Subcritical Fluctuations at the Electroweak Phase Transition

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Abstract

We study the importance of thermal fluctuations during the electroweak phase transition. We evaluate in detail the equilibrium number density of large amplitude subcritical fluctuations and discuss the importance of phase mixing to the dynamics of the phase transition. Our results show that, for realistic Higgs masses, the phase transition can be completed by the percolation of the true vacuum, induced by the presence of subcritical fluctuations.

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The dynamics of the electroweak phase transition has received much attention in recent years [1]. The main reason for this interest is that, within the context of big-bang cosmology, the standard electroweak model can, in principle, satisfy the three conditions obtained by Sakharov for the generation of the observed baryon asymmetry in the Universe [2]: (a) baryon number violation; (b) CP violation, and; (c) nonequilibrium dynamics. To date, most mechanisms invoked to satisfy the third condition make use of a first order phase transition. [For mechanisms based on cosmic strings see Ref. [3].] This is also true for extensions of the standard model, which are currently favored by most authors, due to difficulties in generating sufficient baryon number within the minimal standard model [4].

Based on estimates of the quantum-corrected effective potential, nonequilibrium conditions are generated by the motion of critical bubbles of the broken (true) phase, which are nucleated within the symmetric phase. For the several baryogenesis models which rely on extensions of the standard model, the parameter space is large enough to justify the assumption of a fairly strong first order transition. However, analyses based on the 1-loop finite temperature effective potential for the minimal standard model [5–7] and its improved versions, show that the phase transition is very weak, within the current lower bounds for the Higgs mass, $m_H \gtrsim 60 \text{ GeV}$ [8], or even of second order for larger Higgs masses, $m_H \gtrsim 90 \text{ GeV}$ [9]. The weakness of the transition is further supported by nonperturbative lattice computations [10].

Our interest in the present paper is to further investigate [5,11] the possible consequences of having a weak first order phase transition at the electroweak scale. Although we will restrict our analysis to the standard electroweak model, our results can be adapted to any of its extensions. In fact, we will show that the strength of the transition can be used as a new constraint on the parameters of the model, always a welcome addition to the often large parameter space of extensions to the minimal standard model.

The interest in exploiting the dynamics of weak first order transitions goes beyond its
potential relevance for baryogenesis. As recent results have shown, a sufficiently weak first order transition will exhibit a different dynamics from the usual homogeneous nucleation results [12]; the extra free energy available in large amplitude fluctuations which are not included in the Gaussian computation for the nucleation rate will act to decrease the decay barrier, suppressing supercooling, and speeding up the completion of the phase transition. For even weaker transitions, critical bubble nucleation may be completely absent. Clearly, the information from the effective potential is not sufficient to determine the details of the transition; mean-field theory breaks down in the presence of large infrared corrections.

In order to quantify the above statements, we will make use of the subcritical bubbles method [5,11]. That is, we will model large amplitude thermal fluctuations by Gaussian-shaped bubbles of approximately correlation volume. Previous results based on a kinetic approach, have indicated that such fluctuations can destroy the first-order character of the transition for Higgs masses of order $m_H \gtrsim 55$ GeV [6,5]. Here we would like to complement this calculation by computing in detail the nucleation rate for such configurations, which was previously assumed for simplicity to be $\Gamma = A m^4(T) \exp[-B/T]$, where $A$ is a constant of order unity, $m^4(T)$ is the curvature of the potential in the symmetric phase, and $B$ is the free energy of the Gaussian configuration. Note that this is a nontrivial exercise, as these configurations are not solutions to the Euclidean equations of motion. Within reasonable approximations, we will be able to obtain the equilibrium number density of these configurations as a function of the tree-level Higgs mass, to show how the weakness of the transition is closely related to the breakdown of the dilute gas approximation.

The remainder of this paper is organized as follows. In Sec. II, we give a brief review of subcritical bubbles in the context of the electroweak standard model. In Sec. III, we compute the equilibrium number density of subcritical fluctuations and discuss the choice of parameters in the subcritical bubble configuration. There we also discuss the validity of the approximations taken and the range of applicability of our results. In Sec. IV, we discuss how the weakness of the electroweak phase transition is related to the breakdown of the dilute gas approximation; for large enough Higgs masses, subcritical bubbles percolate,
completing the transition. In Sec. V, we have our main conclusions. Some technical details are left for an Appendix.

II. SUBCRITICAL BUBBLES

Following the work of Ref. [11], large amplitude fluctuations describing thermal fluctuations are parameterized as,

\[ \varphi_{sc}(r) = \varphi_A(T) \exp \left( -\frac{r^2}{R^2(T)} \right), \tag{2.1} \]

where \( \varphi_{sc}(r) \) describes (spherically symmetric) fluctuations in the scalar field, with amplitude \( \varphi_A \) and radius given by \( R(T) \). The minimum value for \( R(T) \) should be compatible with the coarse-graining scale of the model. Later, in section III, we will come back to the parameters in (2.1) and discuss our choice for \( \varphi_A \) and \( R \). The fluctuations described by (2.1) approximate rather well the relevant field configurations, as recent work has shown [13].

A. Free Energy for Electroweak Subcritical Bubbles

Let us estimate the free energy associated with the configurations given by (2.1). We shall use, as a particular case, the electroweak model.

