NON-LOCAL BOUNDARY CONDITIONS FOR
MASSLESS SPIN-$\frac{1}{2}$ FIELDS

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Abstract. This paper studies the 1-loop approximation for a massless spin-1/2 field on a flat four-dimensional Euclidean background bounded by two concentric 3-spheres, when non-local boundary conditions of the spectral type are imposed. The use of $\zeta$-function regularization shows that the conformal anomaly vanishes, as in the case when the same field is subject to local boundary conditions involving projectors. A similar analysis of non-local boundary conditions can be performed for massless supergravity models on manifolds with boundary, to study their 1-loop properties.
1. Introduction

The quantum theory of fermionic fields can be expressed, following the ideas of Feynman, in terms of amplitudes of going from suitable fermionic data on a spacelike surface $S_I$, say, to fermionic data on a spacelike surface $S_F$. To make sure that the quantum boundary-value problem is well posed, one has actually to consider the Euclidean formulation, where the boundary 3-surfaces, $\Sigma_I$ and $\Sigma_F$, say, may be regarded as (compact) Riemannian 3-manifolds bounding a Riemannian 4-manifold. In the case of massless spin-1/2 fields, which are the object of our investigation, one thus deals with transition amplitudes

$$A[\text{boundary data}] = \int e^{-I_E} \mathcal{D}\psi \mathcal{D}\tilde{\psi}$$  \hspace{1cm} (1.1)$$

where $I_E$ is the Euclidean action functional, and the integration is over all massless spin-1/2 fields matching the boundary data on $\Sigma_I$ and $\Sigma_F$. The path-integral representation of the quantum amplitude (1.1) is then obtained with the help of Berezin integration rules, and one has a choice of non-local [1] or local [2] boundary conditions. The mathematical foundations of the former lie in the theory of spectral asymmetry and Riemannian geometry [3], and their formulation can be described as follows. In two-component spinor notation, a massless spin-1/2 field in a 4-manifold with positive-definite metric is represented by a pair $(\psi^A, \tilde{\psi}^{A'})$ of independent spinor fields, not related by any spinor conjugation. Suppose now that $\psi^A$ and $\tilde{\psi}^{A'}$ are expanded on a family of concentric 3-spheres as

$$\psi^A = \frac{1}{2\pi} \tau^{-\frac{3}{2}} \sum_{n=0}^{\infty} \sum_{p,q=1}^{(n+1)(n+2)} \alpha_{pq} m_{np}(\tau) \rho^{nqA} + \tilde{\tau}_{np}(\tau) \tilde{\sigma}^{nqA}$$  \hspace{1cm} (1.2)$$
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$$\tilde{\psi}^{A'} = \frac{1}{2\pi} \tau^{-\frac{3}{2}} \sum_{n=0}^{\infty} \sum_{p,q=1}^{(n+1)(n+2)} \alpha_{nq}^{pq} \left[ \tilde{m}_{np}(\tau) \tilde{\rho}^{npqA'} + \tilde{r}_{np}(\tau) \sigma^{npqA'} \right]. \quad (1.3)$$

With a standard notation, $\tau$ is the Euclidean-time coordinate which plays the role of a radial coordinate, and the block-diagonal matrices $\alpha_{nq}^{pq}$ and the $\rho$- and $\sigma$-harmonics are described in detail in [4]. One can now check that the harmonics $\rho^{npqA}$ have positive eigenvalues for the intrinsic three-dimensional Dirac operator on $S^3$:

$$\mathcal{D}_{AB} \equiv e_{nAB'} e^{B'j} (3) D_j \quad (1.4)$$

and similarly for the harmonics $\sigma^{npqA'}$ and the Dirac operator

$$\mathcal{D}_{A'B'} \equiv e_{nBA'} e^{B'j} (3) D_j. \quad (1.5)$$

With our notation, $e_{nAB'}$ is the Euclidean normal to the boundary, $e_{B'j}$ are the spatial components of the two-spinor version of the tetrad, and $(3) D_j$ denotes three-dimensional covariant differentiation on $S^3$ [1, 2, 4]. By contrast, the harmonics $\sigma^{npqA}$ and $\rho^{npqA'}$ have negative eigenvalues for the operators (1.4) and (1.5) respectively.

