ABOUT KINEMATICS AND HYDRODYNAMICS OF SPINNING PARTICLES: SOME SIMPLE CONSIDERATIONS∗

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Submitted 25 November 1995

Abstract. In the first part (Sections 1 and 2) of this paper—starting from the Pauli current, in the ordinary tensorial language—we obtain the decomposition of the non-relativistic field velocity into two orthogonal parts: (i) the “classical” part, that is, the velocity \( \vec{w} = \vec{p}/m \) of the center-of-mass (CM), and (ii) the so-called “quantum” part, that is, the velocity \( \vec{V} \) of the motion in the CM frame (namely, the internal “spin motion” or zitterbewegung). By inserting such a complete, composite expression of the velocity into the kinetic energy term of the non-relativistic classical (i.e., newtonian) lagrangian, we straightforwardly get the appearance of the so-called “quantum potential” associated, as it is known, with the Madelung fluid. This result carries further evidence that the quantum behaviour of micro-systems can be a direct consequence of the fundamental existence of spin. In the second part (Sections 3 and 4), we fix our attention on the total velocity \( \vec{v} = \vec{w} + \vec{V} \), it being now necessary to pass to relativistic (classical) physics; and we show that the proper time entering the definition of the four-velocity \( v^\mu \) for spinning particles has to be the proper time \( \tau \) of the CM frame. Inserting the correct Lorentz factor into the definition of \( v^\mu \) leads to completely new kinematical properties for \( v^2 \). The important constraint \( p_\mu v^\mu = m \), identically true for scalar particles, but just assumed a priori in all previous spinning particle theories, is herein derived in a self-consistent way.


“If a spinning particle is not quite a point particle, nor a solid three dimensional top, what can it be?”

Asim O. Barut

∗ Work partially supported by UNAM, by FAPESP, CNPq, and by INFN, MURST, CNR.
1. Madelung fluid: A variational approach

The lagrangian for a non-relativistic scalar particle may be assumed to be:

\[ L = i\hbar \left( \psi^* \partial_t \psi - \frac{\hbar^2}{2m} \nabla \psi^* \cdot \nabla \psi - U \right) \tag{1} \]

where \( U \) is the external potential energy and the other symbols have the usual meaning. It is known that, by taking the variations of \( L \) with respect to \( \psi, \psi^* \), one can get the Schroedinger equations for \( \psi^* \) and \( \psi \), respectively.

By contrast, since a generic scalar wavefunction \( \psi \in \mathcal{C} \) can be written as

\[ \psi = \sqrt{\rho} \exp \left[ i\varphi/\hbar \right] \tag{2} \]

with \( \rho, \varphi \in \mathbb{R} \), we take the variations of

\[ L = - \left[ \partial_t \varphi + \frac{1}{2m} (\nabla \varphi)^2 + \frac{\hbar^2}{8m} \left( \frac{\nabla \rho}{\rho} \right)^2 + U \right] \rho \tag{3} \]

with respect to \( \rho \) and \( \varphi \). We then obtain[1-3] the two equations for the so-called Madelung fluid[4] (which, taken together, are equivalent to the Schroedinger equation):

\[ \partial_t \varphi + \frac{1}{2m} (\nabla \varphi)^2 + \frac{\hbar^2}{4m} \left[ \frac{1}{2} \left( \frac{\nabla \rho}{\rho} \right)^2 - \frac{\Delta \rho}{\rho} \right] + U = 0 \tag{4} \]

and

\[ \partial_t \rho + \nabla \cdot \left( \rho \nabla \varphi / m \right) = 0 \tag{5} \]

which are the Hamilton–Jacobi and the continuity equation for the “quantum fluid”, respectively; where

\[ \frac{\hbar^2}{8m} \left[ \frac{1}{2} \left( \frac{\nabla \rho}{\rho} \right)^2 - \frac{\Delta \rho}{\rho} \right] = - \frac{\hbar^2}{2m} \left( \frac{\Delta |\psi|}{|\psi|} \right) \tag{6} \]

is often called the “quantum potential”. Such a potential derives from the last-but-one term in the r.h.s. of eq.(3), that is to say, from the (single) “non-classical term”

\[ \frac{\hbar^2}{8m} \left( \frac{\nabla \rho}{\rho} \right)^2 \tag{7} \]

entering our lagrangian \( L \).

