Abstract

Let a “complex probability” be a normalizable complex distribution $P(x)$ defined on $\mathbb{R}^D$. A real and positive probability distribution $p(z)$, defined on the complex plane $\mathbb{C}^D$, is said to be a positive representation of $P(x)$ if $\langle Q(x) \rangle_P = \langle Q(z) \rangle_p$, where $Q(x)$ is any polynomial in $\mathbb{R}^D$ and $Q(z)$ its analytical extension on $\mathbb{C}^D$. In this paper it is shown that every complex probability admits a real representation and a constructive method is given. Among other results, explicit positive representations, in any number of dimensions, are given for any complex distribution of the form Gaussian times polynomial, for any complex distributions with support at one point and for any periodic Gaussian times polynomial.
I. INTRODUCTION

In quantum physics there are instances of averages where the role of probability distribution is played by a distribution taking complex values. Consider the functional integral formulation of field theory [1]. There, the time ordered expectation value of observables takes the form

$$\langle T O[\phi] \rangle = N \int \mathcal{D}\phi(x) e^{iS[\phi]} O[\phi],$$

where $S[\phi]$ is the action functional and $N$ a normalization constant. This is a first instance of a “complex probability distribution”, namely, the Boltzmann weight $P[\phi] = N e^{iS[\phi]}$. In the continuum, such functional integral is not sufficiently well-behaved and only its Euclidean version can be given a rigorous meaning [2]. Within a lattice regularization, the Minkowski version is mathematically well-defined, nevertheless the Wick rotation is performed in this case too. This is because, in most cases, in the Euclidean theory the Boltzmann weight becomes a real and positive probability distribution. This is important in practice since straightforward Monte Carlo is only defined for positive probabilities. There are cases, however, when even Euclidean field theory presents complex actions. Indeed, the statistical interpretation of the quantum theory requires the Boltzmann weight to be reflection positive, but not directly positive [3]. Instances of complex Euclidean actions occur after integration of fermions, since the fermionic determinant is not positive definite; if there are non vanishing chemical potentials; in gauge theories in the presence of Wilson loops or topological $\theta$-terms or in general after inserting projection operators in the path integral to select particular sectors of the theory [4,5,11,7]. Also, two dimensional fermions can be brought to a bosonic complex action form [8].

As we have said, the computation of averages in the presence of a complex probability distribution poses a practical problem, namely, the Monte Carlo method cannot be used directly to sample the probability since this method only makes sense for true, i.e. real and positive, probabilities. The standard approach to complex probabilities in numerical simulations [4,5] is to factorize a real and positive part to be used as input for some Monte Carlo method and include the remainder in the observable. That is, if the complex probability is $P(x) = P_0(x) F(x)$ with $P_0(x)$ positive, the expectation values can be obtained as

$$\langle O(x) \rangle_P = \frac{\langle O(x) F(x) \rangle_{P_0}}{\langle F(x) \rangle_{P_0}}. \tag{1}$$

Of course, the same formula can be used when $P(x)$ itself is positive. The problem with this approach is that it violates the importance sample principle, since we are not sampling the true probability and that increases the dispersion of Monte Carlo data. For instance, $\langle F(x) \rangle_{P_0}$ may be small, thereby introducing large error bars.

An alternative approach is to look for a positive probability $p(z)$ in the complex configuration space which gives the same expectation values as $P(x)$, i.e., $\langle O(x) \rangle_P = \langle O(z) \rangle_p$, where $O(z)$ is the analytical extension of $O(x)$. The usual way of constructing such a probability is by means of the complex Langevin algorithm [9,10]. In this approach the configuration is updated through a standard Langevin algorithm with the complex action. Since the drift term is complex, the complex extension of the configuration space is sampled as well. Whenever the random walk possesses an equilibrium configuration, it is sampling the complex configuration space with a real and positive probability distribution $p(z)$. We have then traded a complex probability $P(x)$ on $\mathbb{R}^D$ by a positive probability $p(z)$ on $\mathbb{C}^D$. If $p(z)$ happens to be equivalent to $P(x)$ in the sense of expectation values, we have succeeded in sampling the complex probability. Successful implementations of the algorithm
have been obtained in some practical cases, such as two-dimensional compact QED with static charges [6]. In general, however, the complex Langevin algorithm poses two problems. First, it not always converges to an equilibrium distribution. Second and more subtle, for some actions it seems to converge to an equilibrium distribution which is not equivalent to the original complex probability [11–13], (see however [14]). Such phenomenon has been found in practically relevant cases such as QCD with a Wilson loop [11,12,15].

In the present paper we consider the problem of constructing a positive representation directly, independently of the Langevin algorithm. Several properties of representations of complex probabilities on R^D by probabilities on C^D are noted. A constructive method is given to obtain real (although not necessarily positive) representations of very general complex probabilities. Positive representations are explicitly constructed for some probabilities which are beyond the present applicability of the complex Langevin algorithm. These include Gaussian times polynomial, distributions with support at one point, and periodic Gaussian times polynomial. In all cases, such representations are not unique.

These results are of great interest from the point of view of applications. This is not because the constructions found here are of direct usefulness to carry out numerical calculations; there are far more natural ways to compute expectations values with complex Gaussian times polynomial distributions. The interest lies in the following. The negative results found up to now with the complex Langevin algorithm in some systems would make one to have reasonable doubts of whether a positive representation exists at all for those systems. Moreover, the momenta of any positive probability on C^D are bounded to satisfy some inequalities among them. It might happen that those bounds were incompatible with the momenta of the given complex probability on R^D in some cases. At present, the necessary and sufficient conditions for a positive representation to exist are not known. The results of this paper suggest, however, that such representation exists quite generally since the set of Gaussian times polynomial is dense in L^2(R^D). Our results tend to support the idea that there is no obstruction of principle for positive representations to exits. This is the main insight of this work.