The so-called spectral boundary conditions rely therefore on a non-local operation, i.e. the separation of the spectrum of a first-order elliptic operator (our (1.4) and (1.5)) into a positive and a negative part. They require that half of the spin-$1/2$ field should vanish on $\Sigma_F$, where this half is given by those modes $m_{np}(\tau)$ and $r_{np}(\tau)$ which multiply harmonics having positive eigenvalues for (1.4) and (1.5) respectively. The remaining half of the field should vanish on $\Sigma_I$, and is given by those modes $\tilde{r}_{np}(\tau)$ and $\tilde{m}_{np}(\tau)$ which
multiply harmonics having negative eigenvalues for (1.4) and (1.5) respectively. One thus writes [4]

\[ \left[ \psi^A_{(+)} \right]_{\Sigma F} = 0 \implies \left[ m_{np} \right]_{\Sigma F} = 0 \]  
\[ (1.6) \]

\[ \left[ \tilde{\psi}^A_{(+)} \right]_{\Sigma F} = 0 \implies \left[ r_{np} \right]_{\Sigma F} = 0 \]  
\[ (1.7) \]

and

\[ \left[ \psi^A_{(-)} \right]_{\Sigma I} = 0 \implies \left[ \tilde{r}_{np} \right]_{\Sigma I} = 0 \]  
\[ (1.8) \]

\[ \left[ \tilde{\psi}^A_{(-)} \right]_{\Sigma I} = 0 \implies \left[ \tilde{m}_{np} \right]_{\Sigma I} = 0. \]  
\[ (1.9) \]

Massless spin-1/2 fields are here studied since they provide an interesting example of conformally invariant field theory for which the spectral boundary conditions (1.6)–(1.9) occur naturally already at the classical level [3].

Section 2 is devoted to the evaluation of the \( \zeta(0) \) value resulting from the boundary conditions (1.6)–(1.9). This yields the 1-loop divergence of the quantum amplitude, and coincides with the conformally anomaly in our model. Concluding remarks are presented in section 3.

2. \( \zeta(0) \) value with non-local boundary conditions

As shown in [1, 2, 4, 5], the modes occurring in the expansions (1.2) and (1.3) obey a coupled set of equations, i.e.

\[ \left( \frac{d}{d\tau} - \frac{n + \frac{3}{2}}{\tau} \right) x_{np} = E_{np} \tilde{x}_{np} \]  
\[ (2.1) \]
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\[
\left( -\frac{d}{d\tau} - \frac{(n + \frac{3}{2})}{\tau} \right) \tilde{x}_{np} = E_{np} x_{np} \tag{2.2}
\]

where $x_{np}$ denotes $m_{np}$ or $r_{np}$, and $\tilde{x}_{np}$ denotes $\tilde{m}_{np}$ or $\tilde{r}_{np}$. Setting $E_{np} = M$ for simplicity of notation one thus finds, for all $n \geq 0$, the solutions of (2.1) and (2.2) in the form

\[
m_{np}(\tau) = \beta_{1,n} \sqrt{\tau} I_{n+1}(M\tau) + \beta_{2,n} \sqrt{\tau} K_{n+1}(M\tau) \tag{2.3}
\]

\[
r_{np}(\tau) = \beta_{1,n} \sqrt{\tau} I_{n+1}(M\tau) + \beta_{2,n} \sqrt{\tau} K_{n+1}(M\tau) \tag{2.4}
\]

\[
\tilde{m}_{np}(\tau) = \beta_{1,n} \sqrt{\tau} I_{n+2}(M\tau) - \beta_{2,n} \sqrt{\tau} K_{n+2}(M\tau) \tag{2.5}
\]

