Lukyanov’s Screening Operators for the Deformed Virasoro Algebra

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Abstract

The BRST property of Lukyanov’s screening operators in the bosonic representation of the deformed Virasoro algebra is proven.

1. There are two basic approaches to investigation of integrable two-dimensional theories. The most general method is based on the Yang–Baxter equation. Because of this equation the theory has an infinite-dimensional abelian symmetry algebra (commuting integrals of motion) and admits studying by use of the Bethe Ansatz technique. Another approach exploits the existence of infinite-dimensional non-abelian symmetries (Virasoro algebra, W algebra, Kac–Moody algebra, etc.) [1]. The second approach gives a more detailed description of the theory, including an explicit computation of correlation functions. However its applicability was restricted to conformal models of the 2D quantum field theory.

It was recognized in the work [2] that the theory of off-critical two-dimensional integrable models of statistical mechanics is tightly connected with the representation theory of infinite-dimensional quantum algebras and one can study such models using ideas of conformal field theory. In particular, the six-vertex model is governed by the infinite-dimensional quantum affine algebra \( U_q(\widehat{sl}_2) \). Namely, the ‘half’ of the space of states of the six-vertex model in the corner transfer matrix approach can be identified with the irreducible representations of \( U_q(\widehat{sl}_2) \). The well-developed representation theory of the latter makes it possible to diagonalize the transfer matrix by the vertex operator technique and to calculate correlation functions of the theory [3].

In the Andrews–Baxter–Forrester restricted solid-on-solid (RSOS) models [4] in the regime III a similar role of dynamical symmetry algebra belongs to the

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deformed Virasoro algebra (DVA) discovered recently [5, 6, 7]. It is expected by construction [8, 9] that the structure of representations of DVA is very similar to that of the Virasoro algebra. However, it has not yet been proved rigorously.

The present letter aims at clearing up some open points in this problem. Recall that, according to [10], the structure of irreducible representations of the Virasoro algebra heavily depends on the arithmetic nature of its central charge. The most complicated (and physically interesting) case corresponds to the central charge

$$c = 1 - \frac{6}{r(r-1)}$$

with integer $r \geq 4$. There are infinitely many singular vectors in the Verma module in this situation. An explicit description of the irreducible representations of the Virasoro algebra with the central charge (1) in terms of the Fock spaces is given by use of the two-sided Felder resolution [11]. We prove two Propositions concerning existence of the Felder complex for DVA as it was claimed in [7]. Namely, we check that certain powers of the deformed screening operators introduced by Lukyanov commute with the generating function of the deformed Virasoro algebra, i.e. they are intertwining operators and define the injection structure of the singular vectors for the latter. We then give a proof that the deformed screening operators satisfy the BRST property, so that the original Felder complex can be deformed to that for DVA.

2. Recall the definition of the deformed Virasoro algebra [6]. It is generated by elements $T_n, n \in \mathbb{Z}$, such that the series $T(z) = \sum_{n \in \mathbb{Z}} T_n z^{-n}$ satisfies the defining relations

$$f \left( \frac{z}{z'} \right) T(z') T(z) - f \left( \frac{z'}{z} \right) T(z) T(z') = (x - x^{-1})[r - 1]_x [r]_x \left( \delta \left( \frac{z}{z' x^2} \right) - \delta \left( \frac{z x^2}{z'} \right) \right).$$

Here $x$ is a real parameter $0 < x < 1$, and the following notations are used:

$$f(z) = 1 - \frac{(x^{2r} z; x^4)_\infty (x^{-2r} z; x^4)_\infty}{(z; x^4)_\infty (x^{-2r+4} z; x^4)_\infty}, \quad (z; p)_\infty = \prod_{n=0}^{\infty} (1 - z p^n),$$

$$\delta(z) = \sum_{n=-\infty}^{\infty} z^n, \quad [n]_x = \frac{x^n - x^{-n}}{x - x^{-1}}.$$

In the limit $x \to 1$ the algebra (2) gives the Virasoro algebra with the central charge (1).