II. REPRESENTATION OF COMPLEX PROBABILITIES

The complex probabilities P(x) to be considered here will be tempered distributions on R^D of a restricted class, namely, those which are the inverse Fourier transform of an ordinary function ˜P(k) (locally integrable and at most of polynomial growth at infinity), with ˜P(k) non vanishing at the origin and analytical at that point. These conditions allow for a natural definition of ∫ x_1^i ··· x_n^i P(x)d^D x through the Taylor expansion of ˜P(k) at k = 0. In particular ∫ P(x)d^D x will be non vanishing. The expectation value associated to P(x) is defined for any polynomial Q(x) as

$$\langle Q(x) \rangle_P = \frac{\int Q(x)P(x)d^D x}{\int P(x)d^D x}.$$  

(2)

Likewise, we can consider complex probabilities on C^D as the class of distributions defined above on R^{2D}. For any such distribution, p(z), the expectation value takes the form
where \( z_j = x_j + iy_j \), \( d^Dz = d^Dxd^Dy \) and \( q(z) \) is an arbitrary polynomial of \( z \) and its complex conjugate \( z^* \).

By definition, \( p(z) \) is a representation of \( P(x) \) if \( \langle Q(x) \rangle_p = \langle Q(z) \rangle_p \), where \( Q(x) \) is any polynomial on \( R^D \) and \( Q(z) \) its analytical extension on \( C^D \). Equivalently, one can demand \( \langle x_{i_1} \cdots x_{i_n} \rangle_p = \langle z_{i_1} \cdots z_{i_n} \rangle_p \) for any set of indices, where \( i_r = 1, \ldots, D \) and \( n = 0, 1, 2, \ldots \). Two complex probabilities on \( C^D \) will be called equivalent if they have the same expectation values on every analytical polynomial. In general, two equivalent probabilities will not coincide on expectation values of non analytical polynomials \( \langle z_{i_1} \cdots z_{i_n} z_{j_1}^* \cdots z_{j_m}^* \rangle \). A representation will be called real if \( p(z) \) is real, positive if \( p(z) \) is non negative and unitary if \( \int p(z)d^Dz = \int P(x)d^Dx \). Our goal is then to find positive representations of complex probabilities.

We will proceed by noting different ways to obtain new representations from known ones. A first obvious way is by means of complex affine transformations. Let \( A \) be a non singular complex \( D \times D \) matrix, and \( a \in C^D \), and assume that \( P_0(z) \) is an analytical function in a region including \( R^D \) and \( AR^D + a \) such that \( P_0(x) \) and \( P(x) = \det(A)P_0(Ax + a) \) are both complex probabilities. Then if \( p_0(z) \) is a unitary representation of \( P_0(x) \) so is \( p(z) = |\det(A)|^2p_0(Az + a) \) of \( P(x) \): for any polynomial \( Q(x) \)

\[
|\det(A)|^2 \int Q(z)p_0(Az + a)d^Dz = \int Q(A^{-1}(z - a))p_0(z)d^Dz \\
= \int Q(A^{-1}(x - a))P_0(x)d^Dx = \det(A)\int Q(x)P_0(Ax + a)d^Dx.
\]

Furthermore, \( p(z) \) is positive if \( p_0(z) \) is positive. Another construction follows from linear combination. If \( p_i(z) \) are unitary representations of \( P_i(x) \), so is \( p(z) = \sum_{i=1}^n b_ip_i(z) \) of \( P(x) = \sum_{i=1}^n b_iP_i(x) \). Again, if \( p_i(z) \) are positive and \( b_i \) non negative, \( p(z) \) is positive too.

Let us define the partial derivatives \( \partial_k \) and \( \partial_k^* \) on a function on \( C^D \) as \((\partial/\partial x_k \mp i\partial/\partial y_k)/2\), respectively and let \( \phi(z) \) be in the class of distributions on \( C^D \) defined above but dropping the restriction \( \int \phi(z)d^Dz \neq 0 \). Then if \( p(z) \) is a probability, \( p(z) + \partial_k^*\phi(z) \) is also a probability and in fact (unitarily) equivalent to \( p(z) \),

\[
\int Q(z)\partial_k^*\phi(z)d^Dz = \int \partial_k^*(Q(z)\phi(z))d^Dz = 0,
\]

where \( Q(z) \) is any analytical polynomial. That is, \( \partial_k^*\phi(z) \) would represent the zero distribution on \( R^D \). Such distributions will be called null distributions. They will prove useful in what follows to obtain positive representations from real ones, namely, by adding null distribution of the form \( \sum_{k=1}^D \partial_k\partial_k^*\phi_k(z) \), for suitably chosen real \( \phi_k(z) \). Note that \( 4\partial_k\partial_k^* \) is just a Laplacian.

Similarly, by proceeding as in eq. (5), it follows that if \( p(z) \) represents \( P(x) \), the following relations hold

\[
\int Q(z)\partial_kp(z)d^Dz = \int Q(x)\partial_kP(x)d^Dx \\
\int Q(z)R(z)p(z)d^Dz = \int Q(x)R(x)P(x)d^Dx
\]
where $Q(x)$ and $R(x)$ are arbitrary polynomials. That is, $\partial_k$ on $C^D$ represents $\partial_k$ on $\mathbb{R}^D$ and multiplication by an analytical polynomial $R(z)$ represents multiplication by $R(x)$.