\[
\tilde{r}_{np}(\tau) = \beta_{1,n} \sqrt{\tau} I_{n+2}(M\tau) - \beta_{2,n} \sqrt{\tau} K_{n+2}(M\tau) \tag{2.6}
\]

where $\beta_{1,n}$ and $\beta_{2,n}$ are some constants. The insertion of (2.3)–(2.6) into the boundary conditions (1.6)–(1.9) leads to the equations (hereafter $b$ and $a$ are the radii of the two concentric 3-sphere boundaries, with $b > a$, and we define $\beta_n \equiv \beta_{2,n}/\beta_{1,n}$)

\[
I_{n+1}(Mb) + \beta_n K_{n+1}(Mb) = 0 \tag{2.7}
\]

for $m_{np}$ and $r_{np}$ modes, and

\[
I_{n+2}(Ma) - \beta_n K_{n+2}(Ma) = 0 \tag{2.8}
\]

for $\tilde{m}_{np}$ and $\tilde{r}_{np}$ modes, with the same value of $M$ [1]. One thus finds two equivalent formulae for $\beta_n$:

\[
\beta_n = -\frac{I_{n+1}(Mb)}{K_{n+1}(Mb)} = \frac{I_{n+2}(Ma)}{K_{n+2}(Ma)} \tag{2.9}
\]
which lead to the eigenvalue condition

\[ I_{n+1}(Mb)K_{n+2}(Ma) + I_{n+2}(Ma)K_{n+1}(Mb) = 0. \]  \hspace{1cm} (2.10)

The full degeneracy is \(2(n + 1)(n + 2)\), for all \(n \geq 0\), since each set of modes contributes to (2.7) and (2.8) with degeneracy \((n + 1)(n + 2)\) [1].

We can now apply \(\zeta\)-function regularization to evaluate the resulting conformal anomaly [6], following the algorithm developed in [7, 8] and applied several times in the recent literature [9–16]. The basic properties are as follows. Let us denote by \(f_n\) the function occurring in the equation obeyed by the eigenvalues by virtue of boundary conditions, after taking out fake roots (e.g. \(x = 0\) is a fake root of order \(\nu\) of the Bessel function \(I_\nu(x)\)). Let \(d(n)\) be the degeneracy of the eigenvalues parametrized by the integer \(n\). One can then define the function

\[ I(M^2, s) \equiv \sum_{n=n_0}^\infty d(n)n^{-2s}\log f_n(M^2) \]  \hspace{1cm} (2.11)

and the work in [7, 8] shows that such a function admits an analytic continuation to the complex-\(s\) plane as a meromorphic function with a simple pole at \(s = 0\), in the form

\[ "I(M^2, s)" = \frac{I_{\text{pole}}(M^2)}{s} + I^R(M^2) + O(s). \]  \hspace{1cm} (2.12)

The function \(I_{\text{pole}}(M^2)\) is the residue at \(s = 0\), and makes it possible to obtain the \(\zeta(0)\) value as

\[ \zeta(0) = I_{\log} + I_{\text{pole}}(M^2 = \infty) - I_{\text{pole}}(M^2 = 0) \]  \hspace{1cm} (2.13)

where \(I_{\log}\) is the coefficient of the \(\log(M)\) term in \(I^R\) as \(M \to \infty\). The contributions \(I_{\log}\) and \(I_{\text{pole}}(\infty)\) are obtained from the uniform asymptotic expansions of basis functions as
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$M \to \infty$ and their order $n \to \infty$, whilst $I_{\text{pole}}(0)$ is obtained by taking the $M \to 0$ limit of the eigenvalue condition, and then studying the asymptotics as $n \to \infty$. More precisely, $I_{\text{pole}}(\infty)$ coincides with the coefficient of $\frac{1}{n}$ in the expansion as $n \to \infty$ of

$$\frac{1}{2} d(n) \log \left[ \rho_\infty(n) \right]$$

where $\rho_\infty(n)$ is the $n$-dependent term in the eigenvalue condition as $M \to \infty$ and $n \to \infty$.

