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**QUANTUM - CLASSICAL CORRESPONDENCE
USING PROJECTION OPERATORS**

by

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Abstract

Classical properties, associated with a large phase space domain with a smooth boundary as compared to the Planck constant, are shown to be expressed quantum-mechanically by a family of quantum projection operators with the help of known theorems in microlocal analysis. Under the conditions of Egorov's theorem, the conservation of this correspondence under classical / quantum dynamics is asserted.

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to estimate the quantity k_1 at a given point (x, p) in the margin of the cell C , one may perform a linear canonical change of coordinates to bring one coordinate axis along the direction of the normal to ∂C containing (x, p) , i. e. the direction where the derivatives of f_1 are greatest. Since k_1 is invariant under this transformation, one finds that it is of the order of $\hbar/LP$. Denoting by $k_{\min}$ the minimum of $k_1(x, p)$, the symbol $k_2(x, p) = (LP/\hbar) (k_1(x, p) - k_{\min})$ is positive and $O(1)$.

Considering m vectors $(t_1, t_2, \dots, t_m)$ in the tangent space of phase space, one can introduce the m -th differentials of k_2 as defined in Reference 5, Section 18.4 by

$$|k_2|_m^g = \sup_{\{t_j\}} |k_2^{(m)}(x, p, t_1, t_2, \dots, t_m)| \prod_{j=1}^m g(t_j)^{1/2}.$$

They are $O(1)$ with our choice of metric. One can then use the sharp Gårding inequality, for instance in the form given by Lemma 18.6. 10 in Reference 5, which shows that the operator K_2 with symbol $k_2(x, p)$ is bounded by $K_2 \geq -C I$ with $C = O(1)$. The inequality $-C (\hbar/LP) \leq F$, with another constant $O(1)$ follows immediately. The inequality $F \leq I (1 + C (\hbar/LP))$ is obtained similarly by considering $I - F$ in place of F .

Nota

Some of the proofs in the rest of the article must be slightly modified to take into account the modification of Lemma 1, but the results are unchanged.

1. Introduction

One of the keystones of quantum mechanics is Bohr's correspondence principle. Although often simplified as an asymptotic limit of quantum dynamics towards classical dynamics when $\hbar \rightarrow 0$, it was never so bluntly formulated by its author. Our purpose in the present paper is another aspect of it, namely to provide a common framework for quantum properties (associated since von Neumann [1] with projection operators in Hilbert space) and classical properties giving the position and momentum coordinates of a classical system in phase space within some error bounds. Weyl made a major advance in this direction [2] when, extending an idea due to Wigner [3], he associated a quantum observable A with a function $a(q, p)$ in phase space. Since then, this "Weyl calculus" has become an important part of mathematics included in microlocal analysis, or pseudo-differential calculus [4, 5]. One can then guess that, among the various techniques in semiclassical physics (WKB, coherent states, ...) this one is best suited to the problem at hand.

Another hint in the same direction has come from the interpretation of quantum mechanics by consistent histories [6-8] (see [9] for a review). It suggested that a classical property may be associated with an equivalence class of projection operators in Hilbert space, this equivalence expressing an insensitivity at the quantum level to a too sharp precision in the boundary of a phase space domain. It also suggested which topology should be used to express that these operators are near each other : the trace norm, which is central in microlocal analysis but poorly controlled by algebraic methods.

These ideas are further developed in the present paper. The main results were proposed some time ago [10, 9], the priority being then to develop the consistent history framework. Physical hints were essential but proofs needed improvement, to say the least. Better ones will be given here, which is possible because the physically interesting results turn out to be corollaries of known mathematical theorems. The approach will also be different. One introduced formerly some convenient operators (the so-called "quasi-projectors"), which were then directly used in quantum considerations but were puzzling some physicists. One will therefore make a point to work really with classes of projection operators (to be called also projectors), the quasi-projectors being an intermediate step in the construction of the relevant projectors.

Section 2 defines the relevant operators. Section 3 recalls some formulas and theorems from microlocal analysis. Section 4 shows how some regularising symbols, well known in mathematics, generate the operators we need in physics. Section 5 considers "exclusion", i.e. how two different phase space regions can be very sharply distinguished. Section 6 deals with the dynamical aspects of correspondence, which enters under the heading of Egorov's theorem [11, 5, 12]. These ideas have been found essential for reconciling quantum probabilism with classical determinism [9].

In an accompanying paper, the present techniques are applied to a very different problem, which is how to express mathematically the space localisation of a relativistic particle.