In the electroweak model, for a Higgs self-coupling \( \lambda \ll g^2 \) (with \( g \) denoting a generic gauge coupling), contributions due to scalar loops to the finite temperature effective potential are small compared to the contribution due to gauge fields and fermions. This is a common approximation employed in the literature which results in a 1-loop finite temperature effective potential given by [14]

\[ V(\phi, T) = D(T^2 - T_0^2)\phi^2 - ET\phi^3 + \frac{\lambda T}{4} \phi^4, \tag{2.2} \]

where \( D \) and \( E \) are constants given in terms of the \( W \) and \( Z \) boson masses and of the top quark mass as:
\[ D = \frac{1}{24} \left[ 6 \left( \frac{m_W}{\sigma} \right)^2 + 3 \left( \frac{m_Z}{\sigma} \right)^2 + 6 \left( \frac{m_t}{\sigma} \right)^2 \right] \simeq 0.169 \quad (2.3) \]

and

\[ E = \frac{1}{12\pi} \left[ 6 \left( \frac{m_W}{\sigma} \right)^3 + 3 \left( \frac{m_Z}{\sigma} \right)^3 \right] \simeq 10^{-2}, \quad (2.4) \]

where \( \sigma \simeq 246 \text{ GeV} \) is the vacuum expectation value of the Higgs field. \( m_W = 80.6 \text{ GeV}, m_Z = 91.2 \text{ GeV} \) and we use \( m_t \sim 174 \text{GeV} \) \[15,8\]. \( T_2 \) in \( V(\phi, T) \) (the spinodal instability temperature) is given by

\[ T_2 = \sqrt{\frac{m_H^2 - 8B\sigma^2}{4D}}, \quad (2.5) \]

where \( m_H^2 = (2\lambda + 12B)\sigma^2 \) is the physical Higgs mass and \( B = \frac{1}{64\pi^2\sigma^4}(6m_W^4 + 3m_Z^4 - 12m_t^4) \simeq -0.00456 \). \( \lambda_T \) in \( V(\phi, T) \) is the effective Higgs self-coupling (at 1-loop) given by

\[ \lambda_T = \lambda - \frac{1}{16\pi^2} \left[ \sum_b g_b \left( \frac{m_b}{\sigma} \right)^4 \ln \left( \frac{m_b^2}{c_bT^2} \right) - \sum_f g_f \left( \frac{m_f}{\sigma} \right)^4 \ln \left( \frac{m_f^2}{c_fT^2} \right) \right], \quad (2.6) \]

where the sums are performed over bosons and fermions, with degrees of freedom \( g_b \) and \( g_f \), respectively. In (2.6), \( \ln c_b = 5.41 \) and \( \ln c_f = 2.64 \).

For temperatures \( T < T_1 \), where \( T_1 \) is given by the solution of \( E^2T_1^2 = \frac{8}{9}D \left( T_1^2 - T_2^2 \right) \lambda_T \), \( V(\phi, T) \) has minima at \( \phi = 0 \) and at

\[ \varphi_+ (T) = \frac{1}{2\lambda_T} \left[ 3ET + \sqrt{9E^2T^2 - 8D(T^2 - T_2^2)\lambda_T} \right]. \quad (2.7) \]

At the critical temperature \( T_c \),

\[ T_c^2 = \frac{T_2^2}{1 - \frac{E^2}{4\lambda_T D}}, \quad (2.8) \]

we have \( V(\phi = 0, T_c) = V(\phi = \varphi_+, T_c) \) and below \( T_c, \phi = \varphi_f = 0 \) describes the metastable phase (the false vacuum), while \( \phi = \varphi_+(T) \) is the stable phase (the true vacuum).

An important effect from higher loop corrections to \( V(\phi, T) \) in (2.2) is the reduction of the coefficient of the cubic term \( \phi^3 \) by \( 2/3 \) \[16\]. However, we are mostly interested in a toy model computation of the number density of subcritical fluctuations, and will adopt
the simplest 1-loop potential. For recent considerations on improving the 1-loop effective potential by resumming the most important infrared contributions and higher order graphs see, for instance, Refs. [9,17,18].

The free energy for a given field configuration $\varphi_{sc}(T)$ is,

$$F(T) = \int d^3x \left[ \frac{1}{2} (\vec{\nabla} \varphi_{sc})^2 + V(\varphi_{sc}, T) \right].$$  

(2.9)

From (2.1) and (2.2), we get for $F(T)$ the expression

$$F(T) = \alpha(\varphi_A)R(T) + \beta(\varphi_A)R^3(T),$$

(2.10)

where

$$\alpha(\varphi_A) = \frac{3\sqrt{2}\pi^\frac{3}{2}}{8} \varphi_A^2(T),$$

(2.11)

$$\beta(\varphi_A) = \frac{\sqrt{2}\pi^\frac{3}{2}}{4} D(T^2 - T_2^2)\varphi_A^2(T) - \frac{\sqrt{3}\pi^\frac{3}{2}}{9} ET\varphi_A^3(T) + \frac{\pi^3}{32} \lambda_T \varphi_A^4(T).$$

III. NUMBER DENSITY OF SUBCRITICAL BUBBLES

A. Partition Function for the Scalar Field in the Electroweak Model

Let us define in the Electroweak model the partition function

$$Z = \int D\phi D\chi_i e^{-\int_0^\beta dr \int d^3x \mathcal{L}_{\text{Ew}}(\phi, \chi_i)},$$

(3.1)

where $\chi_i$ denotes gauge and fermions fields (and ghost fields) and $\phi$ is the SU(2) doublet

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix},$$

(3.2)

where $\phi_i (i = 1, \ldots, 4)$ are real scalar fields. The tree level potential for the complex scalar field $\phi$, given by
\[ V_0(|\phi|) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2, \quad (3.3) \]

for \( \mu^2 > 0 \), \( \phi \) acquires a nonvanishing vacuum expectation value \( \langle |\phi| \rangle = \sigma \), which one assumes real and along, for example, the real component \( \phi_3 \) of \( \phi \). Thus, in the broken phase, we define \( \phi' = \phi + \sigma \) and \( \phi_1, \phi_2 \) and \( \phi_4 \) are the three Goldstone bosons.

