (T_0 - T_1), \]

\[ \tilde{L}_n = \int_0^{2\pi} d\sigma e^{i n \sigma} \frac{1}{2} (T_0 + T_1). \]  

(15)

Direct calculation of commutators of the operators (15) gives the following algebra:

\[ [L_n, L_m] = \hbar (n - m) L_{n+m} + \hbar^2 \delta_{n,-m} \left( \frac{D}{12} n(n^2 - 1) + 2\alpha(0)n \right) \]

\[ - \left( 4\pi \alpha' \right)^{-1} \hbar (n - m) \int_0^{2\pi} d\sigma e^{-i(n+m)\sigma} e^{\gamma(\tau, \sigma)} \left( 1 + \alpha' \hbar \partial^2 / 4 \right) Q(X), \]

\[ [\tilde{L}_n, \tilde{L}_m] = \hbar (n - m) \tilde{L}_{n+m} + \hbar^2 \delta_{n,-m} \left( \frac{D}{12} n(n^2 - 1) + 2\beta(0)n \right) \]

\[ - \left( 4\pi \alpha' \right)^{-1} \hbar (n - m) \int_0^{2\pi} d\sigma e^{i(n+m)\sigma} e^{\gamma(\tau, \sigma)} \left( 1 + \alpha' \hbar \partial^2 / 4 \right) Q(X), \]

\[ [L_n, \tilde{L}_m] = - \left( 4\pi \alpha' \right)^{-1} \hbar (n - m) \int_0^{2\pi} d\sigma e^{i(n-m)\sigma} e^{\gamma(\tau, \sigma)} \left( 1 + \alpha' \hbar \partial^2 / 4 \right) Q(X) \]

\[ - \left( 4\pi \alpha' \right)^{-1} i \hbar \int_0^{2\pi} d\sigma e^{i(n-m)\sigma} e^{\gamma(\tau, \sigma)} \gamma'(\tau, \sigma) Q(X), \]  

(16)
where $\alpha(0)$, $\beta(0)$ are constant parameters describing ordering ambiguity. We see that the only quantum contributions to the comutators are these of the second order in $\hbar$, all the higher contributions equal zero.

Eqs.(16) define the explicit form of the function $A_{\alpha\beta}$ (4) in the string theory with tachyon. Ghost contribution $G_{\alpha\beta}$ has the same structure as in free string theory and cancels the $c-$valued terms in (16) provided that $D = 26$ and $\alpha(0) = \beta(0) = 1$.

As a result the eqs.(8) for string theory coupled to tachyon appear as

$$D = 26, \quad \alpha(0) = 1, \quad \beta(0) = 1,$$

$$\left(\dot{\gamma}^2 + 4/\alpha' h\right)Q(X) = 0$$

$$\dot{\gamma}(\tau, \sigma) = 0, \quad \gamma'(\tau, \sigma) = 0.$$  

The eqs.(18) is mass-shell condition for free tachyon. It is linear and so effects of tachyonic interaction are not reproduced within perturbative calculation of the conformal algebra (16). This is quite natural because taking into account tachyonic interaction is known to require non-perturbative approaches [13].

The eqs.(19) show that the operators $L_n, \bar{L}_n$ form conformal algebra only in the case $\gamma = \text{const}$. This does not contradict the corresponding results of covariant approaches. After integration over $X^\mu$ the effective action does not depend on conformal factor provided effective equations of motion for background fields are fulfilled. It means that any physical results (e.g. values of correlation functions) do not depend on a gauge choice of $\gamma$. Specifically, $\gamma$ can be choosen to be constant and this situation is reproduced on our case. It should be noted that our approach is not restricted by flat world sheets. Quantum theory can be formulated for any functions $\gamma$ but it is conformal invariant only for constant $\gamma$.

The described example demonstrates a possibility to construct canonical formulation of quantum theory invariant under gauge transformations that are absent at the classical level. The proposed method opens up a possibility for deriving interacting effective equations of motion for massive and massless background fields within the framework of canonical formulation of string models and provides a justification of covariant functional approach to string theory.

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