The bosonic representation of DVA is formulated in terms of the Heisenberg algebra generators $\beta_m, m \in \mathbb{Z} \setminus \{0\}$, and the zero mode operators $\mathcal{P}$ and $\mathcal{Q}$ with the relations

$$[\beta_m, \beta_{m'}] = m \frac{[m]_x}{[2m]_x} \frac{[(r - 1)m]_x}{[rm]_x} \delta_{m+m',0}, \quad [\mathcal{P}, \mathcal{Q}] = -i.$$
Let us define also the Fock modules \( \mathcal{F}_{l,k} \) generated by \( \beta_m, m > 0 \) from the highest weight vector \( |l,k\rangle \):

\[
\mathcal{P}|l,k\rangle = \left( \sqrt{\frac{r}{2(r-1)}} l - \sqrt{\frac{r-1}{2r}} k \right) |l,k\rangle, \quad \beta_m |l,k\rangle = 0, \quad m > 0.
\]

For notational convenience we shall use also the operator \( \hat{\pi} = \sqrt{2r(r-1)}\mathcal{P} \).

The generating function \( T(z) \) of the deformed Virasoro algebra is given by

\[
T(z) = \Lambda_+(x^{-1}z) + \Lambda_-(xz) = \sum_{n \in \mathbb{Z}} T_n z^{-n},
\]

\[
\Lambda_{\pm}(z) = x_{\pm}^{{\pm}\hat{\pi}} :e^{-\sum_{m \neq 0} \frac{1}{m}(x^m - x^{-m})\beta_m z^{-m}}::.
\]

The colons hereafter mean the Wick ordering both in the oscillator and zero modes.

Consider now Lukyanov’s screening operator [8, 7]

\[
X = \oint_{|z|=1} \frac{dz}{2\pi i z} \xi(v) \frac{[v + \frac{1}{2} - \hat{\pi}]}{[v - \frac{1}{2}]},
\]

\[
\xi(v) = z^{\frac{1}{r-1}} :e^{i\sqrt{2(r-1)}(Q-iP \log z) + \sum_{m \neq 0} \frac{1}{m}(x^m + x^{-m})} z^{-m}::,
\]

where we set \( z = x^{2v} \). The notation \([v]\) stands for the theta function

\[
[v] = x^{v^2/r-v(x^{2r};x^{2r})_\infty(x^{2(r-v)};x^{2r})_\infty(x^{2r};x^{2r})_\infty,
\]

having the real half period \( r \)

\[
[v] = -[v + r].
\]

Note that the integrand of the above expression for \( X \) contains the factor [7]

\[
F(v;\hat{\pi}) = [v + \frac{1}{2} - \hat{\pi}]/[v - \frac{1}{2}]
\]

which has no counterpart in the representation theory of the Virasoro algebra. This factor ensures the integrand to be single-valued in \( z \) and the integration is taken over a closed contour. In the conformal limit \( x \to 1 \) with \( z = x^{2v} \) fixed, the function \( F(v;\hat{\pi}) \) tends to a constant as a function of \( z \) for \( \text{Arg } z \neq 0 \) while the poles and zeros of this function condense to a cut along the positive real axis in the \( z \)-plane.

3. It was shown in [6] that the screening current (5) commutes with the deformed Virasoro algebra generators up to a total difference:

**Lemma 1**

\[
[T_n, \xi(v)] = (x^{r-1} - x^{-r+1}) \left( A_n \left( v + \frac{r}{2} \right) - A_n \left( v - \frac{r}{2} \right) \right),
\]

\[
A_n(v) = z^{n+\frac{1}{r-1}} :e^{i\sqrt{2(r-1)}(Q-iP \log z) + \sum_{m \neq 0} \frac{1}{m}(x^{(r-1)m} + x^{(-r+1)m})} z^{-m}::.
\]
If the function $F(v; \hat{\pi})$ did not have poles, Lemma 1 would imply the commutativity of $T_n$ and $X$.¹ In the general case one needs to be more careful with the definition of spaces where the screening operators act. We claim that the following Proposition holds:

**Proposition 2**

For $1 \leq k \leq r$, the operator $X^k$ commutes with generators $T_n$ on the space $\mathcal{F}_{l,k'}$ provided $k' \equiv k \pmod{r}$

$$[T_n, X^k]_{\mathcal{F}_{l,k'}} = 0, \quad k' \equiv k \pmod{r}. \quad (9)$$

This Proposition means that in the bosonic realization the operators $X^k$ acting on the Fock space $\mathcal{F}_{l,k'}$ are intertwining operators for the deformed Virasoro algebra.