Let us denote by \( g_i \) the coupling of the field \( \phi \) with the \( \chi_i \) fields. If \( \lambda \ll g_i^2 \), i.e., the interactions among the \( \phi \) field are weak compared with the \( \phi - \chi_i \) interactions, then we may formally integrate out the \( \chi \) fields in (3.1) to obtain

\[ Z = \int D\phi e^{-W(\phi)}, \quad (3.4) \]

where

\[ W(\phi) = -\ln \int D\chi_i e^{-\int_0^\beta d\tau \int d^3 x L_{\text{Eucl}}(\phi, \chi_i)} \quad (3.5) \]

For vector fields, the integration measure above includes the gauge fixing and ghost terms.

We choose to work in the Landau gauge, which is the one usually used in the studies of the electroweak phase transition. Expanding \( W(\phi) \) in a derivative expansion,

\[ W(\phi) = \int_0^\beta d\tau \int d^3 x \left[ V_0(|\phi|) + V_\beta(|\phi|) + \hat{Z}(|\phi|)(\partial_\mu \phi)^\dagger (\partial^\mu \phi) + \ldots \right], \quad (3.6) \]

where \( V_0(|\phi|) \) is the tree level potential (3.3) and \( V_\beta(|\phi|) \) is the contribution of the \( \chi_i \) loops, coming from the integration over the \( \chi_i \) fields in (3.1), with the scalar field \( \phi \) in the external legs. Since \( V(\phi, T) \), Eq. (2.2), is obtained by neglecting scalar-boson contributions, which is analogous of just making a functional integration on vector and fermion fields at the 1-loop approximation, \( V_0 + V_\beta \) above, at the 1-loop approximation for the \( \chi_i \) fields, can be written as in Eq. (2.2). \( \hat{Z}(|\phi|) \) is the wave-function renormalization factor. \( \hat{Z}(|\phi|) \) has already been evaluated by many authors, in the determination of an effective action for the Higgs field with the objective to study corrections to critical bubbles nucleation rates [see, for instance [19–21]], where it is utilized a process of integration of fields, or field degrees of freedom, analogous to the one done above to determine \( W(\phi) \). Although \( \hat{Z}(|\phi|) \) receives nonvanishing 1-loop contributions, it has also been shown in [19] that these contributions are expected to

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yield only small corrections to the effective action and, therefore, these corrections could be treated as perturbations. Later, in section III.D, we show that the same approximation can be taken here, at least for the range of temperatures and higgs masses we are interested in. Therefore, as a first approximation, we will neglect all wave function corrections and take $\hat{Z}(|\phi|) \simeq 1$, for simplicity. Thus, at 1-loop order, we write the “effective” action $W(\phi)$ as

$$W_1(\phi) \simeq \int_0^\beta d\tau \int d^3x \left[ |\partial_\mu \phi|^2 + V(|\phi|, T) \right].$$

(3.7)

**B. Number Density for Subcritical Fluctuations**

Well-known results show that, in a dilute gas approximation, the average number of extended objects (for example, topological defects) described by some field configuration $\varphi_c$ can be given by [22–24]

$$N_c = \frac{Z(\varphi_c)}{Z(\varphi_v)},$$

(3.8)

where $Z(\varphi_c)$ [$Z(\varphi_v)$] is the partition function of the system computed by expanding the scalar field $\phi$ around the field (vacuum) configuration $\varphi_c$ ($\varphi_v$). For stable configurations the ratio in (3.8) is real. However, for unstable configurations (like, for example the sphaleron, the critical bubble or bounce configuration and also for $\varphi_{sc}$, given by (2.1)), the ratio in (3.8) is complex due to the existence of negative eigenvalues, associated with the instability of the configuration. This is the case for subcritical fluctuations. In this case, we will adopt the procedure of Arnold and McLerran in [25] for the case of sphaleron configurations, where they associated the average number of sphalerons to the total rate of transitions multiplied by the time of a single transition, giving, in the dilute gas approximation,

$$N_c \sim \Gamma \frac{2\pi}{\omega_-} \sim \frac{\text{Im} \left\{ \frac{Z(\varphi_c)}{Z(\varphi_v)} \right\}}{\omega_-},$$

(3.9)

where $\Gamma$ is the transition rate given by $\Gamma \simeq \frac{\omega_-}{\pi} \text{Im} \left\{ \frac{Z(\varphi_c)}{Z(\varphi_v)} \right\}$ and $\omega_-$ is the negative eigenvalue. Note that in this case the final result for (3.9) will not depend on the negative eigenvalue, as shown in [25]. Taking (3.9) as also valid for the case of subcritical fluctuations, we can
associate \( \Gamma \) to the nucleation rate of subcritical fluctuations and the possible negative eigenvalues, which must appear in (3.8), are the ones associated with the collapse mode of the fluctuation. We will, therefore, evaluate \( N_c \) for subcritical fluctuations not taking into account the possible imaginary eigenvalues associated with the instability of the configuration, that is, we will adopt a procedure similar to the one expressed by (3.9).

Computing the partition functions will give us the equilibrium values for relevant physical quantities. In particular, the equilibrium number density density of subcritical fluctuations, we will thus write as

\[
\begin{align*}
n_{sc} &= \frac{N_{sc}}{V} = \frac{1}{V} \int D\phi e^{-W(\phi \rightarrow \varphi_{sc} + \eta)} \int D\phi e^{-W(\phi \rightarrow \varphi_{v} + \zeta)} ,
\end{align*}
\]

(3.10)

where \( \eta \equiv \eta(\vec{x}, \tau) \) and \( \xi \equiv \xi(\vec{x}, \tau) \) are small perturbations around the configurations \( \varphi_{sc} \) and \( \varphi_{v} \) (which we take as the false vacuum configuration \( \varphi_{f} = 0 \)), respectively. In the following section we will discuss the limits of applicability of Eq. (3.10) within the standard electroweak model and in other situations applicable to phase transitions in general.