2. Orientation and definitions

Cells

The propositions occurring in classical physics can most often be expressed by stating that a representative point (q, p) of a system is in some phase space cell C . A cell will be defined as a connected simply connected bounded set, big enough for the quantum uncertainty relations to be of no importance.

To be precise about the last point, let the (finite) dimension of phase space be $2n$. The position coordinates q are supposed to have the dimension of a length. One introduces two scales (L, P) for length and momentum, to be chosen later. When necessary, a simple metric $L^{-2} dq^2 + P^{-2} dp^2$ on an open set in phase space containing C will be introduced. The variables $(q', p') = (q/L, p/P)$ are non-dimensional. Other scale parameters that will be used frequently are defined by

$$\sigma = L (\hbar / 2LP)^{1/2}, \quad \tau = P (\hbar / 2LP)^{1/2}, \quad \sigma\tau = \hbar / 2, \quad \sigma/L = \tau/P. \quad (2.1)$$

The "number of semi-classical states" in C is $N(C) = V(C) (LP/\hbar)^n$, where $V(C)$ is the dimensionless phase space volume of C in the metrics $dq'^2 + dp'^2$. It is also given by $\mu(C)/(2\pi\hbar)^n$ where $\mu(C)$ is the Liouville volume of C . Let $A(\partial C)$ denote the $(2n-1)$ -dimensional area of the boundary ∂C in the dimensionless metrics. The geometric ratio $\theta = A(\partial C)/V(C)$ will play an important role in many estimates and the scales (L, P) may be chosen so that $N(C) = (LP/2\pi\hbar)^n$ and the value of θ is minimal..

The cell will be said to be *regular* (one might also say "classical") if

$$N^{1/n} = LP/(2\pi\hbar) \gg 1, \quad \theta \ll (LP/\hbar)^{1/2}. \quad (2.2)$$

The first condition, expressing that the cell is big enough, is obviously classical. The second condition imposes some loose constraints on the shape of C : It should be bulky enough and its boundary smooth enough well above the scale of Planck's constant. This condition will cease to be satisfied for instance if the cell C represents the outcome of a chaotic classical motion having reached the scale of $\hbar$.

Families of projection operators

Two kinds of norms will be used for an operator A in $L^2(\mathbb{R}^n)$: the usual Hilbert norm $\|A\|$ and the trace norm

$$\|A\|_{\text{tr}} = \text{Tr} |A|, \text{ with } |A| = (AA^\dagger)^{1/2}. \quad (2.3)$$

They are linked by the inequality

$$|\text{Tr}(AB)| \leq \|A\| \cdot \|B\|_{\text{tr}} \quad (2.4)$$

which is very useful when one applies the consistent history formalism to classical physics.

Some families of projectors will be used. Let E be a projector with $\text{Tr}E = N \gg 1$. Another projector E' is said to be equivalent to E in the trace class (N, ϵ) , $\epsilon \ll 1$, if

$$\text{Tr}E' = N(1 + O(\epsilon)), \quad \text{Tr}|E - E'| = N O(\epsilon). \quad (2.5)$$

(One could of course replace the sloppy notation $O(\epsilon)$ by some inequalities involving definite constants of the order of 1). The subspaces associated with two equivalent projectors are near each other in the following sense :

- (i) Their dimensions are comparable, both of them being of the order of N .
- (ii) One can find two bases in these subspaces where about N pairs of vectors, one in each basis, are very near each whereas typically $N\epsilon$ vectors in one basis have no neighbour in the other basis.

An operator F is a quasi-projector in the class of E if it is self-adjoint, of trace class, compact (so that its spectrum is discrete), and furthermore

$$0 \leq F \leq I, \quad (2.6a)$$

$$\text{Tr}F = N(1 + O(\epsilon)), \quad (2.6b)$$

$$\text{Tr}(F - F^2) = N O(\epsilon), \quad (2.6c)$$

$$\text{Tr}|E - F| = N O(\epsilon). \quad (2.6d)$$

Two quasi-projectors F and F' equivalent to E also belong to the equivalence class. Given a quasi-projector F in the class, one can construct a projector in the class by using a basis of eigenstates of F , considering the eigenvalues $\{\lambda_k\}$ of F and replacing λ_k by 1 (resp. 0) if $\lambda_k \geq 1/2$ (resp. $< 1/2$).

The basic trick in the construction of a quantum logic for classical physics [9] consists in using quasi-projectors, which are easy to construct analytically, and to obtain from them true projection operators. Each one of them expresses a specific quantum property whereas all of them express equivalently the same classical property, up to unessential details occurring at the scale of $\hbar$.