Another interesting construction is related to convolutions. The convolution exist for any two complex probabilities since it can be defined through the product of their Fourier transforms which are regular distributions. If $p_1(z)$ and $p_2(z)$ are unitary representations of $P_1(x)$ and $P_2(x)$ respectively, their convolution $p_1 \ast p_2$ is a unitary representation of $P_1 \otimes P_2$. Indeed, $p_1 \otimes p_2$ is a unitary representation of $P_1 \otimes P_2$ and

$$
\langle z_1 \cdots z_n \rangle_{p_1 \ast p_2} = \langle (z_{i_1}^{(1)} + z_{i_1}^{(2)}) \cdots (z_{i_n}^{(1)} + z_{i_n}^{(2)}) \rangle_{p_1 \otimes p_2}
$$

Furthermore, if $p_1(z)$ and $p_2(z)$ are positive, $p_1 \ast p_2$ is positive too. In particular, this allows for obtaining equivalent representations of known ones: if $p(z)$ is a unitary representation of $P(x)$ and $C(z)$ is a unitary representation of $\delta(x)$, the $D$-dimensional Dirac delta function, $p \ast C$ will be unitarily equivalent to $p(z)$, since $P \ast \delta = P$. Any probability $C(z)$ normalized to one defines a unitary representation of $\delta(x)$ if it is invariant under global phase rotations, i.e., $C(e^{i\varphi}z) = C(z)$ for any $\varphi \in \mathbb{R}$. In this case

$$
\int z_1 \cdots z_n C(z) d^{2D}z = \delta_{n,0},
$$

since the angular average of $z_1 \cdots z_n$ vanishes for $n > 0$. In fact this construction can be regarded as adding a Laplacian, namely, $p \ast C - p$, as it is easily seen after Fourier transform. This procedure can be used to obtain positive representations from real ones. On the other hand, it shows that if a complex probability admits a unitary positive representation it is not unique.

A unitary representation can always be obtained for any $P(x)$ by taking $p(z) = P(x)\delta(y)$. If $P(x)$ is positive so will be $p(z)$. This can be generalized as follows. Let $P_0(x)$ be positive and $P(x) = P_0(x - it)$, $t \in \mathbb{R}^D$ (i.e., a complex translation under the conditions considered above for affine transformations). Then $p(z) = P_0(x)\delta(y - t)$ is a unitary positive representation of $P(x)$. If we allow $P_0$ to depend on $t$, taking linear combinations we obtain that $p(z) = p(x, y)$ is a unitary representation of

$$
P(x) = \int p(x - iy, y)d^Dy.
$$

This relation has been noted before in the literature [16,13], considered as a projection from probabilities on $C^D$ to probabilities on $\mathbb{R}^D$. Note, however, that when this relation can be applied it gives just one of the $P(x)$ represented by $p(z)$. In fact, since the momenta of $P(x)$ are the Taylor expansion coefficients of its Fourier transform, there are many complex probabilities characterized by the same momenta. As we have seen, under this projection, the operation $\partial^*_i$ is mapped to zero. Similarly, $\partial_i$ is mapped to $\partial/\partial x_i$, and multiplication by an analytical polynomial $Q(z)$ is mapped to multiplication by $Q(x)$.

As an immediate application of eq. (9), we find that for $m_{ij}$ real, symmetric and positive definite, the probability

$$
P(x) = \tilde{f}(x) \exp\left(-\frac{1}{2}m_{ij}x_ix_j\right)
$$
is represented by

\[ p(z) = \det(m) f(my) \exp\left(-\frac{1}{2} m_{ij} z_i^* z_j^* \right), \quad (11) \]

where \( \tilde{f} \) is the Fourier transform of \( f \) (the repeated index convention will be used in what follows). For example, for \( D = 1 \), and \( \Gamma \) positive, \( P(x) = \cos(x) \exp(-x^2/2\Gamma) \) is represented by the positive probability \( p(z) = \exp(-x^2/2\Gamma)(\delta(y - \Gamma) + \delta(y + \Gamma)) \). Since in this example \( P(x) \) is real but not positive definite, this is an instance where a complex Langevin simulation would fail [12,15,13], yet there is a positive representation.

Next, let us show that every complex probability on \( \mathbb{R}^D \) admits a real representation. Let \( P(x) \) be a complex probability normalized to one and \( \tilde{P}(k) \) its Fourier transform

\[ \tilde{P}(k) = \int e^{ikx} P(x) d^D x \quad (12) \]

where \( kx = k_i x_i \). By definition we have

\[ \tilde{P}(k) = \sum_{n=0}^{\infty} \frac{i^n}{n!} k_i \cdots k_{in} \langle x_{i_1} \cdots x_{i_n} \rangle_P \quad (13) \]

in a neighborhood of \( k = 0 \) since \( \tilde{P}(k) \) is analytic at the origin. Also,

\[ \langle x_{i_1} \cdots x_{i_n} \rangle_P = (-i)^n \partial_{i_1} \cdots \partial_{i_n} \tilde{P}(k) |_{k=0}. \quad (14) \]