**C. 1-loop Evaluation of \( N_{sc} \)**

In the 1-loop approximation for \( W(\phi) \) in (3.10), given by (3.7), \( W_1(\phi) \) can be expanded about a field configuration \( \varphi_c \) as

\[
\begin{align*}
W_1(\phi) &= W_1(\varphi_c) + \int d^4x W'_1(\varphi_c; x) \eta(x) + \\
&+ \frac{1}{2!} \int d^4x d^4x' W''_1(\varphi_c; x, x') \eta(x) \eta(x') + \mathcal{O}(\eta^3) ,
\end{align*}
\]

(3.11)

where \( \eta(x) = \phi(x) - \varphi(x) \) and \( \int d^4x = \int_0^3 d\tau \int d^3x \). Taking \( \eta(x) \) as small perturbations around the field configurations \( \varphi(x) \), the terms of order \( \mathcal{O}(\eta^3) \) and higher can be treated as small perturbations. Note that for the kind of configurations we are dealing with, \( W'_1(\varphi) = \frac{\delta W_1(\phi)}{\delta \phi}|_{\phi=\varphi} \) does not vanish in general \( (\varphi_{sc}(r)) \) since is not a stationary solution of \( W_1(\phi) \).

Using (3.11) in (3.10) we can perform the gaussian functional integrals and since we have already integrated out all other fields interacting with the scalar field \( \phi \), the scalar field
propagators will include loop (quantum) corrections from the other fields coupled to $\phi$. We must, therefore, take some care when performing the gaussian functional integral in $\phi$ in order to avoid possible double-counting. From (3.11) and (3.10), with $\varphi_v = \varphi_f = 0$ and $W'_1(\varphi_v) = 0$, we get
\begin{equation}
\frac{Z(\varphi_{sc})}{Z(\varphi_v)} = \left[ \frac{\det \bar{W}'_1(\varphi_{sc})}{\det W'_1(\varphi_v)} \right]^{-\frac{1}{2}} e^{-\Delta W_1} e^{W'_sc(\bar{W}'_{sc})^{-1} W'_sc}, \tag{3.12}
\end{equation}
where $\Delta W_1 = W_1(\varphi_{sc}) - W_1(\varphi_v)$ and $\bar{W}'_1$ denotes the correct dressed (including the 1-loop quantum corrections from the $\chi_i$ fields in (3.1)) inverse propagator for the $\phi$ field:
\begin{equation}
\bar{W}'_1(\varphi) = -\Box + m^2_\beta(\varphi), \tag{3.13}
\end{equation}
with $m^2_\beta(\varphi) = 2D(T^2-T^2_0)+3\lambda_T\varphi^2$. Note that $m_\beta(\varphi)$ is the classical Higgs mass corrected by the self-energy ($T \neq 0$) corrections coming from the $\chi_i$ fields (fermions and gauge bosons). $m^2_\beta(\varphi)$ does not include the term proportional to $E$ of (2.2), which would give origin to a linear term in (3.13). This term is absent since it would not appear in the self-energy corrections to the scalar field $\phi$ and also that the presence of such a term in (3.13) is well known to lead to a wrong counting of loop corrections to the scalar field $\phi$ effective potential (see for instance, ref. [16]).

In (3.12) we also have that
\begin{equation}
W'_sc(\bar{W}'_sc)^{-1} W'_sc = \int d^4x d^4x' W'_1(\varphi_{sc}; x)W'_1(\varphi_{sc}; x') \langle x | \bar{W}'_1(\varphi_{sc}; x, x')^{-1} | x' \rangle, \tag{3.14}
\end{equation}
where
\begin{equation}
\langle x | \bar{W}'_1(\varphi_{sc}; x, x')^{-1} | x' \rangle = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\omega_n(\tau-\tau') + i\vec{k}.(\vec{x}-\vec{x}')}}{\omega^2_n + \vec{k}^2 + m^2_\beta(\varphi_{sc})}, \tag{3.15}
\end{equation}
where $\omega_n = \frac{2\pi n}{\beta}$, ($n = 0, \pm 1, \pm 2, \ldots$) are the Matsubara frequencies. In appendix A we solve (3.14) explicitly.

In order to compute the determinant ratio in (3.12), we must first isolate the possible zero modes in $\det \bar{W}'(\varphi_{sc}) = \det[-\Box + m^2_\beta(\varphi_{sc})]$. Note that, as exposed above, here we will not take into account possible imaginary eigenvalues. Noting that $W_1(\phi)$, Eq. (3.7), and the
corresponding field equation, \( \frac{\delta W_1}{\delta \phi} = 0 \), are translational invariant. Therefore, even though if \( \phi_{sc} \) is not a solution of the field equation, there must be three zero eigenvalues associate to \( \phi_{sc} \), related to the three translational modes. We also have three more zero modes that are related to the rotational symmetry \( SU(2) \), associated with the three Goldstone bosons in the broken phase. We handle these zero modes by the standard way, by introducing collective coordinates, such that the determinantal ratio in (3.12) can be written as:

\[
\left[ \frac{\det \bar{W}_1''(\phi_{sc})}{\det \bar{W}_1''(\phi_v)} \right]^{-\frac{1}{2}} = \Omega_{\text{trans.}} \Omega_{\text{rot.}} \left[ \frac{\det' \bar{W}_1''(\phi_{sc})}{\det \bar{W}_1''(\phi_v)} \right]^{-\frac{1}{2}},
\]

(3.16)

where \( \Omega_{\text{trans.}} \) is the usual factor coming from the translational modes, given by

\[
\Omega_{\text{trans.}} = \frac{\Delta W_1}{2\pi} \right)^{\frac{3}{2}} V
\]

(3.17)

and \( \Omega_{\text{rot.}} \), due to the rotational modes, has been explicitly obtained in [19], for the case of fluctuations around the critical bubble configuration, and in our case we can write the analogous expression for the \( \phi_{sc} \) configuration as

\[
\Omega_{\text{rot.}} = \frac{\pi^2}{2} \right)^{\frac{3}{2}} \left[ \frac{\beta}{2\pi} \int d^3 x \phi_{sc}^2(\vec{x}) \right]^{\frac{3}{2}},
\]