3. Some results in microlocal analysis

Symbols

We shall use the Weyl calculus for pseudodifferential operators [7], [8, Chapter 18.5], which has been shown to provide the only sensible correspondence between quantum observables and classical dynamical variables [4]. A pseudo-differential operator A with symbol $a(q,p)$ in C^∞ acts on a function u in L^2 by

$$Au(q) = \int a\left(\frac{1}{2}(q+q'), p\right) e^{i p \cdot (q-q')/\hbar} u(q') dq' dp (2\pi\hbar)^{-n}. \quad (3.1)$$

It should be recalled at this point that, though one is using Fourier integrals, one needs not to assume that the configuration q -space is $\mathbb{R}^n$. This is one of the great "microlocal" virtues of pseudo-differential calculus.

We shall be most often interested in symbols belonging to the Schwartz class C_0^∞ so that the precise definition of the various symbol classes will not be essential. One can use however for definiteness the seminorms

$$C_{\alpha\beta} = \sup_{(q,p)} [(1+L^{-2}q^2 + P^{-2}p^2)^{(|\alpha|+|\beta|-m)/2} \times L^{|\alpha|} P^{|\beta|} |\partial_q^\alpha \partial_p^\beta a(q,p)|], \quad (3.2a)$$

or the more conventional ones [12]

$$C_{\alpha\beta} = \sup_{(q,p)} [(1+P^{-1}|p|)^{(\rho|\beta|-\delta|\alpha|-m)} \times L^{|\alpha|} P^{|\beta|} |\partial_q^\alpha \partial_p^\beta a(q,p)|], \quad (3.2b)$$

each one of them defining a specific symbol class S^m as a Fréchet space.

Products

It will be convenient to use a special notation for Poisson parentheses and their iterations. The phase space coordinates (q,p) can be denoted by $X = \{X^\mu\}$ with $\mu = (j, \alpha)$, $j \in \{1, 2, \dots, n\}$, $\alpha \in \{1, 2\}$, $X^{j1} = p^j$, $X^{j2} = q^j$. Defining $\varepsilon^{\mu\nu} = \varepsilon^{\alpha\beta} \delta^{jk}$ for $\nu = (k, \beta)$, the ordinary Poisson parenthesis $\{a, b\}$ can be written as

$$\{a, b\} = \varepsilon^{\mu\nu} \partial_\mu a \partial_\nu b \quad (3.3)$$

and iterated Poisson parentheses, to be written as $\{a\}^r b$, are defined by iteration of the operator $\{\}$ acting on a couple (a, b) .

Consider the product $C = AB$ of two operators A and B with symbols a and b , respectively in the classes S^m and $S^{m'}$. Its symbol is

$$c(q, p) = \int a(q+q', p+p') b(q+q'', p+p'') \exp[2i(q'' \cdot p' - q' \cdot p'')/\hbar] \times \\ dq' dp' dq'' dp'' (\pi\hbar)^{-2n} \quad (3.4)$$

and its kernel

$$K_C(q, q') = \int a(\frac{1}{2}(q+q''), p) b(\frac{1}{2}(q'+q''), p') \exp[i(q \cdot p - q' \cdot p' - iq'' \cdot p + iq'' \cdot p')/\hbar] \times \\ dp dp' dq'' (2\pi\hbar)^{-2n}. \quad (3.5)$$

Semi-classical developments in powers of $\hbar$ are obtained from

$$c(q, p) = \sum_{k=0}^{s-1} (-i\hbar/2)^k a^{(k)}(q, p) b^{(k)}(q', p') + r_s, \quad (3.6)$$

$$|r_s(q, p)| \leq K(\hbar/LP)^s (1 + L^{-2}q^2 + P^{-2}p^2)^{(m+m'-s)/2} \quad (3.7a)$$

$$K \leq C \frac{1}{s!} \sup |L(|\alpha| + |\alpha'|) P(|\beta| + |\beta'|) \partial_q^\alpha \partial_{q'}^{\alpha'} \partial_p^\beta \partial_{p'}^{\beta'} \hbar^s \times \\ \epsilon^{\mu_1 \nu_1} \dots \epsilon^{\mu_s \nu_s} \partial_{\mu_1} \dots \partial_{\mu_s} a(q, p) \partial_{\nu_1} \dots \partial_{\nu_s} b(q', p'), \quad (3.7b)$$

C being a constant of the order of 1. Similar expressions hold with the seminorms (3. 2b). The supremum in (3. 8) is taken over $(q, p, q', p') \in \mathbb{R}^{4n}$ and the derivation indices are such that $0 \leq |\alpha| + |\alpha'| + |\beta| + |\beta'| \leq n + 1$.