The $I_{\text{pole}}(0)$ value is instead obtained as the coefficient of $\frac{1}{n}$ in the expansion as $n \to \infty$ of

$$\frac{1}{2} d(n) \log \left[ \rho_0(n) \right]$$

where $\rho_0(n)$ is the $n$-dependent term in the eigenvalue condition as $M \to 0$ and $n \to \infty$ [7, 8, 14].

In our problem, using the limiting form of Bessel functions when the argument tends to zero [17], one finds that the left-hand side of (2.10) is proportional to $M^{-1}$ as $M \to 0$. Hence one has to multiply by $M$ to get rid of fake roots. Moreover, in the uniform asymptotic expansion of Bessel functions as $M \to \infty$ and $n \to \infty$, both $I$ and $K$ functions contribute a $\frac{1}{\sqrt{M}}$ factor. These properties imply that $I_{\text{log}}$ vanishes:

$$I_{\text{log}} = \frac{1}{2} \sum_{l=1}^{\infty} 2l(l+1) \left( 1 - \frac{1}{2} - \frac{1}{2} \right) = 0. \quad (2.14)$$

Moreover,

$$I_{\text{pole}}(\infty) = 0 \quad (2.15)$$
since there is no $n$-dependent coefficient in the uniform asymptotic expansion of (2.10) [7–16]. Last, one finds

$$I_{\text{pole}}(0) = 0$$

(2.16)

since the limiting form of (2.10) as $M \to 0$ and $n \to \infty$ is

$$\frac{2}{Ma} (b/a)^{n+1}.$$

The results (2.14)–(2.16), jointly with the general formula (2.13), lead to a vanishing value of the 1-loop divergence:

$$\zeta(0) = 0.$$

(2.17)

3. Concluding remarks

To our knowledge, the analysis leading to (2.17) in the spectral case, had not been performed in the current literature. Our detailed calculation shows that, in flat Euclidean 4-space, the conformal anomaly for a massless spin-1/2 field subject to non-local boundary conditions of the spectral type on two concentric 3-spheres vanishes, as in the case when the same field is subject to the local boundary conditions

$$\sqrt{2} \epsilon n_A A' \psi^A = \pm \tilde{\psi}^{A'} \text{ on } \Sigma_I \text{ and } \Sigma_F.$$  

(3.1)

If (3.1) holds and the spin-1/2 field is massless, the work in [10] shows in fact that $\zeta(0) = 0$.

Backgrounds given by flat Euclidean 4-space bounded by two concentric 3-spheres are not the ones occurring in the Hartle-Hawking proposal for quantum cosmology, where the initial 3-surface $\Sigma_I$ shrinks to a point [18]. Nevertheless, they are relevant for the
quantization programme of gauge fields and gravitation in the presence of boundaries [11, 12]. In particular, similar techniques have been used in section 5 of [16] to study a two-boundary problem for simple supergravity subject to spectral boundary conditions in the axial gauge. One then finds the eigenvalue condition

$$I_{n+2}(Mb)K_{n+3}(Ma) + I_{n+3}(Ma)K_{n+2}(Mb) = 0 \quad (3.2)$$

for all $n \geq 0$. The analysis of (3.2) along the same lines of section 2 shows that transverse-traceless gravitino modes yield a vanishing contribution to $\zeta(0)$, unlike transverse-traceless modes for gravitons, which instead contribute $-5$ to $\zeta(0)$ [12, 16].

Thus, the results in [16] seem to show that, at least in finite regions bounded by one 3-sphere or two concentric 3-spheres, simple supergravity is not one-loop finite in the presence of boundaries. Of course, more work is in order to check this property, and then compare it with the finiteness of scattering problems suggested in [19]. Further progress is thus likely to occur by virtue of the fertile interplay of geometric and analytic techniques [20–24] in the investigation of heat-kernel asymptotics and (1-loop) quantum cosmology.

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