(3.18)

In (3.16) the prime in the determinant is to indicate that the six zeroes modes of (3.19), associated with the three Goldstone bosons of the Higgs doublet, in the broken phase, and the three translational modes, we obtain

\[
\left[ \frac{\det \bar{W}_1''(\phi_{sc})}{\det \bar{W}_1''(\phi_v)} \right]^{-\frac{1}{2}} = \Omega_{\text{trans.}} \Omega_{\text{rot.}} \exp \left\{ -\frac{1}{2} \ln \left[ \frac{\Pi_{n=\infty}^{\infty} \Pi_{l=\infty}^{\infty} (\omega_n^2 + E_j^2(\phi_{sc}))}{\Pi_{n=\infty}^{\infty} \Pi_{l=\infty}^{\infty} (\omega_n^2 + E_j^2(\phi_v))} \right] \right\},
\]

(3.19)

where the productories in \( i \) and \( j \) are, formally, over the eigenvalues \( E_j^2(\phi_{sc}) \) and \( E_j^2(\phi_v) \), respectively. In (3.19), we have used the identity \( \ln \det \hat{O} = \text{tr} \ln \hat{O} \). Taking into account the six zeroes modes of (3.19), associated with the three Goldstone bosons of the Higgs doublet, in the broken phase, and the three translational modes, we obtain

\[
\left[ \frac{\det \bar{W}_1''(\phi_{sc})}{\det \bar{W}_1''(\phi_v)} \right]^{-\frac{1}{2}} = \Omega_{\text{trans.}} \Omega_{\text{rot.}} \exp \left\{ -\frac{1}{2} \ln \left[ \frac{(\Pi_{n=\infty}^{\infty} \omega_n^2)^{12} \Pi_{n=\infty}^{\infty} (\omega_n^2 + E_j^2(\phi_{sc}))}{\Pi_{l=\infty}^{\infty} (\omega_l^2 + E_j^2(\phi_v))} \right] \right\},
\]

(3.20)
where the factors $\Omega_{\text{trans}}$ and $\Omega_{\text{rot}}$ are given by (3.17) and (3.18), respectively. The prime in $\prod_i$ in (3.20) is a reminder that those zero modes, associated with (3.17) and (3.18) have been excluded from the product of eigenvalues.

Using the identity:
\[
\prod_{n=1}^{+\infty} \left( 1 + \frac{a^2}{(2\pi n)^2} \right) = \frac{\sinh(a/2)}{a/2},
\]
we get for (3.20) the expression
\[
\left[ \frac{\det W''_1(\varphi_{\text{sc}})}{\det W''_1(\varphi_v)} \right]^{-\frac{1}{2}} = \Omega_{\text{trans}}. \Omega_{\text{rot}}. \exp \left\{ -\frac{1}{2} \left[ 12 \ln \beta + 2 \sum_i \left[ \frac{\beta}{2} E_i(\varphi_{\text{sc}}) + \ln \left( 1 - e^{-\beta E_i(\varphi_{\text{sc}})} \right) \right] - 2 \sum_j \left[ \frac{\beta}{2} E_j(\varphi_v) + \ln \left( 1 - e^{-\beta E_j(\varphi_v)} \right) \right] \right\}.
\]

For the field configuration $\varphi_{\text{sc}}$, given by (2.1), and the vacuum configuration $\varphi_v$ (the false vacuum, $\varphi_v = 0$), from (3.7), we can write that
\[
\sum_{(\text{continuum})} \left[ \frac{\beta}{2} E(\varphi) + \ln \left( 1 - e^{-\beta E(\varphi)} \right) \right] \simeq \int d^3 x \int \frac{d^3 k}{(2\pi)^3} \left[ \frac{\beta}{2} \sqrt{k^2 + m^2_\beta(\varphi)} + \ln \left( 1 - e^{-\beta \sqrt{k^2 + m^2_\beta(\varphi)}} \right) \right],
\]
where $m^2_\beta(\varphi) = 2D(T^2 - T_2^2) + 3\lambda T \varphi^2$. Note that the terms like $\int d^3 k \sqrt{k^2 + m^2}$ in (3.23), that are ultraviolet divergent, can be subtracted by introducing the usual counterterms of renormalization for the scalar field loops, rendering the exponent in (3.22) finite.

Using (3.23) in (3.22), Eq. (3.12) can therefore be written as
\[
\frac{Z(\varphi_{\text{sc}})}{Z(\varphi_v)} \overset{1-\text{loop approx.}}{\simeq} \Omega_{\text{trans}}. \Omega_{\text{rot}}. T^6 \exp \left[ -\frac{\Delta F_{\text{eff}}(T)}{T} \right],
\]
where $\Delta F_{\text{eff}}(T)$ denotes an effective free energy for subcritical bubbles, given by
\[
\Delta F_{\text{eff}}(T) = F(T) + \int d^3 x \left[ V_\phi(\varphi_{\text{sc}}, T) - V_\phi(\varphi_v, T) \right] - T \int d^4 x d^4 x' W_1'(\varphi_{\text{sc}}; \vec{x}) W_1'(\varphi_{\text{sc}}; \vec{x}') \langle x | \left[ W_1''(\varphi_{\text{sc}}; x, x') \right]^{-1} | x' \rangle,
\]
where $F(T)$ is given by (2.10). The scalar field quantum contribution in (3.25), in the high $T$ limit, is given by
\[
\int d^3x [V(\varphi_{sc}, T) - V(\varphi_v, T)] \approx 4\pi \int_0^\infty drr^2 \frac{T^2}{24} 3\lambda T \varphi_{sc}(r) = \frac{\sqrt{2\pi}}{32} \lambda T \varphi_A^2(T) T^2 R^3(T),
\]

(3.26)