Traces

The trace of A is simply given by

$$\text{Tr } A = \int a(q, p) dq dp (2\pi\hbar)^{-n}. \quad (3.8)$$

When estimating the trace norm of an operator A with symbol $a(q, p)$, it is convenient to introduce the function

$$\delta a(q, p) = \sup |\partial_q^\alpha \partial_p^\beta a(q+q', p+p')|, \quad (3.9)$$

with $L^{-2}q^2 + P^{-2}p^2 \leq \hbar^2$, $|\alpha| + |\beta| = n+1$. Then one has [13]

$$\text{Tr } |A| \leq C_1 \|a\|_{L_1} + C_2 \|\delta a\|_{L_1}, \quad (3.10)$$

C_1 and C_2 being two constants $O(1)$.

4. Constructing projection operators

The aim of this Section is to construct explicitly a family of equivalent projectors for a regular cell. For this equivalence class (N, ε) , N is the "number of semiclassical states" defined in (2. 2) and $\varepsilon = \theta (\hbar/LP)^{1/2}$, θ being the "area to volume ratio" previously introduced. As an intermediate step in the construction, one must introduce some convenient quasi-projectors. This can be done with the help of coherent states or more directly by defining convenient symbols. Both techniques are useful in applications.

A coherent state is a gaussian wave function g_{qp} normalised in L^2 with average values q and p for position and momentum. One will take σ as defined in Eq.(2. 1) to be the uncertainty in the position coordinates (and therefore τ for the momentum uncertainty). A quasi-projector can then be directly defined by

$$F_g = \int_C |g_{qp}\rangle \langle g_{qp}| dq dp (\pi\hbar)^{-n}. \quad (4.1)$$

As an instructive example due to I. Daubechies, take $n = 1$ and C as the region $q^2/L^2 + p^2/P^2 \leq R^2$. F_g commutes with the oscillator hamiltonian $q^2/L^2 + (\hbar/i) \partial/\partial q)^2/P^2$ and its eigenfunctions are therefore Hermite functions ϕ_n . Evaluating explicitly $(\phi_n, F_g \phi_n)$, one obtains for the n -th eigenvalue of F_g the incomplete gamma function

$$\lambda_n = (1/n!) \int_0^{R^2/2} t^n \exp(-t) dt.$$

It is thus found that only a fraction $1/\sqrt{N}$ of the eigenvalues depart from a close neighbourhood of the set $\{1, 0\}$ with two elements, $N = \pi R^2 LP/\hbar$ being the number of eigenvalues close to 1.

One can also define a quasi-projector F through its symbol $f(q, p)$: Using non-dimensional coordinates, one defines a *margin* M of C as a region inside C at a distance from ∂C at most equal to $(\hbar/LP)^{1/2}$. The symbol $f(q, p)$ is a function belonging to S^∞ , equal to 1 in $C \setminus M$ and to 0 outside C . A more special but similar definition consists in introducing the characteristic function χ (equal to 1 in C and to 0 outside) and a function $\phi(q, p)$ in S^∞ having a compact support of radius σ and such that $\int \phi dq dp = 1$. The convolution $\phi * \chi$ is then a convenient symbol for a quasi-projector. One can recover F_g as a special case by using instead

$$\phi(q, p) = (\pi\hbar)^{-n} \exp[- q^2/(2\sigma^2) - p^2/(2\tau^2)]. \quad (4.2)$$

It should be noticed that these definitions always give rise to a compact self-adjoint operator so that its spectrum is discrete. This is because F_g is a Hilbert-Schmidt operator and, for F , because its symbol is in S^∞ .

The various definitions of quasi-projectors are (N, ε) equivalent, as shown by the two following lemmas:

Lemma 1

$$0 \leq F_g \leq I, \quad 0 \leq F \leq I. \quad (4.3)$$

Lemma 2

$$\text{Tr } F_g = N [1 + O(\varepsilon)], \quad (4.4)$$

$$\text{Tr } F = N [1 + O(\varepsilon)], \quad (4.5)$$

$$\text{Tr } |F - F_g| = N O(\varepsilon). \quad (4.6)$$

Proof of Lemma 1.

$F_g \geq 0$ is obvious. Replacing the integration domain C by R^{2n} in the integral (2. 1), one obtains the identity operator. $I - F_g$ is therefore given by a similar integral over $R^{2n} \setminus C$, from which $F_g \leq I$ follows.