It is important that the Lukyanov’s screening operators satisfy also the following property

**Proposition 3**

The operator $X$ is nilpotent:

$$X^r = 0. \quad (10)$$

Due to this Proposition one can regard the intertwining operators given by powers of the screening operators as differentials of the two-sided Felder complex and the Eq. (10) can be treated as the BRST property of screening operators.

For the proof of these two Propositions we prepare some lemmas. In general, consider an expression of the form

$$\Xi(F) = \oint \cdots \oint \frac{dz_1}{2\pi i z_1} \cdots \frac{dz_k}{2\pi i z_k} \xi(v_1) \cdots \xi(v_k) F(v_1, \ldots, v_k).$$

Symmetrizing the integrand and using the commutation relation

$$\xi(v)\xi(v') = h(v - v')\xi(v')\xi(v), \quad h(v) = \frac{[v - 1]}{[v + 1]},$$

we can rewrite $\Xi(F)$ into the form $\Xi(\text{Sym } F)$, where

$$\text{Sym } F(v_1, \cdots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} F(v_{\sigma(1)}, \cdots, v_{\sigma(k)}) \prod_{i < j, \sigma(i) > \sigma(j)} h(v_{\sigma(i)} - v_{\sigma(j)}).$$

Set $\hat{F}(v_1, \cdots, v_k) = \text{Sym } F(v_1, \cdots, v_k)$. In view of the properties $h(v)h(-v) = 1$ and $h(0) = -1$, it is easy to see that

¹In particular, this happens when the screening operator (5) and the generators of DVA (3) act on the Fock space $\mathcal{F}_{l,1}$ where the additional factor $F(v; \hat{\pi})$ becomes simply $\pm 1$ [6].
(a) \( \hat{F}(v_1, \ldots, v_{i+1}, v_i, \ldots, v_k) = \hat{F}(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k) h(v_i - v_{i+1}) \),

(b) \( \hat{F}(v_1, \ldots, v_k) \) has a zero on \( v_i = v_j \) (\( i < j \)).\(^2\)

The proof of the Propositions will be based on the following identity of theta functions.

**Lemma 4**

\[
\text{Sym} \prod_{i=1}^{k} [v_i - 2i + 2] = \frac{[k]!}{k! [1]^k} \prod_{i<j} \frac{[v_i - v_j]}{[v_i - v_j - 1]} \prod_{i=1}^{k} [v_i - k + 1]. \tag{11}
\]

Here \([k]! = \prod_{i=1}^{k} [i]\).

**Proof.** Denote by \( \hat{F}(v_1, \ldots, v_k) \) the left hand side. Explicitly this function reads

\[
\hat{F}(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \prod_{i=1}^{k} [v_{\sigma(i)} - 2i + 2] \prod_{\sigma(i) > \sigma(j)} \frac{[v_{\sigma(i)} - v_{\sigma(j)} - 1]}{[v_{\sigma(i)} - v_{\sigma(j)} + 1]}. \tag{12}
\]

In the right hand side of (12), the summand has the same quasi-periodicity property in each variable \( v_i \). This can be shown by noting that, for any \( i \leq 1 \leq k \) and \( \sigma \in S_k \), we have

\[
\#\{j \mid j > i, \; \sigma(j) < \sigma(i)\} - \#\{j \mid j < i, \; \sigma(j) > \sigma(i)\} = \sigma(i) - i.
\]

From the remark made above, (12) has zeroes on \( v_i = v_j \) (\( i < j \)). Therefore it can be written as

\[
\hat{F}(v_1, \ldots, v_k) = \prod_{i<j} \frac{[v_i - v_j]}{[v_i - v_j - 1]} G(v_1, \ldots, v_k)
\]

with some holomorphic function \( G(v_1, \ldots, v_k) \).

Comparing the quasi-periodicity, we conclude that

\[
G(v_1, \ldots, v_k) = C \prod_{i=1}^{k} [v_i - k + 1]
\]

with some constant \( C \). The constant can be determined by setting \( v_i = i + k - 1 \) to be

\[
C' = \frac{[k]!}{k! [1]^k}. \tag{13}
\]

\(^2\)We assume in (b) that \( F(v_1, \ldots, v_k) \) has no poles on \( v_i = v_j \).
Lemma 5

\[ X^k = \frac{[k]!}{k!} \int \frac{dz_1}{2\pi i z_1} \cdots \frac{dz_k}{2\pi i z_k} \xi(v_1) \cdots \xi(v_k) \]
\[ \times \prod_{i<j} \frac{|v_i - v_j|}{|v_i - v_j - 1|} \prod_{i=1}^{k} \frac{|v_i - \frac{1}{2} + k - \pi|}{|v_i - \frac{1}{2}|} . \]

