May 1996 INFNCA-TH9611

Multi-black holes and instantons in effective string theory

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The effective action for string theory which takes into account non-minimal coupling of moduli admits multi-black hole solutions. The euclidean continuation of these solutions can be interpreted as an instanton mediating the splitting and recombination of the throat of extremal magnetically charged black holes.

1. Introduction

String theories are presently considered one of the most promising candidates for a theory of quantum gravity. It is therefore natural that their implications on the theory of gravity and especially on black holes had been widely investigated. Since at the moment the formalism of string theories cannot deal directly with these problems, this has usually been done by means of field theoretical low-energy effective actions, which correct the action of general relativity by the addition of non-minimally coupled scalar fields, such as the dilaton or moduli fields. On the other hand, if the gravitational field is described by a quantum theory, its topology should fluctuate at Planck length scales [1]. A first approximation to these effects should therefore be obtained in the context of a euclidean path integral formalism [2]. In this formalism a fundamental role is played by finite action regular solutions of the classical field equations, interpolating between different topological sectors. These configurations give the main contribution to the path integral in the context of a semiclassical approximation.

It is then worthwhile to consider this approach in the context of effective string theory. Unfortunately, however, very little is known about gravitational instanton solutions of effective string actions [3-4]. In general relativity, an interesting case of gravitational instanton is given by the euclidean multi-black hole solutions in the presence of a Maxwell field. This class of instantons possesses several asymptotic regions which approach Bertotti-Robinson universes. Brill [5] has exploited this property for the calculation of the probability of the splitting of a Bertotti-Robinson universe into two or more. This is especially relevant because a Bertotti-Robinson universe approximates very well the throat of an extremal Reissner-Nordström black hole near the horizon and the probability should therefore approximate the splitting or recombination ratio of extremal Reissner-Nordström black holes.

In this paper, we discuss the generalization of the multi-black hole solutions, with lorentzian or euclidean signature, to the case of a one-parameter effective string action which includes the non-minimal coupling of both the dilaton and a modulus field and interpolates between general relativity and the standard GHS model [6]. We show that the euclidean action vanishes for these instantons, except in the GR limit, according to the fact that in the string case both the initial and final states have vanishing entropy.

2. The multi-black hole solutions

In [7] an effective action for the heterotic string which took into account the non-minimal coupling to gravity of a modulus field due to one-loop threshold effects was introduced. In terms of the string metric, the action read:

$$I = \frac{1}{16\pi}\int\sqrt{-g}\; d^4x\; e^{-2\phi}\left[R + 4(\nabla\phi)^2 - \frac{2}{3}(\nabla\psi)^2 - \left(1 + e^{2(\phi - q\psi/3)}\right)F^2\right] \tag{1}$$

where ϕ is the dilaton, ψ a modulus, F^2 is the Maxwell field strength and q a coupling constant. It was then shown that exact solutions of the classical field equations can be found if $e^{-2\phi} = \frac{q^2}{3}e^{-2q\psi/3}$. In this case, the action simplifies to [4]:

$$I = \frac{1}{16\pi}\int\sqrt{-g}\; d^4x\; e^{-2\phi}\left[R - \frac{8k}{1-k}(\nabla\phi)^2 - \frac{3+k}{1-k}F^2\right] \tag{2}$$

where $k = \frac{3-2q^2}{3+2q^2}$ and $-1 \le k \le 1$. In particular, for $k = -1$, the action reduces to the standard string action in absence of modulus coupling [6], while in the singular limit $k = 1$, the dilaton decouples and one can recover the Einstein-Maxwell theory.