Consider now F . One can use Eqs.(3. 4) and (3. 8) to obtain

$$\text{Tr } F^2 = \int f^2(q, p) dq dp (2\pi\hbar)^{-n} \leq N.$$

Iterating, one finds that all the even powers F^{2N} have uniformly bounded traces, so that the eigenvalues of F^2 are smaller than 1 and therefore $\|F\| \leq 1$. To show that $F \geq 0$, consider $f_\lambda = f(q/\lambda, p/\lambda)$, $\lambda > 1$. One has $0 \leq f_\lambda - f \leq 1$.

Approximating a wave function in L^2 by a function in C_0^∞ , one easily shows that the operators F_λ with symbol f_λ converge strongly to I when λ tends to infinity. Since $\|F_\lambda - F\| \leq 1$ by the previous argument, one has $\|I - F\| \leq 1$, from which $0 \leq F \leq I$ follows.

Proof of Lemma 2

One has

$$\langle g_{qp} | g_{q'p'} \rangle = \exp[- (q - q')^2 / (4 \sigma^2) - 2(p - p')^2 / \tau^2 + i (p' - p) \cdot (q' + q) / (2\hbar)]. \quad (4.7)$$

To get Eq. (4. 4), one writes

$$\text{Tr } F_g = \int_{C \times C} (2\pi\hbar)^{-2n} dq dp dq' dp' \langle g_{qp} | g_{q'p'} \rangle \quad (4.8)$$

Replace $C \times C$ by $C \times C'$ where $C' = C \cup M'$, M' being a margin of ∂C similar to M but outside C . It is easily shown from Eq. (4.7) that the integral over $C \times C'$ differs from the integral over $C \times \mathbb{R}^{2n}$ by a quantity of order $\exp(-\hbar)/(LP)$. The integral over $C \times \mathbb{R}^{2n}$ gives N . The integral over $C \times M'$ is bounded by $N(\hbar/LP)^{1/2}\theta$, from which the estimate (4.4) follows.

To prove the estimate (4.5), one uses Hörmander's formula (3.10), the calculations being straightforward. To prove Eq. (4.6), one introduces the intermediate operator

$$F_1 = \int |g_{qp}\rangle \langle g_{qp}| f(q, p) dq dp (2\pi\hbar)^{-n}.$$

One has trivially $F_g - F_1 \geq 0$ so that $\text{Tr}|F_g - F_1| = \text{Tr}(F_g - F_1) = N O(\varepsilon)$ by the same proof as for (4.4). But an explicit calculation shows that the symbol of f_1 is $f * \phi$, with ϕ as given in (4.2). Estimating $\text{Tr}|F - F_1|$ by (3.10), one finds that $\|\delta(f - f_1)\|_{L_1}$ in the second term of the estimate (3.10) is negligible whereas the first term is $O(N\varepsilon)$.

It remains to show that the operators we have introduced are quasi-projectors, i. e.

Lemma 3.

$$\text{Tr}|F - F^2| = N O(\varepsilon), \text{Tr}|F_g - F_g^2| = N O(\varepsilon). \quad (4.9)$$

Proof. Since, $0 \leq F \leq I$, $\text{Tr}|F - F^2| = \text{Tr}(F - F^2)$, which is given by the integral of $f - f^2$ for which the first estimate (4.9) holds. The second estimate can easily be obtained from $0 \leq F_g \leq I$ and Eq. (4.7).

Remarks

1. The choice of ε one started with is essential for the equivalence between the two natural definitions of a quasi-projector. It was also shown previously to be the best possible one [10]. It is often called the "classicality" parameter. One will come back in next Section to the fact that it is only algebraically small in $\hbar^{1/2}$, which is what one would expect [14] but not really extremely small.

2. The metric that has been used is rather arbitrary. One may wonder whether the results depend upon this choice. There are strong indications that the true minimum of θ would be obtained through a more involved process where one covers ∂C by a family of neighbourhoods, then use local coordinates where ∂C is represented by an equation $X^V = \text{constant}$ and

introduce finally convenient *local* metrics similar to our one. From this one could deduce that under a canonical change of coordinates $X \rightarrow X'$, one has $\theta' < C \theta$, C being an upper bound on $|\partial X^\mu / \partial X'^\nu|$. This has not been proved however in detail.

5. Exclusion

Quasi-projectors have apparently a serious inconvenience because the associated errors in $\hbar^{1/2}$ may be not small enough from a conservative standpoint, so that the logical propositions of classical physics would not be precise enough consequences of quantum logic. One should however consider how these propositions are used in practice. It is true that the proposition " (q, p) is in C " excludes its contrary " (q, p) is in $\mathbb{R}^{2n} \setminus C$ " with a probability of order ε , small but not extremely small. This is due to the fact that one cannot sharply exclude the points (q, p) very close to ∂C outside C . When one considers however two classical statements that must be sharply exclusive, they correspond to two cells C and C' rather far from each other. One may think for instance of two different experimental results indicated by two classically static (or still moving) positions of a pointer. This is typically the case where the two properties should be strongly exclusive and the corresponding projectors should satisfy $E E' = E' E = 0$ with a very good accuracy. This is what one is going to show.