For a probability \( p(z) \) on \( \mathbb{C}^D \), the Fourier transform is defined similarly,

\[ \tilde{p}(\sigma) = \int e^{ikx+iry} p(z) d^{2D} z = \int e^{i(\sigma z^* + z\sigma^*)/2} p(z) d^{2D} z, \quad (15) \]

where \( \sigma_i = k_i + ir_i \). Assuming that \( p(z) \) is normalized to one, its momenta are obtained through

\[ \langle z_{i_1} \cdots z_{i_n} z_{j_1}^* \cdots z_{j_m}^* \rangle_p = (-2i)^{n+m} \partial_{i_1}^* \cdots \partial_{i_n}^* \partial_{j_1} \cdots \partial_{j_m} \tilde{p}(\sigma) |_{\sigma=0} \quad (16) \]

where \( \partial_i \) refers to \( \sigma_i \) and \( \partial_i^* \) to \( \sigma_i^* \). Consider the following probability,

\[ \tilde{p}(\sigma) = \tilde{C}(\sigma) \tilde{P} \left( \frac{\sigma^*}{2} \right) \tilde{P} \left( -\frac{\sigma^*}{2} \right)^*. \quad (17) \]

Here \( C(z) \) is one of the real unitary representations of \( \delta(x) \) above mentioned. Thus \( \tilde{C}(\sigma) \) is analytical at the origin as a function of \( k_i \) and \( r_i \) and is invariant under global phase rotations of \( \sigma \). \( \tilde{P}(\sigma) \) stands for the analytical extension of \( \tilde{P}(k) \) in a neighborhood of the origin. Beyond the analyticity circle (if it is finite) we can choose \( \tilde{C}(\sigma) \) equal to zero so that \( \tilde{p}(\sigma) \) exists. By construction, \( \tilde{p}(\sigma) \) is unity at the origin and analytical there. Also it is locally integrable and, with a suitable choice of \( \tilde{C}(\sigma) \), grows at most polynomially at infinity, therefore it defines a probability \( p(z) \) on \( \mathbb{C}^D \). Furthermore, \( p(z) \) is real since \( C(z) \) is real and \( (\tilde{p}(\sigma))^* = \tilde{p}(-\sigma) \). It remains to show that it is a representation of \( P(x) \),

\[ \langle z_{i_1} \cdots z_{i_n} \rangle_p = (-2i)^n \partial_{i_1}^* \cdots \partial_{i_n}^* \tilde{p}(\sigma) |_{\sigma=0} = (-2i)^n \partial_{i_1}^* \cdots \partial_{i_n}^* \tilde{P} \left( \frac{\sigma^*}{2} \right) |_{\sigma=0} = (-i)^n \partial_{i_1} \cdots \partial_{i_n} \tilde{P}(k) |_{k=0} = \langle x_{i_1} \cdots x_{i_n} \rangle_P, \quad (18) \]
where it has been used that \( \partial_1^* \cdots \partial_n^* \hat{C}(\sigma) |_{\sigma=0} \) vanishes for \( n > 0 \). That is, we have given a constructive method, eq. (17), to obtain a real representation of any complex probability within the class of complex probabilities considered.

As an illustration, consider \( D = 1 \) and

\[
P(x) = \delta(x) + a \delta'(x), \quad a = a_R + i a_I \in \mathbb{C}.
\]

In this case \( \hat{P}(\sigma) = 1 - i a \sigma \) is a polynomial, thus it is entire and well-behaved at infinity and we can take \( \hat{C}(\sigma) = 1 \), i.e., \( C(z) = \delta(x) \delta(y) \). With this choice

\[
\tilde{p}(\sigma) = 1 - \frac{1}{4} |a|^2 |\sigma|^2 - \frac{i}{2} (a^* \sigma + a \sigma^*)
\]

and

\[
p(z) = \delta(x) \delta(y) + a_R \delta'(x) \delta(y) + a_I \delta(x) \delta''(y) + \frac{1}{4} |a|^2 (\delta''(x) \delta(y) + \delta(x) \delta''(y)).
\]

One can easily check that this is a real distribution which represents \( P(x) \), however it is not positive. We can find a positive representation by first applying a convolution (i.e., a better choice of \( C(z) \)) and then adding a suitable Laplacian. Furthermore, it can be done for an arbitrary distribution of support at zero in any number of dimensions. Rather than showing this in detail here, it will be obtained as a byproduct in the next section. There we will obtain positive representations of Gaussian functions times polynomials.

By formally undoing the Fourier transform of \( \tilde{p}(\sigma) \) in eq. (17), the following explicit form of \( p(z) \) is obtained

\[
p(z) = \int C_0(x - \frac{x_1 + x_2}{2}, y - \frac{x_1 - x_2}{2i}) P(x_1) P^*(x_2) d^D x_1 d^D x_2,
\]

where \( C_0(z_1, z_2) \) is the analytical extension of \( C_0(x, y) = C(x + iy), \) with \( x \) and \( y \) real. In order for this formula to make sense, we should require \( C_0(z_1, z_2) \) to be entire on \( \mathbb{C}^D \) and further the integrand should be sufficiently convergent so as to define a probability on \( \mathbb{C}^D \). Such probability is real by construction, since \( C(z) \) is real, however it will not be positive in general even if \( C(z) \) is positive since such property is lost after analytical extension. The interest of this relation, as compared, for instance with that in eq. (9), is that it is constructive.