The field equations stemming from (2) are:

$$\begin{aligned}
&\nabla_\mu(e^{-2\phi}F^{\mu\nu}) = 0 \\
&R = \frac{8k}{1-k}\left[-\nabla^2\phi + (\nabla\phi)^2\right] + \frac{3+k}{1-k}F^2 \\
&R_{\mu\nu} = 4\frac{1+k}{1-k}\nabla_\mu\phi\nabla_\nu\phi - 2\nabla_\mu\nabla_\nu\phi + [2(\nabla\phi)^2 - \nabla^2\phi]g_{\mu\nu} + 2\frac{3+k}{1-k}\left[F_{\mu\rho}F^\rho_{\ \nu} - \frac{1}{4}F^2 g_{\mu\nu}\right]
\end{aligned} \tag{3}$$

These equations admit black hole solutions with magnetic monopole configurations of the Maxwell field given by [7]:

$$ds^2 = -\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right)^k dt^2 + \left(1 - \frac{r_+}{r}\right)^{-1}\left(1 - \frac{r_-}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$e^{-2\phi} = \left(1 - \frac{r_-}{r}\right)^{(1-k)/2} \qquad\qquad F_{\hat\imath\hat\jmath} = \frac{Q}{r^2}\epsilon_{\hat\imath\hat\jmath} \tag{4}$$

where $\hat\imath,\ \hat\jmath = 2, 3$. The two parameters r_+ and r_- are related to the magnetic charge Q and the ADM mass M of the solution by the relations:

$$M = \frac{1}{2}r_+ + \frac{3-k}{4}r_- \qquad\qquad Q^2 = \frac{1-k}{4}r_+r_- \tag{5}$$

while the temperature T and the entropy S of the black hole are

$$T = \frac{1}{4\pi r_+}\left(1 - \frac{r_-}{r_+}\right)^{(1+k)/2} \qquad\qquad S = \pi r_+^2\left(1 - \frac{r_-}{r_+}\right)^{(1-k)/2} \tag{6}$$

(These formulae correct some errors present in ref. [7]).

Of special interest are the solutions corresponding to extremal black hole with $r_+ = r_- = \alpha$, i.e. $Q^2 = 4\frac{1-k}{(5-k)^2}M^2$. In this limit, the solution (4) can be written as

$$ds^2 = -\left(1 + \frac{\alpha}{\rho}\right)^{-(1+k)} dt^2 + \left(1 + \frac{\alpha}{\rho}\right)^2 (d\rho^2 + \rho^2 d\Omega^2)$$

$$e^{-2\phi} = \left(1 + \frac{\alpha}{\rho}\right)^{(k-1)/2} \qquad\qquad F_{ij} = \frac{\sqrt{1-k}}{2}\frac{\alpha}{(\rho+\alpha)^2}\epsilon_{ij} \tag{7}$$

where $\rho = r - \alpha$. These metrics are regular everywhere in the range $0 < \rho < \infty$. Besides the horizon at $\rho = 0$, the g_{00} component of the metric and the dilaton field become complex for non-integer k and hence the metric cannot be continued in that region. The surfaces $\rho = 0$ are regular horizons placed at infinite spatial distance. However, the timelike and lightlike geodesics have a finite extent except for $k = -1$, and hence the metrics are not geodesically complete in general. For a more detailed discussion on this point, see ref. [8].

Near the horizon at $\rho = 0$, the metric takes the form of a Bertotti-Robinson universe [9], namely, it is the direct product of 2-d anti-de Sitter spacetime (or flat space if $k = -1$) and a 2-sphere of constant radius:

$$ds^2 = -\left(\frac{\rho}{\alpha}\right)^{k+1} dt^2 + \alpha^2\frac{d\rho^2}{\rho^2} + \alpha^2 d\Omega^2 \tag{8}$$

with constant magnetic field: $F_{\hat{i}\hat{j}} = \frac{\sqrt{1-k}}{2\alpha}\epsilon_{\hat{i}\hat{j}}$.

It is easy to generalize these solutions to the case of many black holes. In fact, if one inserts into the field equations (3) the ansatz, suggested by (7),

$$ds^2 = -V^{-(1+k)}dt^2 + V^2 d\mathbf{x}\cdot d\mathbf{x}$$

$$e^{-2\phi} = V^{(k-1)/2} \qquad\qquad F_{ij} = \frac{\sqrt{1-k}}{2}\epsilon_{ijk}\partial_k V \tag{9}$$

where $\mathbf{x} = (x_1, x_2, x_3)$, $i, j = 1, 2, 3$, one can check that they are satisfied if $\nabla^2 V = 0$. In particular, for a multi-black hole solution, V takes the form:

$$V = c + \sum_{i=1}^{N} \frac{\alpha_i}{|\mathbf{x} - \mathbf{x}_i|} \tag{10}$$

If one requires that the solutions be asymptotically flat , $c = 1$ and for $N = 1$ one recovers (7). If $c = 0$, instead, one obtains solutions which are asymptotically Bertotti-Robinson also at infinity. The asymptotically flat solutions describe a distribution of extremal black holes with masses $M_i = \frac{5-k}{4}\alpha_i$, magnetic charges $Q_i = \frac{\sqrt{1-k}}{2}\alpha_i$, and dilatonic charges $\Sigma_i = \frac{1-k}{4}\alpha_i$, in equilibrium, and generalizes the Majumdar-Papapetrou solution of general relativity [10], which is recovered in the limit $k = 1$. The properties of the metric near $\mathbf{x} = \mathbf{x}_i$ are analogous to those of the single extremal black holes described above, with $\alpha = \alpha_i$. In particular, the surfaces at $\mathbf{x} = \mathbf{x}_i$ are event horizons and the curvature is regular there.

By duality, one can also obtain electrically charged solutions. In this case, the metric, the dilaton and the Maxwell field are

$$ds^2 = -V^{-2}dt^2 + V^{1+k}d\mathbf{x} \cdot d\mathbf{x}$$

$$e^{-2\phi} = V^{(1-k)/2} \qquad\qquad F_{0i} = \frac{\sqrt{1-k}}{2}\partial_i(V^{-1}) \tag{11}$$

where V is given by (10). These solutions describe a distribution of masses with electric and dilatonic charge in equilibrium, but display naked singularities at $\mathbf{x} = \mathbf{x}_i$, except for $k = 1$.

The solutions (9) and (11) are also related by a conformal transformation to those discussed by Shiraishi [11]. In fact, in terms of the "canonical" metric $\hat{g}_{\mu\nu} = e^{-2\phi}g_{\mu\nu}$, the action (2) becomes:

$$I = \frac{1}{16\pi}\int \sqrt{-\hat{g}}\; d^4x \left[\hat{R} - 2(\hat{\nabla}\hat{\phi})^2 - e^{-2a\hat{\phi}}\hat{F}^2\right] \tag{12}$$

with $a = \sqrt{\frac{1-k}{3+k}}$, $\phi = a\hat{\phi}$, $F = a\hat{F}$ and line element

$$ds^2 = -V^{-(k+3)/2}dt^2 + V^{(k+3)/2}d\mathbf{x} \cdot d\mathbf{x} \tag{13}$$

Also in this case, however, the surfaces at $\mathbf{x} = \mathbf{x}_i$ are naked singularities, except in the Einstein limit $k = 1$.

3. The euclidean continuation

The analytical continuation of the solutions (9) to imaginary time can be interpreted as gravitational instantons. In particular, in the case $c = 0$, one obtains the generalization to the string case of the instanton mediating the splitting and recombination of several Bertotti-Robinson universes, introduced by Brill [5]. As we have seen before, the throat of an extremal magnetically charged string black hole has the form of a Bertotti-Robinson universe, and therefore our instantons can be considered as an approximation of those mediating the splitting of extremal black holes.

Let us proceed to the extension of the Brill instanton to our models. The euclidean metric is given by

$$ds^2 = V^{-(1+k)} d\tau^2 + V^2 d\mathbf{x} \cdot d\mathbf{x} \tag{14.a}$$

with dilaton and Maxwell field

$$e^{-2\phi} = V^{(k-1)/2} \qquad\qquad F_{ij} = \frac{\sqrt{1-k}}{2} \epsilon_{ijk} \partial_k V \tag{14.b}$$

where

$$V = \sum_{i=1}^{N} \frac{\alpha_i}{|\mathbf{x} - \mathbf{x}_i|}$$

The metric is geodesically complete. When $\mathbf{x}$ approaches $\mathbf{x}_j$, V approaches $\frac{\alpha_j}{|\mathbf{x}-\mathbf{x}_j|}$ and the metric takes the Bertotti-Robinson form in this limit. Similarly, in the limit $\mathbf{x} \to \infty$, $V \to \frac{\alpha_\infty}{|\mathbf{x}|}$, with $\alpha_\infty = \sum \alpha_i$, which is also of the Bertotti-Robinson form. Thus the geometry interpolates between the $N + 1$ Bertotti-Robinson universes corresponding to its $N + 1$ asymptotic regions.