One will use for simplicity the same metric for the two cells with fixed scales (L, P) . To define the distance between C and C' ($C \cap C' = \emptyset$), let $(q, p) \in \partial C$ and $(q', p') \in \partial C'$ be two points such that $D^2 = L^{-2} (q - q')^2 + P^{-2} (p - p')^2$ is minimal. Microlocal analysis [5] shows in general that $\|E E'\|$ decreases more rapidly than any power of $D\hbar/(LP)$. This rapid decrease expresses exclusion between E and E' but its precise form depends on the Schwartz functions entering the symbols from which E and E' are constructed. This looks somewhat obscure for down-to-earth physicists who prefer a more intuitive statement of exclusion, even if less general. Here is a convenient example : Let two quasi-projectors F_g and F'_g for C and C' be given by integrals (4.1). The symbol $c(q, p)$ of $F_g F'_g$ can be computed after a few gaussian integrations as

$$c(q_0, p_0) = \int_{C \times C'} (2\pi\hbar)^{-2n} dq dq' dp dp' (\pi\hbar)^{-n} \exp(-\Phi), \quad (5.1)$$

$$\Phi = (2\sigma^2)^{-1} [q_0 - (q + q')/2]^2 + (2\tau^2)^{-1} [p_0 - (p + p')/2]^2 + (8\sigma^2)^{-1} (q - q')^2 \\ + (8\tau^2)^{-1} (p - p')^2 - i p_0 (q - q')/\hbar + i q_0 (p - p')/\hbar.$$

Using (3.10), this implies

$$\text{Tr} |F_g F'_g| \leq C \exp(-D^2 LP/4\hbar). \quad (5.2)$$

with $C = O(1)$.

This bound upon quasi-projectors can be extended to a bound involving true projectors E and E' that are generated by F_g and F'_g . Let λ_k, ϕ_k (resp. λ'_j, ϕ'_j) denote the eigenvalues and eigenvectors of F_g (resp. F'_g). Let Σ_k denote a summation over all values of k and $\Sigma_{k'}$ a summation over the indices k such that $\lambda_k \geq 1/2$. One has

$$\text{Tr}(F_g F'_g) = \sum_{k,j} \lambda_k \lambda'_j |(\phi'_j, \phi_k)|^2 \leq \text{Tr}|F_g F'_g|,$$

$$\|EE'u\|^2 = \sum_{j,k'} |(\phi'_j, \phi_k)|^2 |(\phi_k, u)|^2 \leq \left(\sum_{j,k'} |(\phi'_j, \phi_k)|^2 \right) \|u\|^2,$$

so that

$$\|EE'\|^2 \leq \sum_{j,k'} |(\phi'_j, \phi_k)|^2 \leq 4 \sum_{k,j} \lambda_k \lambda'_j |(\phi'_j, \phi_k)|^2,$$

and therefore (with $C = O(1)$),

$$\|EE'\| \leq C \exp(-D^2 LP/2\hbar), \quad (5.3)$$

showing how good exclusion can really be.

6. Dynamics

The correspondence between canonical transformations in phase space and unitary transformations in Hilbert space goes back to Dirac and it was made more explicit by Egorov [11]. Egorov's theorem [11, 5, 12] is concerned with a canonical transformation $(q', p') = \Phi(q, p)$. One considers an operator A with symbol $a(q, p) \in S^m$ and the operator A' having a symbol that is obtained from $a(q, p)$ by a canonical change of coordinates : $a' = a \circ \Phi(q, p) = a(q'(q, p), p'(q, p))$. Egorov's theorem gives conditions under which there exists a unitary transformation U in L^2 such that

$$A' = U A U^{-1} + \text{lower-order terms.} \quad (6.4)$$

The lower-order terms are operators with lower-order symbols. The correspondence is exact (no lower-order terms) in two cases : for linear canonical transformations and for a C^∞ change of coordinates $q' = \zeta(q)$. In both cases the result follows from elementary algebra. Egorov's theorem takes of course explicit care of the neighbourhoods in which (q, p) and (q', p') must stand, the kind of symbols to be used, conditions on Φ , general properties of the lower-order terms and so on. The trouble is that these

restrictions can be chosen in various ways and there are quite a few theorems of that type.