Notice that we got the present hydrodynamical reformulation of the Schroedinger theory directly from a variational approach.[3] This procedure, as we are going to see, offers us a physical interpretation of the non-classical terms appearing in eqs.(3) or (4). On the contrary, eqs.(4-5) are ordinarily obtained by inserting relation (2) into the Schroedinger equation, and then separating the real and the imaginary part: a rather formal procedure, that does not shed light on the underlying physics.

Let us recall that an early physical interpretation of the so-called “quantum” potential, that is to say, of term (6) was forwarded by de Broglie’s pilot–wave
theory\cite{5}; in the fifties, Bohm\cite{6} revisited and completed de Broglie’s approach in a systematic way [and, sometimes, Bohm’s theoretical formalism is referred to as the “Bohm formulation of quantum mechanics”, alternative and complementary to Heisenberg’s (matrices and Hilbert spaces), Schroedinger’s (wave-functions), and Feynman’s (path integrals) theory]. From Bohm’s up to our days, several conjectures about the origin of that mysterious potential have been put forth, by postulating “subquantal” forces, the presence of an ether, and so on. Well-known are also the derivations of the Madelung fluid within the stochastic mechanics framework:\cite{7,2} in those theories, the origin of the non-classical term (6) appears as substantially kinematical. In the non-markovian approaches,\cite{2} for instance, after having assumed the existence of the so-called zitterbewegung, a spinning particle appears as an extended-like object, while the “quantum” potential is tentatively related to an internal motion.

But we do not need following any stochastic approach, even if our phylosophical starting point is the recognition of the existence\cite{8-12} of a zitterbewegung (zbw) or diffusive or internal motion [i.e., of a motion observed in the center-of-mass (CM) frame, which is the one where $p = 0$ by definition], besides of the [external, or drift, or translational, or convective] motion of the CM. In fact, the existence of such an internal motion is denounced, besides by the mere presence of spin, by the remarkable fact that in the standard Dirac theory the particle impulse $p$ is in general not parallel to the velocity: $v \neq \frac{p}{m}$; moreover, while $[\hat{p}, \hat{H}] = 0$ so that $p$ is a conserved quantity, quantity $v$ is not a constant of the motion: $[\hat{v}, \hat{H}] \neq 0$ ($\hat{v} \equiv \alpha \equiv \gamma^0 \gamma$ being the usual vector matrix of Dirac theory). Let us explicitly notice, moreover, that for dealing with the zbw it is highly convenient\cite{10,12} to split the motion variables as follows (the dot meaning derivation with respect to time):

$$x = \xi + X; \quad \dot{x} \equiv v = w + V,$$

where $\xi$ and $w \equiv \dot{\xi}$ describe the motion of the CM in the chosen reference frame, whilst $X$ and $V \equiv \dot{X}$ describe the internal motion referred to the CM frame (CMF). [Notice that what is called the “diffusion velocity” $v_{\text{diff}}$ in the stochastic approaches is nothing but our $V$]. From a dynamical point of view, the conserved electric current is associated with the helical trajectories\cite{8-10} of the electric charge (i.e., with $x$ and $v \equiv \dot{x}$), whilst the center of the particle coulombian field is associated with the geometrical center of such trajectories (i.e., with $\xi$ and $w \equiv \dot{\xi} = \frac{p}{m}$).