An example of application of this formula is provided by

\[
P(x) = \sum_{i=1}^{N} a_i \delta(x - x^{(i)}), \quad C(z) = \exp \left( -\frac{z j z^*}{2 \Gamma} \right),
\]

which gives

\[
p(z) = \sum_{i,j=1}^{N} a_i a_j^* \exp \left( -\frac{1}{2 \Gamma} \left( \left( x - \frac{x^{(i)} + x^{(j)}}{2} \right)^2 + \left( y - \frac{x^{(i)} - x^{(j)}}{2i} \right)^2 \right) \right).
\]

Another application is when \( P(x) \) is a finite linear combination of Gaussian distributions centered anywhere in the complex plane and with arbitrary complex widths, provided we choose \( \Gamma > |\Gamma_i|, i = 1, \ldots, N. \)
III. POSITIVE REPRESENTATIONS OF GAUSSIAN DISTRIBUTIONS

A Gaussian complex probability takes the general form

\[ G(x) = N_G \exp\left(-\frac{1}{2} m_{ij} x_i x_j - b_i x_i\right), \]
\[ N_G = (2\pi)^{-D/2} (\det(m))^{1/2} \exp\left(-\frac{1}{2} (m^{-1})_{ij} b_i b_j\right) \quad (25) \]

where \( m_{ij} \) is a symmetric complex matrix with positive definite real part to ensure normalizability. As a consequence \( m_{ij} \) is non singular and can be written as \( A_k A_i \). This allows to set \( m_{ij} = \delta_{ij} \) and \( b_i = 0 \) by means of a complex affine transformation. That is, we will consider only

\[ G(x) = (2\pi)^{-D/2} \exp\left(-\frac{1}{2} x_i x_i\right) \quad (26) \]

and the general case can be obtained a posteriori as \( G(A x + A^{-1} b) \). A positive representation of \( G(x) \) is simply \( G(x) \delta(y) \). A more general representation \( g(z) \) is obtained by convolution with \( C(z) = (2\pi \eta)^{-D} \exp\left(-\frac{1}{2\eta} z_i z_i^*\right) \), where \( \eta \) is positive. This gives

\[ g(z) = N_g \exp\left(-\frac{1}{2(\eta + 1)} x_i x_i - \frac{1}{2\eta} y_i y_i\right) = N_g \exp\left(-\bar{\gamma} z_i z_i^* + \frac{1}{2} \gamma z_i z_i + \frac{1}{2} \gamma z_i^* z_i^*\right), \quad (27) \]

where the normalization constant is \( N_g = \left(2\pi \sqrt{\eta(\eta + 1)}\right)^{-D} \) and we have introduced the positive numbers

\[ \gamma = \frac{1}{4\eta(\eta + 1)}, \quad \bar{\gamma} = \frac{2\eta + 1}{4\eta(\eta + 1)}. \quad (28) \]

The same representation is obtained by following the method of eq. (22). The value of the parameter \( \eta \), or equivalently \( \gamma \), will be fixed below.

The set of probabilities to be considered is \( P(x) = Q(x) G(x) \), where \( Q(x) \) is a complex polynomial of degree \( N \). \( P(x) \) can always be written as

\[ P(x) = \sum_{n=0}^{N} \frac{1}{n!} a_{i_1 \ldots i_n} \partial_{i_1} \ldots \partial_{i_n} G(x), \quad (29) \]

where \( a_{i_1 \ldots i_n} \) is completely symmetric and the zeroth order coefficient \( a_0 \) must not vanish (in fact, is unity if \( P(x) \) is normalized). A real representation of \( P(x) \) is given by

\[ p_0(z) = (|a_0|^2 + a_0^* \sum_{n=1}^{N} \frac{1}{n!} a_{i_1 \ldots i_n} \partial_{i_1} \ldots \partial_{i_n} + a_0 \sum_{n=1}^{N} \frac{1}{n!} a_{i_1 \ldots i_n} \partial_{i_1}^* \ldots \partial_{i_n}^*) g(z), \quad (30) \]

since the terms with \( \partial^* \) do not contribute and \( \partial / \partial z \) is mapped to \( \partial / \partial x \) under projection.

It is convenient to introduce the polynomials...
\[ Q_{i_1...i_n}(z) = g(z)^{-1} \partial_{i_1} \cdots \partial_{i_n} g(z). \]  

(31)

They can be computed recursively by means of the formula

\[ Q_0(z) = 1, \quad Q_{i_1...i_n}(z) = (\partial_{i_n} + \omega_{i_n})Q_{i_1...i_{n-1}}(z), \]

(32)

where we have introduced the variable

\[ \omega_i = \gamma z_i - \bar{\gamma} z_i^* . \]

(33)

The functions \( Q_{i_1...i_n}(z) \) are polynomials of degree \( n \) in \( \omega_i \), with coefficients depending only on \( \gamma \). With this notation, \( p_0(z) \) can be rewritten as

\[
p_0(z) = \left( |a_0|^2 + a_0^* \sum_{n=1}^{N} \frac{1}{n!} a_{i_1...i_n} Q_{i_1...i_n}(z) + a_0 \sum_{n=1}^{N} \frac{1}{n!} a_{i_1...i_n}^* Q_{i_1...i_n}^*(z) \right) g(z).
\]

(34)

In order to obtain a positive representation, \( p(z) \) can be further cast in the form

\[
p_0(z) = \left( |a_0|^2 + \sum_{n=1}^{N} \frac{1}{n!} \beta_n |Q_{i_1...i_n}(z)|^2 + \beta_n^{-1} a_0 a_{i_1...i_n}^* |a_n|^2 \right) g(z),
\]