Proof. Noting that \( \hat{\pi} \xi(v) = \xi(v) (\hat{\pi} - 2 + 2r) \), we see that

\[ X^k = \int \frac{dz_1}{2\pi i z_1} \cdots \frac{dz_k}{2\pi i z_k} \xi(v_1) \cdots \xi(v_k) \prod_{i=1}^{k} \frac{|v_i + \frac{1}{2} - \hat{\pi} + 2k - 2i|}{|v_i - \frac{1}{2}|} . \]

The assertion follows by applying Lemma 4. \( \square \)

Proof of Proposition 2. Note first that \( |v_i - \frac{1}{2} + k - \pi| = \pm |v_i - \frac{1}{2}| \) on \( F_{l,k'} \), provided \( k' \equiv k (\mod r) \). Under this circumstance, we get from Lemmas 1 and 5 that

\[ [T_n, X^k]|_{F_{l,k'}} \]
\[ \propto \sum_{s=1}^{k} \int \frac{dz_1}{2\pi i z_1} \cdots \frac{dz_k}{2\pi i z_k} \xi(v_1) \cdots (A_n(v_s + r/2) - A_n(v_s - r/2)) \cdots \xi(v_k) \]
\[ \times \prod_{i<j} \frac{|v_i - v_j|}{|v_i - v_j - 1|} . \]

(14)

The integral is taken over the contours \(|z_1| = \cdots = |z_k| = 1\). With the change of variable \( z_s \to x^{-2r} z_s \), each term in the right hand side of (14) formally vanishes. It remains to show that the poles \(|v_i - v_j - 1| = 0\) do not affect the procedure.

To see this, note the following normal ordering rules

\[ \xi(v_1) A_n(v_2) = z_1^{2(r-1)/r} \left( x^{-r+2} z_2 / z_1 ; x^{2r} \right)_\infty ; \xi(v_1) A_n(v_2) ; \]
\[ A_n(v_2) \xi(v_1) = z_2^{2(r-1)/r} \left( x^{-r+2} z_1 / z_2 ; x^{2r} \right)_\infty ; \xi(v_1) A_n(v_2) ; \]

Using these equations we find that the expressions

\[ \frac{|v_i - v_s|}{|v_i - v_s - 1|} \xi(v_i) A_n(v_s + r/2) \quad (i < s), \quad \frac{|v_s - v_i|}{|v_s - v_i - 1|} A_n(v_s + r/2) \xi(v_i) \quad (i > s) \]

have poles only at \( z_s = x^{2rj-2} z_i \) (\( j = 1, 2, \cdots \)) and \( z_s = x^{-2rj+2} z_i \) (\( j = 2, 3, \cdots \)). Therefore, the contour for \( z_s \) in the (14) can be shifted from \(|z_s| = 1\) to \(|z_s| = x^{-2r}\) without affecting the integral. This completes the proof of Proposition 2. \( \square \)
Proof of Proposition 3. This is an immediate consequence of Lemma 5. Indeed, taking $k = r$ one obtains that the constant factor in (14) vanishes since $[r] = 0$.

Remark 1. Because of the $r \to 1 - r$ symmetry in the DVA defining relations (2) there is the second screening operator defined by

$$\tilde{X} = \oint_{|z| = 1} \frac{dz}{2\pi i z} \tilde{\xi}(v) \frac{[v - \frac{1}{2} + \hat{\pi}']}{[v + \frac{1}{2}]}$$

where

$$\tilde{\xi}(v) = e^{-i \sqrt{2r} (Q-iP \log z)} e^{-\sum_{m \neq 0} \frac{1}{m} (x^m + x^{-m}) \tilde{\beta}_m x^{-m}};$$

$$\tilde{\beta}_m = \frac{[rm]_x}{[(r-1)m]_x} \beta_m,$$

and

$$[v]' = x^{v^2 - v} (x^{2v}; x^{2r-2})_\infty (x^{2r(v-1)}; x^{2r-2})_\infty (x^{2r-2}; x^{2r-2})_\infty.$$  

It can be shown that

(a) $[X, \tilde{X}] = 0$,

(b) $\tilde{X}^{r-1} = 0$,

(c) $[T_n, \tilde{X}'] = 0$ on $\mathcal{F}_{l,k}$ provided $l' \equiv l \pmod{r-1}$.