Going back to lagrangian (3), it is now possible to attempt an interpretation\cite{3} of the non-classical term $\frac{\hbar^2}{8m} (\nabla \rho/\rho)^2$ appearing therein. So, the first term in the r.h.s. of eq.(3) represents, apart from the sign, the total energy

$$\partial_t \varphi = - E;$$

whereas the second term is recognized to be the kinetic energy $\frac{p^2}{2m}$ of the CM, if one assumes that

$$p = - \nabla \varphi.$$

The third term, that gives origin to the quantum potential, will be shown below to be interpretable as the kinetic energy in the CMF, that is, the internal energy due to the zbw motion. It will be soon realized, therefore, that in lagrangian (3)
the sum of the two kinetic energy terms, $p^2/2m$ and $\frac{1}{2}mV^2$, is nothing but a mere application of the König theorem. We are not going to exploit, as often done, the arrival point, i.e. the Schroedinger equation; by contrast, we are going to exploit a non-relativistic (NR) analogue of the Gordon decomposition\[13\] of the Dirac current: namely, a suitable decomposition of the Pauli current.\[14\] In so doing, we shall meet an interesting relation between zbw and spin.

2. The “quantum” potential as a mere consequence of spin and zbw

During the last thirty years Hestenes\[15\] did systematically employ the Clifford algebras language in the description of the geometrical, kinematical and hydrodynamical (i.e., field) properties of spinning particles, both in relativistic and NR physics, i.e., both for Dirac theory and for Schroedinger–Pauli theory. In the small-velocity limit of the Dirac equation, or directly from Pauli equation, Hestenes got the following decomposition of the particle velocity:

$$v = \frac{p - eA}{m} + \frac{\nabla \wedge (\rho s)}{m\rho}$$  \hspace{1cm} (11)

where the light speed $c$ is assumed equal to 1, quantity $e$ is the electric charge, $A$ is the external electromagnetic vector potential, $s$ is the spin vector $s \equiv \rho^{-1}\psi^\dagger \hat{s}\psi$, and $\hat{s}$ is the spin operator usually represented in terms of Pauli matrices as

$$\hat{s} \equiv \frac{\hbar}{2}(\sigma_x; \sigma_y; \sigma_z).$$  \hspace{1cm} (12)

[Hereafter, every quantity is a local or field quantity: $v \equiv v(x; t)$; $p \equiv p(x; t)$; $s \equiv s(x; t)$; and so on]. As a consequence, the internal (zbw) velocity reads:

$$V \equiv \frac{\nabla \wedge (\rho s)}{m\rho}.$$  \hspace{1cm} (13)

Let us repeat the previous derivation —by making now recourse to the ordinary tensor language— from the familiar expression of the Pauli current\[14\] (i.e., from the Gordon decomposition of the Dirac current in the NR limit):

$$j = \frac{i\hbar}{2m}[(\nabla \psi^\dagger)\psi - \psi^\dagger \nabla \psi] - \frac{eA}{m}\psi^\dagger \psi + \frac{1}{m}\nabla \wedge (\psi^\dagger \hat{s}\psi).$$  \hspace{1cm} (14)

A spinning NR particle can be simply factorized into

$$\psi \equiv \sqrt{\rho} \Phi,$$  \hspace{1cm} (15)

$\Phi$ being a Pauli 2-component spinor, which has to obey the normalization constraint

$$\Phi^\dagger \Phi = 1$$

if we want to have $|\psi|^2 = \rho$.

By definition $\rho s \equiv \psi^\dagger \hat{s}\psi \equiv \rho \Phi^\dagger \hat{s}\Phi$; therefore, introducing the factorization $\psi \equiv \sqrt{\rho} \Phi$ into the above expression (14) for the Pauli current, one just obtains:\[3\]

$$j \equiv \rho v = \rho \frac{p - eA}{m} + \frac{\nabla \wedge (\rho s)}{m}$$  \hspace{1cm} (16)
which is nothing but Hestenes’ decomposition (11) of $v$.