Take for instance a canonical transformation with a generating function $\phi(q', p)$ so that $p' = \partial \phi / \partial q'$, $q = \partial \phi / \partial p$. It may result from a hamiltonian flow giving at a given time the action $S(q, q')$, from which $\phi(q', p)$ is obtained by a Legendre transform : $\phi = p' \cdot q - S$. Among the cases where Egorov's theorem is most conveniently stated [12], the canonical transformation is homogeneous, i. e. $q'(q, \lambda p) = q'(q, p)$, $p'(q, \lambda p) = \lambda p'(q, p)$, $\lambda > 0$. This occurs with a purely kinetic hamiltonian $h(q, p) = g_{jk}(x) p_j p_k$. There are many other cases for which the theorem holds, though apparently an exhaustive list does not exist. In some cases, one can be sure *a priori* that the theorem cannot hold : for instance if Φ is obtained from a chaotic motion extending down to scales of order $\hbar$. Another example occurs when there is a potential barrier and some classical trajectories Γ in phase space starting from C just touch the barrier. Other trajectories initially near Γ will strongly diverge since they can be reflected or not and there is no theorem for C . Similarly, suppose there is again a barrier, apparently not dangerous because all the classical trajectories starting from C are reflected. Suppose however that the barrier is extremely narrow. This means that some derivatives of the potential are very large and, via seminorms of symbols, the so-called "lower-order" terms will blow up.

The symbols we are interested in are of course associated with quasi-projectors. It may happen that a coherent state approach provides a convenient set-up, for instance with the Hamilton function $h(q, p) = p^2/2m + V(q)$. The results by G. Hagedorn [15] are very useful in that case and they can be taken as a substitute for Egorov's theorem. Anyway, the general status of quasiclassical physics is essentially this : One has the feeling that the results (amounting to a "correspondence principle") should be very general except when "physical" reasons forbid it. But the mathematical techniques, however powerful they may be, do not cover all the cases one encounters in real situations (or at least we cannot make sure of that with full confidence).

A conservative approach consists in finding at least some estimates for the validity of Bohr's "correspondence" that can be easily applied. One therefore considers a hamiltonian H with a (Weyl) symbol $h(q, p)$ belonging to S^2 . As a Hamilton function, $h(q, p)$ defines a hamiltonian flow : $(q, p) \rightarrow (q', p')$ after a time t . One also considers an initial regular cell C , with a definite symbol $f(q, p)$ giving a representative quasi-projector F . One assumes that the hamiltonian flow $C \rightarrow C'$ generates a cell C' that is also regular and has a representative quasi-projector F' . One makes the strong assumption that some version of Egorov's theorem insures that $a'(q', p') = a(q', p')$, $p(q', p')$ is a S^∞ symbol. The task is then reduced to an estimate of the errors in "correspondence", as they are expressed by the difference between F' and $F(t) = U^{-1}(t) F U(t)$, where $U(t) = \exp(-iHt/\hbar)$. Note that this the transformation law of a state operator and not Heisenberg operators.

The scales (L, P) will be fixed once for all. Liouville's theorem shows that $N = N'$. One will use the overall coordinates $X = \{X^\mu\} = (q, p)$ with greek indices, $X' = \{X^a\} = (q', p')$ with latin indices. One denotes $\partial_a \partial_b \dots X^\mu$ by $A_{ab\dots}^\mu$. Bounds $B_1, B_2, \dots$ on these derivatives will be defined by, for instance,

$$B_3 = \sup (|A_a^\lambda A_c^\mu A_e^\nu|, |A_a^\lambda A_{bc}^\mu|, |A_{abc}^\lambda|).$$

The Heisenberg equation $i\hbar \partial F(t)/\partial t = -[H, F(t)]$ becomes in view of Eq.(3.6) an equation for the symbol $f(t)$

$$\partial f(q, p, t')/\partial t + \{h, f(t')\} = -(\hbar^2/24) h \{ \}^3 f(t') \quad (6.2)$$

to be used for $0 \leq t' \leq t$, higher order terms being neglected. Putting the right-hand side of Eq. (6.2) equal to 0, one obtains a Liouville equation with the simple solution $f_0(X^a, t') = f(X^\mu(X^a))$, f being the symbol of the initial F and the reference time being t' . Notice that $h(X^\mu) = h(X^a)$. One then look for a solution $f(t') = f_0(t') + f_1(t')$ of Eq. (6.2). Up to terms of third order in $\hbar$, one has

$$\partial f_1(X^a, t')/\partial t' + \varepsilon^{ab} \partial_a h(X^a) \partial_b f_1(X^a, t') = -(\hbar^2/24) h \{ \}^3 f_0(t'). \quad (6.3)$$