(35)

where the indices \( i_1 \ldots i_n \) are summed over, \( \beta_1, \ldots, \beta_N \) are arbitrary positive numbers which value is to be specified below and we have defined the quantity \( |a_n| \) as

\[
|a_n|^2 = \frac{1}{n!} a_{i_1...i_n} a_{i_1...i_n}^* .
\]

(36)

We will assume that \( |a_n| \) is non vanishing, since the vanishing case is trivial. In Appendix A it is shown that

\[
\phi_n(z) = \left( \frac{1}{n!} Q_{i_1...i_n}(z) Q_{i_1...i_n}^*(z) - \bar{\gamma}^n K_n(D) \right) g(z)
\]

(37)

is a null distribution, where

\[
K_n(D) = \frac{(D+n-1)!}{n!(D-1)!}.
\]

(38)

By removing \( \phi_n(z) \) from \( p_0(z) \) we obtain an equivalent representation \( p(z) \), namely,

\[
p(z) = \left[ \sum_{n=1}^{N} \frac{1}{n!} \beta_n |Q_{i_1...i_n}(z)|^2 + |a_0|^2 - \sum_{n=1}^{N} (\beta_n \bar{\gamma}^n K_n(D) + \beta_n^{-1} |a_0|^2 |a_n|^2) \right] g(z).
\]

(39)

To ensure positivity of \( p(z) \) we require

\[
\sum_{n=1}^{N} (\beta_n \bar{\gamma}^n K_n(D) + \beta_n^{-1} |a_0|^2 |a_n|^2) \leq |a_0|^2 .
\]

(40)
This can be achieved by choosing the positive coefficients $\beta_n$ so as to minimize the left-hand side,

$$\beta_n = \frac{|a_0| \cdot |a_n|}{\sqrt{\gamma^n K_n(D)}}. \quad (41)$$

In this way the inequality is satisfied for any $\bar{\gamma}$ smaller than the unique positive solution of

$$\sum_{n=1}^{N} \sqrt{K_n(D)} |a_n| \bar{\gamma}^{n/2} = \frac{1}{2} |a_0|. \quad (42)$$

For this choice of $\bar{\gamma}$, $p(z)$ takes the simple form

$$p(z) = \sum_{n=1}^{N} \frac{1}{n!} \beta_n |Q_{i_1 \cdots i_n}(z)| + \beta^{-1}_n a_0 a^*_{i_1 \cdots i_n} |2g(z)|. \quad (43)$$

To summarize, any Gaussian times polynomial complex probability, eq. (29), admits a positive representation, namely, $p(z)$ in eq. (43), with $\beta_n$ given by eq. (41), and $\bar{\gamma}$ given by eq. (42).

Incidentally, let us note that from a computational point of view, it is convenient to minimize the width of $p(z)$ in the complex plane (e.g., if $P(x)$ is already positive, the best choice is $P(x)\delta(y)$), since this reduces the dispersion of points in the sample. In the family of probabilities described by the expression of $p(z)$ in eq. (39), this minimization corresponds to our choice of $\beta_n$ in eq. (41) and $\bar{\gamma}$ in eq. (42). In general, however, this needs not be best equivalent positive representation of $P(x)$. The construction presented above corresponds to adding to $p_0(z)$ a Laplacian of the form $\partial_{i_1} \cdots \partial_{i_n} \partial^*_{i_1} \cdots \partial^*_{i_n} g(z)$ (as can be seen using the formulas of Appendix A). More generally, one could add terms of the form $b_{i_1 \cdots i_n ; j_1 \cdots j_n} \partial_{i_1} \cdots \partial_{i_n} \partial^*_{j_1} \cdots \partial^*_{j_n} g(z)$, with $b$ self-adjoint, in order to optimize $p(z)$, or even more general terms so long as they have a $\partial^*_{j}$ and are real.

Let us now come back to the problem of finding positive representations of complex distributions with support at 0. Such distributions take the form

$$P(x) = \sum_{n=0}^{N} \frac{1}{n!} a_{i_1 \cdots i_n} \partial_{i_1} \cdots \partial_{i_n} \delta(x). \quad (44)$$

This distribution can be considered as the zero width limit of the Gaussian times polynomial distribution.

$$P(x) = \lim_{\lambda \to 0^+} P_{\lambda}(x), \quad P_{\lambda}(x) = \sum_{n=0}^{N} \frac{1}{n!} a_{i_1 \cdots i_n} \partial_{i_1} \cdots \partial_{i_n} (\lambda^{-D} G(x/\lambda)). \quad (45)$$

Naming $P(x; a)$ the probability in eq. (29), we find

$$P_{\lambda}(x) = \lambda^{-D} P(x/\lambda; a^\lambda), \quad a^\lambda_{i_1 \cdots i_n} = \lambda^{-n} a_{i_1 \cdots i_n}. \quad (46)$$

Therefore, the positive representation of $P(x; a)$, namely, $p(z; a)$ in eq. (43), provides a positive representation of $P_{\lambda}(x)$,
\[ p_\lambda(z) = \lambda^{-2D} p(z/\lambda; a^\lambda) . \]  