The Schroedinger subcase [i.e., the case in which the vector spin field $s = s(x,t)$ is constant in time and uniform in space] corresponds to spin eigenstates; so that we need now a wave-function factorizable into the product of a “non-spin” part $\sqrt{\rho} e^{i\phi}$ (scalar) and of a “spin” part $\chi$ (Pauli spinor):

$$\psi = \sqrt{\rho} e^{i\phi} \chi,$$

(17)

$\chi$ being constant in time and space. Therefore, when $s$ has no precession (and no external field is present: $A = 0$), we have $s \equiv \chi^\dagger \hat{s} \chi = \text{constant}$, and

$$V = \frac{\nabla \rho \wedge s}{m\rho} \neq 0.$$  \hspace{1cm} (Schroedinger case)

(18)

One can notice that, even in the Schroedinger theoretical framework, the zbw does not vanish, except for plane waves, i.e., for the non-physical case of $p$-eigenfunctions, when not only $s$, but also $\rho$ is constant and uniform, so that $\nabla \rho = 0$. (Notice also that the continuity equation (6), $\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$, can be still rewritten in the ordinary way $\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$. In fact, quantity $\nabla \cdot \mathbf{V} \equiv \nabla \cdot (\nabla \wedge (\rho s))$ is identically zero, it being the divergence of a rotor, so that $\nabla \cdot (p/m) = \nabla \cdot \mathbf{v}$).

But let us go on. We may now write

$$V^2 = \left( \frac{\nabla \rho \wedge s}{m\rho} \right)^2 = \frac{(\nabla \rho)^2 s^2 - (\nabla \rho \cdot s)^2}{(m\rho)^2}$$  \hspace{1cm} (19)

since in general it holds

$$(a \wedge b)^2 = a^2 b^2 - (a \cdot b)^2.$$  \hspace{1cm} (20)

Let us now put into equation (19), for instance, Hestenes’ constraint ($\beta$ being the Takabayasi angle[16]): $\nabla \cdot (\rho s) = -m\rho \sin \beta$, which in the NR limit yields $\beta = 0$ (“pure electron”) or $\beta = \pi$ (“pure positron”), so that: $\nabla \cdot (\rho s) = 0$ and in the Schroedinger case $|s = \text{constant}; \nabla \cdot s = 0|$ becomes

$$\nabla \rho \cdot s = 0.$$  \hspace{1cm} (21)

Then, eq.(19) does assume[3] the important form

$$V^2 = s^2 \left( \frac{\nabla \rho}{m\rho} \right)^2,$$  \hspace{1cm} (22)

which does finally allow us to attribute to the so-called “non-classical” term, eq.(7), of our lagrangian (3) the simple meaning of kinetic energy of the internal (zbw) motion [i.e., of kinetic energy associated with the internal (zbw) velocity $V$], provided that

$$\hbar = 2s.$$  \hspace{1cm} (23)

In agreement with the already mentioned König theorem, such an internal kinetic energy does appear, in lagrangian (3), as correctly added to the (external) kinetic
energy $p^2/2m$ of the CM [besides to the total energy (9) and the external potential energy $U$].

Vice-versa, if we assume (within a zbw philosophy) that $V$, eq.(22), is the velocity attached to the kinetic energy term (7), then we can deduce eq.(23), i.e., we deduce that actually:

$$|s| = \frac{1}{2} \hbar.$$  

Let us mention, by the way, that in the stochastic approaches the (“non-classical”) stochastic, diffusion velocity is $V \equiv v_{\text{diff}} = \nu (\nabla \rho/\rho)$, quantity $\nu$ being the diffusion coefficient of the “quantum” medium. In those approaches, however, one has to postulate that $\nu \equiv \hbar/2m$. In our approach, on the contrary, if we just adopted for a moment the stochastic language, by comparison of our eqs.(7), (22) and (23) we would immediately deduce that $\nu = \hbar/2m$ and therefore the interesting relation

$$\nu = \frac{|s|}{m}. \quad (24)$$

Let us explicitly remark that, because of eq.(22), in the Madelung fluid equation (and therefore in the Schroedinger equation) quantity $\hbar$ is naturally replaced by $2|s|$, the presence itself of the former quantity being no longer needed; in a way, we might say that it is more appropriate to write $\hbar = 2|s|$, rather than $|s| = \hbar/2 \ldots$!