We point out that the euclidean metrics (14.a) are regular without any need to impose a periodicity in the time coordinate. This is confirmed by the fact that extremal black holes (7) have vanishing temperature for any value of k.

In order to calculate the transition probability, one has to evaluate the euclidean action, which is the dominant term in a semiclassical approximation of the path integral.

This is given by:

$$
\begin{aligned}
I_E = &-\frac{1}{16\pi}\int_M \sqrt{g}\ d^4x\ e^{-2\phi}\left[R - \frac{8k}{1-k}(\nabla\phi)^2 - \frac{3+k}{1-k}F^2\right] \\
&- \frac{1}{8\pi}\int_{\partial M} \sqrt{{}^3g}\ d^3x\ e^{-2\phi}K
\end{aligned}
\tag{15}
$$

where K is the trace of the second fundamental form and the boundary term is needed to ensure the unitarity of the theory.

By using the field equations, the volume integral can be converted into a surface integral, so that the action can be written:

$$
I_E = -\frac{1}{8\pi}\int_{\partial M} \sqrt{{}^3g}\ d^3x\ e^{-2\phi}\left[\frac{4k}{1-k}n\cdot\nabla\phi + K\right] \tag{16}
$$

For the boundary 3-surfaces, one can choose the cylinder with "mantles" $|\mathbf{x}| = P$, $\tau \in (-T, T)$ and $|\mathbf{x} - \mathbf{x}_j| = P_j$, $\tau \in (-T, T)$ and basis $|\mathbf{x}| \leq P$, $|\mathbf{x} - \mathbf{x}_j| \geq P_j$, $\tau = T\ (-T)$, in the limit $P \to \infty$, $P_j \to 0$, $T \to \infty$ [5]. On the bases the integral vanishes, while on the mantle surfaces it is given by

$$
\frac{k-1}{2}P_iT \tag{17}
$$

This vanishes identically for $k = 1$, while is undetermined otherwise. One has therefore to carefully choose a regularization prescription, which can be fixed by requiring that one obtain zero when the action is evaluated for the initial, single Bertotti-Robinson universe. This can be achieved by taking the limits $P_i \to 0$, before the limit $T \to \infty$ [5]. With this prescription, one obtains a null contribution from the mantle surfaces at finite distance. For $x \to P$, instead, the integral diverges, but the action has to be renormalized by subtracting the contribution of the single initial Bertotti-Robinson universe, which exactly cancels the diverging contribution.

The net result is that the action vanishes. However, there is another contribution to the action coming from the cylinder two-dimensional edges at $|\mathbf{x}| = P$, $|\mathbf{x} - \mathbf{x}_j| = P_j$, $\tau = \pm T$, where the extrinsic curvature has a delta-function behaviour [5]. The contribution to (15) from the jth pair of edges is given by [12]:

$$
-2\frac{e^{-2\phi(P_j)}}{8\pi}\frac{\pi}{2}A_j = -\frac{\pi}{2}P_j^{(1-k)/2}\alpha_j^{(3+k)/2} \tag{18}
$$

where A_j is the area of the edges, given by $4\pi\alpha_j^2$. One must again subtract the contribution of the initial Bertotti-Robinson universe. The final result is

$$I_E = \frac{\pi}{2}\left(\left(\sum \alpha_i\right)^2 - \sum \alpha_i^2\right) \tag{19}$$

for $k = 1$, while for $k \neq 1$ it vanishes as $P_j^{(1-k)/2}$, since it is suppressed by the dilaton†.