In order to take the limit, we should consider how the different variables scale. We already have the scaling law of \( z \) and of the coefficients \( a_{i_1 \ldots i_n} \). From eqs. (41,42) \( \beta_n^\lambda \) is found to scale as \( \lambda^{-2n} \beta_n \) and \( \bar{\gamma}^\lambda \) as \( \lambda^2 \bar{\gamma} \). From eqs. (28), \( \eta^\lambda \) is given in leading order by \( \lambda^{-2} \eta \) with \( \eta = 1/(2\bar{\gamma}) \) and \( \bar{\gamma}^\lambda \) is of order \( \lambda^4 \) and can be neglected. Therefore, in leading order \( \lambda^{-2D} g(z/\lambda; \bar{\gamma}^\lambda) \) becomes

\[ g^0(z; \bar{\gamma}) = (2\pi \eta)^{-D} \exp(-\bar{\gamma}z^*_i z^*_i), \quad \eta = \frac{1}{2\bar{\gamma}} \]  

and is independent of \( \lambda \). This results is to be used in eq. (43). Finally, in leading order, \( Q_{i_1 \ldots i_n}(z/\lambda; \bar{\gamma}^\lambda) \) becomes \( \lambda^\alpha Q_{i_1 \ldots i_n}^0(z; \bar{\gamma}) \) with

\[ Q_{i_1 \ldots i_n}^0(z; \bar{\gamma}) = g^0(z; \bar{\gamma})^{-1} \partial_{i_1} \cdots \partial_{i_n} g^0(z; \bar{\gamma}) = (-\bar{\gamma})^n z^*_i \cdots z^*_i . \]  

To summarize, any complex distribution with support at a single point, eq. (44), admits a positive representation, namely,

\[ p(z) = \sum_{n=1}^{N} \frac{1}{n!} \beta_n |Q_{i_1 \ldots i_n}^0(z)| + \beta_n^{-1} a_0 a_{i_1}^* \cdots a_{i_n}^* |^2 g^0(z) . \]  

with \( \beta_n \) given by eq. (41), and \( \bar{\gamma} \) given by eq. (42).

As an illustration we can consider again the distribution of eq. (19). In this case we find \( \bar{\gamma} = (4|a|^2)^{-1} \) and \( \eta = \beta_1 = 2|a|^2 \), and thus

\[ p(z) = |z - 2a|^2 \exp\left(-\frac{|z|^2}{4|a|^2}\right) . \]  

As a final application of the results of this section, we can consider periodic probabilities. Such probabilities correspond to variables effectively defined in a compact domain and find application in the context of compact gauge theories on the lattice. They satisfy, \( P(x) = P(x - na) \) with \( (na)_i = n_i a_i \) where \( n \in \mathbb{Z}^D \) is arbitrary and \( a \in \mathbb{R}^D_+ \) is characteristic of \( P(x) \). Without loss of generality, we may choose \( a_i = 2\pi \). These probabilities do not belong to the class previously considered. The normalization as well as the expectation values should be taken on a lattice cell \( \{x, 0 \leq x_i < 2\pi, i = 1, \ldots, D\} \). The test functions should be periodic and the concept of representation should be modified accordingly: \( p(z) \) is periodic on the real axis, \( x \) is to be integrated on the periodic cell and \( y \) on \( \mathbb{R}^D \). Also instead of equality of expectation values of polynomials we demand \( \langle \exp(in_j x_j) \rangle_P = \langle \exp(in_j z_j) \rangle_p \) for any integers \( n_j, j = 1, \ldots, D \). Assume now that the periodic distribution is a function of the form

\[ P(x) = \sum_{n \in \mathbb{Z}^D} P_0(x - 2\pi n), \]  

where the series is uniformly convergent. Let \( p_0(z) \) be a function which is a positive representation of \( P_0(x) \) not only on polynomials but also on exponential test functions, and such that
\[ p(z) = \sum_{n \in \mathbb{Z}^D} p_0(z - 2\pi n). \]  

(53)

is uniformly convergent. Then, \( p(z) \) is a positive representation of \( P(x) \), as is readily shown.

In particular, \( P_0(x) \) may be a Gaussian times polynomial and \( p_0(z) \) its positive representation found above, since these functions are sufficiently convergent at infinity. Therefore the construction given above provides a positive representation for this case too. Another example is the periodic version of the one dimensional Gaussian times cosine considered above after eq. (11):

\[ P(x) = \cos(x) \sum_{n \in \mathbb{Z}} \exp \left( -\frac{(x - 2\pi n)^2}{2\Gamma} \right), \]

\[ p(z) = (\delta(y - \Gamma) + \delta(y + \Gamma)) \sum_{n \in \mathbb{Z}} \exp \left( -\frac{(x - 2\pi n)^2}{2\Gamma} \right). \]

(54)

This example is interesting since it is similar to simplified probabilities considered in the literature [12,15] to model the SU(2) gauge theory in the presence of a Wilson loop, for which the complex Langevin algorithm did not work.

**IV. CONCLUDING REMARKS**

We have studied the problem of representation of complex distributions by distributions on the analytically extended complex plane. The positive representation problem is of immediate interest in some areas of physics: field theory and statistical mechanics. On the other hand it also seems a new and interesting field from the mathematical point of view. One could consider extending the particular class of complex distributions studied here, namely, Fourier transforms of regular distributions analytical at the origin, by allowing as well for adding non regular distributions with support outside the origin. Perhaps more interesting, and in the opposite direction, one could extend the set of test functions in the definition of representation beyond polynomials to insure, for instance, that each probability on \( C^D \) is at most the representation of one probability on \( R^D \). From the viewpoint of applications it would also be interesting to extend the concept of representations to distributions defined on group manifolds since they appear naturally in lattice gauge theories. Our discussion on periodic distributions corresponds in fact to the manifold of the direct product of \( D \) U(1) factors.

**V. ACKNOWLEDGMENTS**

I would like to thank C. García-Recio for comments on the manuscript.