Let us conclude the first part of the present contribution by stressing the following. We first achieved a non-relativistic, Gordon-like decomposition of the field velocity within the ordinary tensorial language. Secondly, we derived the “quantum” potential (without the postulates and assumptions of stochastic quantum mechanics) by simply relating the “non-classical” energy term to zbw and spin. Such results carry further evidence that the quantum behaviour of micro-systems may be a direct consequence of the existence of spin. In fact, when $s = 0$, the quantum potential does vanish in the Hamilton–Jacobi equation, which then becomes a totally classical and newtonian equation. We have also seen that quantity $\hbar$ itself enters the Schroedinger equation owing to the presence of spin. We are easily induced to conjecture that no scalar quantum particles exist that are really elementary; but that scalar particles are always constituted by spinning objects endowed with zbw.

3. About the kinematics of spinning particles

In the first part of this paper, we addressed ourselves to spin, zbw and Madelung fluid in (non-relativistic) physics. The previous analysis led us, in particular, to fix our attention on the internal velocity $V$ of the spinning particle, besides on its external velocity $w = p/m$. In the second part of this article, we want to fix our attention on the total velocity $v = w + V$. It is now essential to allow $w$ assume any value, and therefore to pass to relativistic physics. In what follows our considerations will be essentially classical, while the quantum side of these last Sections will be studied in the next contribution to this Volume.[17]

Before going on, let us make a brief digression by recalling that, since the works by Compton,[8] Uhlenbeck and Goudsmith,[18] Frenkel,[18] and Schrödinger[9] till the present times, many classical theories —often quite different among themselves
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from a physical and formal point of view— have been advanced for spinning particles [for simplicity, we often write “spinning particle” or just “electron” instead of the more pertinent expression “spin-$\frac{1}{2}$ particle”]. Following Bunge,[19] they can be divided into three classes:

I) strictly point-like particle models

II) actual extended–type particle models (“spheres”, “tops”, “gyroscopes”, and so on)

III) mixed models for “extended–like” particles, in which the center of the point-like charge $Q$ results to be spatially distinct from the particle center-of-mass (CM).

Notice that in the theoretical approaches of type III—which, being in the middle between classes I and II, could answer the dilemma posed by Barut at the top of this paper— the motion of $Q$ does not coincide with the motion of the particle CM. This peculiar feature was found to be an actual characteristic[20-22,15,11,10] (just called, as we know, the zbw motion) of spinning particles kinematics. The type III models, therefore, are a priori convenient for describing zbw, spin and intrinsic magnetic moment of the electron, while these properties are hardly predicted by making recourse to the point-like–particle theories of class I. The theories of type III, moreover, are consistent[8-12] with the ordinary quantum theory of the electron: see below. The “extended-like” electron models of class III are at present after fashion also because of their possible generalizations to include supersymmetry and superstrings.[10b] At last, the “mixed” models help bypassing the obvious non-locality problems involved by a relativistic covariant picture for extended–type (in particular rigid) objects of class II. Quite differently, the extended–like (class III) electron is non-rigid and consequently variable in its “shape” and in its characteristic “size”, depending on the considered dynamical situation. This is a priori consistent with the appearance in the literature of many different “radii of the electron” [for instance, in his book,[23] McGregor lists at page 5 seven typical electron radii, from the Compton to the “classical” and to the “magnetic” radius]. Because of all these reasons, therefore, the spinning particle we shall have in mind in the next Section is to be described by class III theories.