The last term in (3.25), from (3.14) and (3.15), can be written as

\[
\int d^3xd^3x' \left[ -\nabla^2 \varphi_{sc}(\vec{x}) + V'(\varphi_{sc}(\vec{x}), T) \right] \left[ -\nabla'^2 \varphi_{sc}(\vec{x}') + V'(\varphi_{sc}(\vec{x}'), T) \right] I(|\vec{x} - \vec{x}'|),
\]

(3.27)

where \( I(|\vec{x} - \vec{x}'|) \), in the high temperature limit, is given by (see the Appendix)

\[
I(|\vec{x} - \vec{x}'|) \sim \frac{\pi}{(2\pi)^2 |\vec{x} - \vec{x}'|} e^{-m_\beta |\vec{x} - \vec{x}'|}.
\]

(3.28)

Using the above expression in (3.27), we can see that the largest contribution will come for values of \(|\vec{x} - \vec{x}'|\) close to the inverse of \(m_\beta\), i.e., for values close to the correlation length \(\xi(T) \sim 1/m_\beta\). We therefore may restrict \(|\vec{x} - \vec{x}'|\) to the size of the subcritical fluctuations and we here consider \(R(T) \sim \xi(T)\), thus obtaining

\[
I(|\vec{x} - \vec{x}'|) \sim \frac{\pi}{(2\pi)^2 TR(T)}.
\]

(3.29)

Substituting \(\varphi_{sc}(\vec{x})\) in (3.27) and using the above expression for \(I(|\vec{x} - \vec{x}'|)\), we get

\[
W'(W'')^{-1}W' \approx \frac{\pi^2 \varphi_A^2(T) R^5(T)}{Te} \left[ D(T^2 - T_0^2) - 3\sqrt{2} \frac{E}{8} T \varphi_A(T) + \frac{3\sqrt{3}}{18} \lambda T \varphi_A^2(T) \right]^2.
\]

(3.30)

Using the expression for \(F(T)\), Eq. (2.10), and Eqs. (3.26) and (3.30), we get for \(\frac{Z(\varphi_{sc})}{Z(\varphi_v)}\), in the high temperature limit and at 1-loop order, the expression

\[
\frac{Z(\varphi_{sc})}{Z(\varphi_v)} = VT^3 A(T) \exp[-B(T)],
\]

(3.31)

where \(A(T)\) and \(B(T)\) are given by

\[
A(T) \sim \left[ R^6(T) \varphi_A^6(T) \right]^{\frac{7}{10}} \left[ \frac{3\sqrt{2}}{8} + \frac{\sqrt{2}}{4} D(T^2 - T_0^2) R^2(T) - \frac{\sqrt{3}}{9} E T \varphi_A(T) R^2(T) + \frac{\lambda T}{32} \varphi_A^2(T) R^2(T) \right]^\frac{7}{2}.
\]

(3.32)

and
\begin{equation}
B(T)^{1-\text{loop, high }T} \approx \frac{\Delta F_{\text{eff}}(T)}{T}, \quad (3.33)
\end{equation}

with \(\Delta F_{\text{eff}}(T)\) given by (3.25) together with Eqs. (2.10), (3.26) and (3.30).

From (3.8), the equilibrium number density of subcritical fluctuations, in the dilute gas approximation, is given by

\begin{equation}
n_{\text{sc}}(T) = \frac{1}{V} \frac{Z(\varphi_{\text{sc}})}{Z(\varphi_v)}. \quad (3.34)
\end{equation}

D. A Discussion of the Validity of the Approximations and the Choice for the Parameters \(\varphi_A\) and \(R\)

We expect that the expression for \(n_{\text{sc}}\), obtained in the dilute gas approximation, be valid as long as the following holds

\begin{equation}
[n_{\text{sc}}(T)]^{\frac{1}{3}} R(T) \ll 1, \quad (3.35)
\end{equation}

We can also express the condition in terms of the volume occupied by the fluctuations described by \(\varphi_{\text{sc}}\), at some fixed temperature \(T\), \(V_{\text{sc}}(T)\),

\begin{equation}
V_{\text{sc}}(T) = \frac{4\pi}{3} R^3(T) n_{\text{sc}}(T)V, \quad (3.36)
\end{equation}

such that (3.35) can be reexpressed as

\begin{equation}
\frac{V_{\text{sc}}(T)}{V} \ll 1, \quad (3.37)
\end{equation}

that is, the dilute gas approximation is a good approximation as long as the volume occupied by the subcritical fluctuations be small.

Another approximation that we considered, the neglected of wave function corrections to the effective action, can be thought to be somewhat problematic. However, we can estimate their relative contribution to our results and inquiry whether our approximation of neglecting these contributions is good enough. For such estimation we may take the leading order correction to \(\Delta F_{\text{eff}}(T)\), due to the wave function correction coming from the functional
integration over the gauge boson fields. As an order of estimative, we may take the result obtained in [19], for the SU(2)-Higgs model, which we write as, in neglecting contributions due to plasma masses,

\[ Z(\phi, T) = \frac{11}{32\pi} \frac{gT}{\phi}, \]  

(3.38)

where \( g = 2m_W/\sigma \approx 0.328 \). An estimate of the relative contribution of the above factor can be given by the ratio (as in [19])

\[ \delta = \frac{\int d^3 x Z(\phi_{sc}, T) \left( \vec{\nabla} \phi_{sc} \right)^2}{\int d^3 x \left( \vec{\nabla} \phi_{sc} \right)^2} = \frac{11g}{4\sqrt{2\pi}} \frac{T}{\phi_A}, \]  

(3.39)

where we have used (2.1) in the rhs of the above equation. It is interesting to note that the above expression do not depend on the subcritical bubble configuration radius \( R \).

The range of applicability of the dilute gas approximation and the validity of the approximation of neglecting wave function contributions is determined once the parameters \( \phi_A \) and \( R \), of our ansatz, Eq. (2.1), are set.