The action of the instanton is therefore null, except for the GR limit. Thus, it appears that in the dilaton-modulus gravity, the probabilities for the splitting of a Bertotti-Robinson universe in two or more does not depend on the parameters α_i. This is in accordance with the thermodynamics: comparing with (6) one sees that the euclidean action is half the difference of the entropies of the extremal black holes whose throats are approximated by these Bertotti-Robinson universes (and which vanish except in the GR limit), so that the semiclassical probability of the transition is equal to the exponential of the difference of the entropy of the initial and final states, which gives the probability of a thermodynamical fluctuation. In particular, if $k \neq 1$, the entropy of the extremal black hole vanishes, and there is no potential barrier to obstruct the process.

4. Final remarks

In a semiclassical approximation, the probability for the splitting of a Bertotti-Robinson universe is given at first order by the exponential of the instanton action $e^{-I_E/\hbar}$ and hence, as first shown in [5], it is suppressed in the GR limit according to the violation of the second law of thermodynamics, while, as we have seen, in the other cases the entropy is null both for the initial and final states and at this order of approximation there is no potential barrier to prevent the splitting when the mass of the black holes is of the order of the Planck scale.

† Some authors [13] have argued that for extremal black holes the time coordinate should be periodically identified. In this case, of course, the region of integration has no edge and the action is always zero. With this prescription, anyway, also the entropy of the $k = 1$ extremal black holes vanishes, in agreement with the thermodynamical argument given below.

This is interesting in view of the fact that Bertotti-Robinson universes approximate the throat of extremal magnetically charged black holes. As is well known, extremal black holes can be considered as the ground state for the Hawking evaporation process of black holes of given charge, and from the results obtained one can deduce that at the last stages of evaporation the extremal string black holes may split into smaller ones [14].

It is also interesting to compare the results obtained here with those for instantons describing pair production of extremal black holes in a background magnetic field [3]. Also in that case, corrections proportional to the black hole entropy are present in the $k = 1$ limit.

Acknowledgements

We wish to thank M. Cadoni for interesting remarks

References

[1] S.W. Hawking, Nucl. Phys. **B144**, 349 (1978);

[2] S.W. Hawking, in *General Relativity, an Einstein centenary survey*, edited by S.W. Hawking and W. Israel (Cambridge Un. Press 1979);

[3] H.F. Dowker, G. Gauntlett, D.A. Kastor and J. Traschen, Phys. Rev. **D49**, 958 (1994); H.F. Dowker, G. Gauntlett, S.B. Giddings and G.T. Horowitz, Phys. Rev. **D50**, 2662 (1994); S.F. Ross, Phys. Rev. **D49**, 6599 (1994);

[4] M. Cadoni and M. Cavaglià, Phys. Rev. **D50**, 6435 (1994);

[5] D.R. Brill, Phys. Rev. **D46**, 1560 (1992);

[6] D. Garfinkle, G.T. Horowitz and A. Strominger, Phys. Rev. **D43**, 3140 (1991); S.B. Giddings and A. Strominger, Phys. Rev. **D46**, 627 (1992);

[7] M. Cadoni and S. Mignemi, Phys. Rev. **D48**, 5536 (1993) and Nucl. Phys. **B427**, 669 (1994);

[8] M. Cadoni and S. Mignemi, Phys. Rev. **D51**, 4319 (1995);

[9] B. Bertotti, Phys. Rev. **116**, 1331 (1959); I. Robinson, Bull. Akad. Polon. **7**, 351 (1955);

[10] S.D. Majumdar, Phys. Rev. **72**, 930 (1947); A. Papapetrou, Proc. Roy. Irish Acad. **A51**, 191 (1947); J.B. Hartle and S. Hawking, Commun. Math. Phys. **26**, 87 (1972);

[11] K. Shiraishi, J. Math. Phys. **34**, 1480 (1993);

[12] G. Hayward, Phys. Rev. **D47**, 3275 (1993);

[13] G.W. Gibbons and R.E. Kallosh, Phys. Rev. **D51**, 2839 (1995); S.W Hawking, G.T. Horowitz and S.F. Ross, Phys. Rev. **D51**,4300 (1995) ; C. Teitelboim, Phys. Rev. **D51**, 4315 (1995);

[14] R. Kallosh, A. Linde, T. Ortin, A. Peet and A. van Proeyen, Phys. Rev. **D46**, 5278 (1992).