Notice that for any pair of functions (f, g) , one has

$$f \{ \}^3 g = \varepsilon^{ab} \varepsilon^{cd} \varepsilon^{ef} \partial_a \partial_c \partial_e f \partial_b \partial_d \partial_f g. \quad (6.4)$$

$$\begin{aligned} &= \varepsilon^{\lambda\mu} \varepsilon^{\nu\rho} \varepsilon^{\sigma\tau} \partial_\lambda \partial_\nu \partial_\sigma f \partial_\mu \partial_\rho \partial_\tau g \\ &+ 3 \varepsilon^{ab} \varepsilon^{cd} A_a^\lambda A_{bc}^\mu A_d^\nu (\partial_\lambda \partial_\nu f \partial_\mu g - \partial_\mu f \partial_\lambda \partial_\nu g) \\ &+ 3 \varepsilon^{ab} \varepsilon^{cd} A_{ac}^\lambda A_{bd}^\mu \varepsilon^{\nu\rho} \partial_\lambda \partial_\nu f \partial_\mu \partial_\rho g \\ &+ 3 \varepsilon^{ab} \varepsilon^{cd} \varepsilon^{ef} A_a^\lambda A_{bc}^\mu A_{de}^\nu A_f^\rho (\partial_\lambda \partial_\nu f \partial_\mu \partial_\rho g - \partial_\mu \partial_\rho f \partial_\lambda \partial_\nu g) \\ &+ 3 \varepsilon^{ab} \varepsilon^{cd} \varepsilon^{ef} A_{ac}^\lambda A_{bde}^\mu A_f^\nu (\partial_\lambda \partial_\nu f \partial_\mu g - \partial_\mu f \partial_\lambda \partial_\nu g). \quad (6.5) \\ &+ \varepsilon^{ab} \varepsilon^{cd} \varepsilon^{ef} A_a^\lambda A_c^\mu A_e^\nu A_{bdf}^\rho (\partial_\lambda \partial_\mu \partial_\nu f \partial_\rho g - \partial_\rho f \partial_\lambda \partial_\mu \partial_\nu g) \\ &+ X^\mu \{ \}^3 X^\nu \partial_\mu f \partial_\nu g \end{aligned}$$

The fact that the first term in Eq. (6.5) has the same formal expression as the right-hand side in Eq.(6.4) is well known. The time-dependent canonical change of coordinates $f_1(X^a, t') = f_1(X^a(X^\mu, t'), t') = g_1(X^\mu, t')$ replaces Eq.(6.3) by

$$\partial g_1/\partial t' = -(\hbar^2/24) h \{ \}^3 f_0(t'),$$

where Eq.(6.5) should be used for the right-hand side. This gives

$$g_1(t) = -(\hbar^2/24) \int_0^t \hbar \{ \}^3 f_0(t') dt'. \quad (6.6)$$

One can now use these relations for an estimate of the relevant quantities, to get, after straightforward and tedious calculations,

$$\text{Tr} | F' - F(t) | = N O(\zeta). \quad (6.7)$$

with

$$\zeta = \sup_{0 \leq t' \leq t} (\epsilon', \epsilon B_1(t), \hbar^2 t (LP)^{-3} (\sup |\partial^3 \hbar(X^a)/\partial^3 X^\mu(t')|) B_3(t')). \quad (6.8)$$

Notice the meaning of the result and particularly when the correspondence breaks down : The occurrence of ϵ' excludes a too evolved chaotic motion. The bounds B_1 and B_3 would also give ζ 's of the order of 1 after a finite time if there is a positive Liapunov exponent (i. e. once again chaos). Assuming that C' is a cell excludes the marginal crossing of a barrier. A very narrow barrier that is not crossed classically but can be quantum-mechanically would manifest itself in $\partial^3 \hbar/\partial^3 X^\mu$. It would be interesting to find other cases of exception.

A remarkable feature of the results is that the validity of the quantum/classical correspondence is completely controlled by classical calculations. Numerical examples will be published elsewhere, showing how the present gross estimates can be improved in practice. The application of these results to the validity of classical logic and determinism, with the corresponding errors attributed to quantum fluctuations, have been given elsewhere [9]. Considerations about exclusion similar to those in Section 5 show that determinism is quite often exponentially valid if a deterministic logical implication " $(q, p) \in C$ at time 0 " $\Rightarrow$ " $(q', p') \in C'_1$ at time t " is applied to a cell $C'_1 \supset C'$ rather than strictly to C' .

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