In the following, we take the amplitude \( \phi_A \) as been given by the temperature dependent broken minimum, \( \phi_+(T) \), Eq. (2.7). For \( R \) we assume it as been given by the correlation length \( \xi(T) \), given by the inverse of the temperature dependent mass in \( V(\phi, T) \). We next justify these assumptions. We expect that fluctuations in the scalar field would appear in the system with arbitrary amplitudes and sizes. However, physically, we can set some limits on these parameters. For example, fluctuations with too small amplitudes are already summed over in the computation of the (coarse-grained) effective potential. The relevant fluctuations for the dynamics of the phase transition can roughly be identified [12,13] as been those with amplitudes \( \phi_A \gtrsim \phi_{\text{max}} \), where \( \phi_{\text{max}} \) is the value of the higgs field at the maximum of the effective potential. Among these fluctuations in the false vacuum, those with \( \phi_A \) close to the local minimum \( \phi_+ \) are certainly the most expected. About the fluctuations radius, \( R \), their minimum radius must be compatible with the coarse-graining scale of the model and the coarse-grained effective potential. In general, we can expect that \( R \gtrsim \xi(T) \). Here, we implicitly assume that the most probable fluctuations in the system are those with
radius close to the correlation length $\xi(T)$. A better approximation would be taking $R$ as the average radius of the fluctuations. Unfortunately, a reliable method to determine this quantity is still lacking. Recently, the authors of ref. [27] propose a method to determine the average radius $\bar{R}$ of fluctuations, considering $R$ as a truly dynamical variable, however, the obtained result for $\bar{R}$ is much smaller than $\xi(T)$. In another recent method proposed by the authors in [12,13], a statistical mechanical method is used for studying the importance of phase mixing in the phase transition, where the volume fraction occupied by fluctuations is determined by summing over all possible fluctuations of different sizes and amplitudes. In the next section, by computing this volume fraction within our approximations, we will be able to compare our results with the ones obtained in the statistical mechanics method.

With $\varphi_A = \varphi_+(T)$ and $R = \xi(T)$, we can immediately evaluate (3.39) and have an estimative of the leading wave function contribution. In Figure 1 we have given $\delta$ as a function of the higgs mass, with $\delta$ computed at the critical temperature $T_c$. Our result for the fraction given by (3.39) is much like the same one as given in [19], where $\delta \geq 1$ for $m_H \gtrsim 80 GeV$. However, for such larger higgs masses it is well known that the perturbative expansion breaks down and the effective potential becomes unreliable [7,9,26]. For the interval of higgs masses we will be interested in, the wave function correction is small and, if higher order corrections are taken into account and plasma masses are included in $Z(\phi, T)$, we expect the wave function contributions be even smaller, as shown in [19].

**IV. PERCOLATION OF SUBCRITICAL FLUCTUATIONS**

Studies of the electroweak phase transition, through the analyses of the effective potential [26] and direct numerical simulations [28], indicate that the nature of the electroweak phase transition is most possible of first order, but it has a excitation barrier to small, qualifying the phase transition as very weakly first order. In these situations it has been argued that the dynamics of the phase transition could be quite different from the usual process of critical bubble nucleation. In fact, due the weak nature of the phase transition, (subcritical)
fluctuations over the barrier can become the main mechanism by which the phase transition can complete. Here, we analyze the importance of a possible large amount of phase mixing at temperatures higher than the nucleation temperature of critical bubbles. In this sense, we argue that if the volume of true vacuum phase fluctuations becomes higher than a certain value, percolation of the fluctuations can occur. We thus have a possible scenario where the phase transition can be completed just by those over the barrier subcritical fluctuations.

A well known result of statistical mechanics of dynamics of cluster systems [29] show that there is a critical probability value $p_c$, with the cluster probability defined by

$$p = \frac{\text{Cluster Volume}}{\text{System Volume}},$$

where, in three space dimensions, this critical percolation probability is roughly $p_c \sim 0.3$ [29] and beyond which clusters are favored to coalesce and grow, filling the whole volume of the system.

From our definition of the volume occupied by the field configurations $\varphi_{sc}$, at a fixed temperature $T$, Eq. (3.36), it is quite natural to define, therefore, a percolation temperature $T_p$ as given by

$$\frac{V_{sc}(T_p)}{V} \sim 0.3.$$  \hspace{1cm} (4.2)

From Eq. (3.36), we obtain

$$\frac{4\pi}{3} R^3(T_p)n_{sc}(T_p) \simeq 0.3.$$  \hspace{1cm} (4.3)

Condition (4.3) gives us the temperature (as a function of the Higgs mass) below which subcritical fluctuations become favorable to coalesce and grow, forming large regions of the true vacuum phase. If $T_p$ is larger than the nucleation temperature $T_N$, that marks the onset of critical bubbles nucleation, then we have an effective mechanism by which the phase transition can complete, as explained above, only due the dynamics of subcritical fluctuations of the stable phase $\varphi_+$, inside the false vacuum phase. Note that at $T_p$, condition (4.2), due to (3.37), will also signal the breakdown of the dilute gas approximation. Thus, for
temperatures too close to $T_p$, the obtained expression for $n_{sc}$, Eq. (3.34), would just give a qualitative indication of the importance of subcritical fluctuations during the phase transition. We must, however, comment on the alternative method of the authors of refs. [12,13], where a statistical mechanical approach is used for studying the dynamics of subcritical bubbles, modeled there as in Eq. (2.1). By constructing a kinetic equation for the (dynamical) number density of fluctuations, they could obtain an analytical expression for $V_{sc}/V$, within appropriate approximations. Their kinetic approach also breaks down for $V_{sc}/V$ close to the critical percolation probability, where fluctuations become more dense and the treatment of the coalescence of fluctuations becomes important. Besides the different approach, the method applied to the electroweak phase transition (see Gleiser’s talk in [13]) show that $V_{sc}/V$ changes sharply for $m_H \gtrsim 60\text{GeV}$, just as the result obtained here for $V_{sc}/V$ shows (Figure 2). Also, recent numerical simulations of very weak first order phase transitions [28] give support to our results. All these dynamical studies indicate the validity of the various approximations here adopted in order to make possible to arrive to an analytical expression for $n_{sc}$.