**APPENDIX A**

In this appendix we will show that \( \phi_n(z) \) defined in eq. (37) is a null distribution. To this end let us introduce the polynomials
\[ Q_{i_1 \ldots i_n j_1 \ldots j_m}(z) = g(z)^{-1} \partial_{i_1} \ldots \partial_{i_n} \partial_{j_1}^* \ldots \partial_{j_m}^* g(z). \] (A1)

They generalize \( Q_{i_1 \ldots i_n}(z) \) and satisfy the relation
\[ Q_{i_1 \ldots i_n j_1 \ldots j_m}(z) = Q_{j_1 \ldots j_m i_1 \ldots i_n}^*(z). \] (A2)

To prove eq. (37), we will use the following Wick theorem:
\[ Q_{i_1 \ldots i_n}(z)Q_{j_1 \ldots j_m}^*(z) = \sum_{[i_1 \ldots i_n; j_1 \ldots j_m]} Q_{i_1 \ldots i_n j_1 \ldots j_m}(z). \] (A3)

where the sum is over all possible sets of contractions of the indices \( i_1 \ldots i_n \) with the indices \( j_1 \ldots j_m \). The contraction of two indices \( i, j \) gives a factor \( \gamma \delta_{ij} \) and removes them from the list, e.g.,
\[ Q_{i_1 i_2}(z)Q_{j_2}^*(z) = Q_{i_1 i_2 j_2}(z) + \gamma \delta_{i_1 j_2} Q_{i_2}(z) + \gamma \delta_{i_2 j_2} Q_{i_1}(z). \] (A4)

In general there are \( n! m! / k!(n - k)! (m - k)! \) terms with \( k \) contractions. Let us apply the Wick theorem to \( Q_{i_1 \ldots i_n}(z)Q_{j_1 \ldots j_m}^*(z)g(z) \). Whenever two indices \( i, j \) are not contracted we will have \( Q_{i \ldots j}(z)g(z) \) which contains \( \partial_j^* \) and hence is a null distribution. Therefore only the terms with all indices contracted contribute and the non null part is
\[ \gamma^n \sum_{p \in S_n} \delta_{i_1 j_{p1}} \cdots \delta_{i_n j_{pn}} g(z), \] (A5)

where the sum runs over all permutations. After contracting the indices we obtain eq. (37). \( K_n(D) \) is the number of ways of choosing \( n \) objects out of \( D \) allowing repetitions.

The Wick theorem can be proven by induction. Defining the operator
\[ D_1 = g^{-1}(z)\partial_i g(z) = \partial_i + \omega_i, \] (A6)

\( (g(z) \) is a multiplicative operator here \) we have
\[ Q_{i_1 \ldots i_n}(z) = D_{i_1} \cdots D_{i_n} Q_0(z), \]
\[ Q_{i_1 \ldots i_n j_1 \ldots j_m}(z) = D_{i_1} \cdots D_{i_n} D_{j_1}^* \cdots D_{j_m}^* Q_0(z), \] (A7)

where \( Q_0(z) = 1 \). Trivially, \( [\partial_i, D_j^*] = -\gamma \delta_{ij} \), thus
\[ \partial_i Q_{j_1 \ldots j_m}^*(z) = - \sum_{k=1}^m \gamma \delta_{ij_k} Q_{j_1 \ldots j_k \ldots j_m}^*(z), \] (A8)

where the hat means that the index has been removed from the list. On the other hand \( D(AB) = (DA)B - A\partial B \). The Wick theorem holds for \( n = m = 0 \). Assuming it has been proven up to some \( (n, m) \),
\[ Q_{i_1 \ldots i_n+1}(z)Q_{j_1 \ldots j_m}^*(z) = (D_{i_1} \cdots D_{i_n} Q_{i_1 \ldots i_n}(z))Q_{j_1 \ldots j_m}^*(z) \]
\[ = D_{i_1} \cdots (Q_{i_1 \ldots i_n}(z)Q_{j_1 \ldots j_m}^*(z)) = Q_{i_1 \ldots i_n}(z)Q_{j_1 \ldots j_m}^*(z) + Q_{i_1 \ldots i_{n+1}}(z)D_{i_{n+1}} Q_{j_1 \ldots j_m}^*(z). \] (A9)

Using that the theorem holds for \( (n, m) \) and eq. (A8),
\[ Q_{i_1 \ldots i_{n+1}}(z)Q^{*}_{j_1 \ldots j_m}(z) = \sum_{[i_1 \ldots i_{n+1}; j_1 \ldots j_m]} Q_{i_1 \ldots i_{n+1}j_1 \ldots j_m}(z) \]
\[ + \sum_{k=1}^{m} \sum_{[i_1 \ldots i_{n}; j_1 \ldots \hat{j}_k \ldots j_m]} \tilde{\gamma}\delta_{i_{n+1}j_k} Q_{i_1 \ldots i_{n}; j_1 \ldots \hat{j}_k \ldots j_m}(z). \] (A10)

The first term contains all the contractions not involving the index \( i_{n+1} \), and the second one all the contractions involving the index \( i_{n+1} \), hence the theorem is proven for \((n + 1, m)\). It is worth noticing that the reverse expansion also holds, i.e.,

\[ Q_{i_1 \ldots i_n;j_1 \ldots j_m}(z) = \sum_{[i_1 \ldots i_n;j_1 \ldots j_m]} Q_{i_1 \ldots i_n}(z)Q^{*}_{j_1 \ldots j_m}(z). \] (A11)

where the contraction of \( ij \) now is \(-\tilde{\gamma}\delta_{ij}\).