We omit the proofs of these statements since they are completely analogous to those provided above.

Remark 2. The proof can be easily generalized to arbitrary real values of $r > 1$. In the same way as above, the operator $X^k$ commutes with $T_n$ on the space $\mathcal{F}_{l,k}$ (where $k$ is a positive integer)

$$[T_n, X^k]|_{\mathcal{F}_{l,k}} = 0. \quad (9')$$

For $r$ irrational, this property is sufficient to construct the BRST complex. For rational $r$,

$$r = \frac{q}{q - p} \quad (15)$$

with coprime positive integers $p$ and $q$, $q > p$, the main propositions take the form

**Proposition 2'**

The operator $X^k$ commutes with $T_n$ on the space $\mathcal{F}_{l,k'}$ provided $k' \equiv k \pmod{q}$:

$$[T_n, X^k]|_{\mathcal{F}_{l,k'}} = 0, \quad k' \equiv k \pmod{q}. \quad (9'')$$
Proposition 3′

The operator \( X \) is nilpotent:

\[
X^q = 0. \tag{10′}
\]

Similar properties hold for \( \tilde{X} \), but \( \tilde{X}^p = 0 \) provided \( p > 1 \).

4. We have proved that Lukyanov’s screening operators satisfy the BRST property. Note that, in the \( x = 1 \) case, the symmetrization of the integrand in \( X^k \) also leads to an appearance of a constant factor

\[
\frac{1}{k!} \prod_{j=1}^{k} e^{\frac{2\pi i j}{r}} - 1 \quad \frac{1}{e^{\frac{2\pi i j}{r}} - 1}
\]

which becomes zero at \( k = r \) and the \( r \) th power of the screening operator vanishes [11]. As it was demonstrated above, the situation in the deformed case turns out to be more complicated and the property (10) of Lukyanov’s screening operators follows as a result of non-trivial theta function identities.

On the basis of Propositions 2 and 3, we can construct a family of BRST complexes of DVA depending on the continuous parameter \( x \):

\[
\cdots \xrightarrow{X^k} \mathcal{F}_{i,2r-k} \xrightarrow{X^{r-k}} \mathcal{F}_{i,k} \xrightarrow{X^k} \mathcal{F}_{i,-k} \xrightarrow{X^{r-k}} \cdots
\]

In the undeformed case \( x = 1 \), it is known that the latter complex has trivial cohomologies except at the term \( \mathcal{F}_{i,k} \) while \( \text{Ker}\mathcal{F}_{i,k} X^k / \text{Im}\mathcal{F}_{i,k} X^{r-k} \) provides an irreducible representation of the Virasoro algebra [11]. Note that the derivation of this result relies entirely on the structure theory of representations of the Virasoro algebra, and no direct proof is available. In the deformed case, we expect that the same is true about the cohomology, at least for generic values of \( x \). However, as very little is known about the representations of DVA, a rigorous proof is still lacking.

Among the open problems we would like to mention the following one. Recall that the braiding matrices of the chiral vertex operators of the Virasoro algebra are constant solutions of the Yang–Baxter equation in the IRF form [12]. This reflects the existence of a hidden quantum group symmetry in the minimal models of conformal field theory. More explicitly, in the bosonic realization, the screening operators can be regarded as giving a representation of nilpotent generators of two quantum groups \( U_q(sl_2) \) with the deformation parameters related by the \( r \rightarrow 1 - r \) transformation [12]. The action of the Virasoro algebra generators preserves the monodromy of matrix elements of vertex operators, and the complete symmetry algebra is a tensor product of the infinite-dimensional part (the

\[\text{Compare with (13).}\]

\[\text{Here we restrict our attention to one chirality of the conformal field theory.}\]
Virasoro algebra) and the finite-dimensional one given by two quantum groups. The monodromy of matrix elements of vertex operators in the ABF models is given by an elliptic solution of the Yang–Baxter equation [13]. Owing to the works [7, 6] it becomes clear that the proper deformation of Virasoro algebra is given by DVA. However, it is not clear yet what is the “elliptic deformation” of the quantum group part of the symmetry algebra of critical theory.

From this point of view our results on the properties of deformed screening operators can be considered as a first step toward understanding this problem.

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