In our analyses it is enough to study the ratio $V_{sc}/V$ at the critical temperature $T_c$, for which we get the plot shown in Figure 2, in terms of the Higgs mass $m_H$, where we have used the approximation for $\lambda_T$,

$$\lambda_T \sim \lambda \sim 0.08 \left( \frac{m_H}{100\text{GeV}} \right)^2. \quad (4.4)$$

From Figure 2 we see that for a Higgs mass of $\sim 60\text{GeV}$, our results indicate that phase mixing is so large that the volume of subcritical fluctuations in the broken phase start becoming relevant for the dynamics of the phase transition. However, due to the condition of validity of the dilute gas approximation used in our evaluation, Eq. (3.37), our results can only be interpreted qualitatively for $m_H \gtrsim 60\text{GeV}$. Besides these limitations, the results obtained here are in accordance to recent results based on dynamical studies of very weak phase transitions, as discussed above.

The main result of this paper is the computation of the equilibrium density for the field
configuration $\varphi_{sc}$, with the full evaluation of the preexponential term $A(T)$ given by (3.32) and the effective free energy $\Delta F(T)$, Eq. (3.25). Previous results have given $n_{sc}$ just in terms of a Boltzman distribution form, with free energy given by (2.10) and prefactor given by just $T^3$. From (3.31) and (3.34) we see that the preexponential factors differ by the factor $A(T)$, which we show in Figure 3, as a function of the Higgs mass, which shows that the prefactor can assume large values for small values of the higgs mass. In Figure 4, we also compare the exponential factor $\Delta F_{\text{eff}}(T)$ with $F(T)$, both compute at $T = T_c$.

V. CONCLUSIONS

We have explicitly evaluated the equilibrium density of subcritical fluctuations and shown that the same can differ substantially from the usual expression. We have also analyzed the relative importance of subcritical fluctuations during the electroweak phase transition, showing that the mechanism for completion of the phase transition can be quite different from the usual critical bubble nucleation in a first order phase transition. For values of the Higgs mass at and higher than the experimental lower bound of $\sim 60\text{GeV}$, our results show that substantial phase mixing can be present prior to critical bubble nucleation, changing the dynamics of the phase transition. Our results for the number density and volume occupied by those subcritical fluctuations can be interpreted as an average value at some fixed temperature, since we expect that in some large volume $V \gg R^3(T) (R(T) \sim \xi(T))$ we have both processes of nucleation of subcritical bubbles and their shrinking, keeping their average density, at that temperature, constant in that large volume.

We must also remember that the results obtained here are in accordance with other studies involving the relevance of thermal fluctuations during the electroweak phase transition [7] and recent studies of the dynamics of very weak first order phase transitions, both by numerical simulations as by a statistical mechanics approach for the dynamics of fluctuations modeled by $\varphi_{sc}$, Eq. (2.1).
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APPENDIX A: EVALUATION OF EQ. (3.28)

In this appendix we evaluate (3.14) and get the expression for \( I(|\vec{x} - \vec{x}'|) \), Eq. (3.28). The term expressed in (3.14),

\[
W_{sc}'(\bar{W}_{sc}'')^{-1}W_{sc}' = \int d^4x d^4x' W_1'(\varphi_{sc}; x)W_1'(\varphi_{sc}; x')\langle x| [\bar{W}_1''(\varphi_{sc}; x, x')]^{-1} |x'\rangle, \quad (A1)
\]

where \( W_1' = \frac{\delta W_1}{\delta \varphi} \), is obtained from Eq. (3.7). Since \( W_1'(\phi) \) computed at \( \phi = \varphi_{sc}(\bar{x}) \) is independent of the Euclidean time, the two Euclidean time integrations in (A1) applied directly to (A1) applied directly to \( \langle x| [\bar{W}_1''(\varphi_{sc}; x, x')]^{-1} |x'\rangle \) which we called \( I(|\vec{x} - \vec{x}'|) \):

\[
I(|\vec{x} - \vec{x}'|) = \int_{0}^{\beta} d\tau \int_{0}^{\beta} d\tau' \langle x| [\bar{W}_1''(\varphi_{sc}; x, x')]^{-1} |x'\rangle . \quad (A2)
\]

Using (3.15), performing the sum in the Matsubara frequencies in (3.15) and using the result in (A2), we can perform the two Euclidean time integrations in (A2) in order to get the final result

\[
I(|\vec{x} - \vec{x}'|) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik.(\vec{x} - \vec{x}')}}{2(k^2 + m_\beta^2)^{\frac{3}{2}}} \sinh \left( \beta \sqrt{k^2 + m_\beta^2} \right) . \quad (A3)
\]

Performing the angular integrations and the \( k \) integration of (A3), we get, in the high temperature limit, \( \beta m_\beta \ll 1 \), the result given in Eq. (3.29).
REFERENCES


Figure 1: An estimate of the relative magnitude of the leading wave function contribution, Eq. (3.39), computed at $T = T_c$. 
Figure 2: The volume fraction $\frac{V_{sc}}{V}$ of true vacuum fluctuations (computed at the critical temperature $T_c$) as a function of the Higgs mass $m_H$ (GeV).
Figure 3: The Prefactor coefficient $A(T)$, computed at the critical temperature $T_c$, as a function of the Higgs mass.
Figure 4: The exponential factor $B(T) = \frac{\Delta F_{\text{eff}}(T)}{T}$ (upper curve) compared with $F(T)/T$ (lower curve), both evaluated at $T_c$, as a function of the Higgs mass.