We have here to rephrase some of the previous considerations in terms of Minkowsky (four-dimensional) vectors. For instance, let us recall again that in the ordinary Dirac theory the particle four-impulse $p^\mu$ is in general not parallel to the four-velocity: $v^\mu \neq p^\mu/m$. Before all, let us repeat that, in order to describe the zbw, in all type III theories it is very convenient[10-12] to split the motion variables as follows (the dot meaning now derivation with respect to the proper time $\tau$):

$$ x^\mu \equiv \xi^\mu + X^\mu ; \quad \dot{x}^\mu \equiv \dot{v}^\mu = w^\mu + V^\mu , $$

(25)

where $\xi^\mu$ and $w^\mu \equiv \dot{\xi}^\mu$ describe as before the external motion, i.e. the motion of the CM, whilst $X^\mu$ and $V^\mu \equiv \dot{X}^\mu$ describe the internal motion. From an electrodynamical point of view, as we know, the conserved electric current is associated with the trajectories of $Q$ (i.e., with $x^\mu$), whilst the center of the particle Coulomb field —obtained,[22] e.g., through a time average over the field generated by the quickly oscillating charge— is associated with the CM (i.e., with $w^\mu$; and then, for free particles, with the geometric center of the internal motion). In such a way, it
is $Q$ which follows the (total) motion, whilst the CM follows the mean motion only. It is worthwhile also to notice that the classical extended–like electron of type III is totally consistent with the standard Dirac theory; in fact, the above decomposition for the total motion is the classical analogue of two well-known quantum-mechanical procedures: i.e., of the Gordon decomposition of the Dirac current, and the (operatorial) decomposition of the Dirac position operator proposed by Schrödinger in his pioneering works.[9] We shall come back to these points below.

The well-known Gordon decomposition of the Dirac current reads[13] (hereafter we shall choose units such that numerically $c = 1$):

$$\bar{\psi}\gamma^\mu\psi = \frac{1}{2m} \left[ \bar{\psi}\hat{p}^\mu\psi - (\hat{p}^\mu\bar{\psi})\psi \right] - \frac{i}{m} \hat{p}_\nu \left( \bar{\psi}S^{\mu\nu}\psi \right), \quad (26)$$

$\bar{\psi}$ being the “adjoint” spinor of $\psi$; quantity $\hat{p}^\mu \equiv i\partial^\mu$ the 4-dimensional impulse operator; and $S^{\mu\nu} \equiv \frac{i}{4} (\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$ the spin-tensor operator. The ordinary interpretation of eq.(26) is in total analogy with the decomposition given in eq.(25). The first term in the r.h.s. results to be associated with the translational motion of the CM (scalar part of the current, corresponding to the traditional Klein–Gordon current). The second term in the r.h.s. results, instead, directly connected with the existence of spin, and describes the zw motion.

In the above-quoted papers, Schrödinger started from the Heisenberg equation for the time evolution of the acceleration operator in Dirac theory $[\dot{v} \equiv \alpha]$

$$a \equiv \frac{dv}{dt} = \frac{i}{\hbar} [H, v] = \frac{2i}{\hbar} (Hv - p), \quad (27)$$

where $H$ is equal as usual to $v\cdot p + \beta m$. Integrating once this operator equation over time, after some algebra one can obtain:

$$v = H^{-1}p - \frac{i}{2}\hbar H^{-1}a, \quad (28)$$

and, integrating it a second time, one obtains[14] just the spatial part of the decomposition:

$$x \equiv \xi + X \quad (29)$$

where (still in the operator formalism) it is

$$\xi = r + H^{-1}pt \quad (30)$$

related to the motion of the CM, and

$$X = \frac{i}{2}\hbar \eta H^{-1}, \quad (\eta \equiv v - H^{-1}p) \quad (31)$$

related to the zw motion.

4. New kinematical properties of the “extended–like” particles

We want now to analyze the formal and conceptual properties of a new definition for the 4-velocity of our extended–like electron. Such a new definition has been
first adopted —but without any emphasis— in the papers by Barut et al. dealing with a successful model for the relativistic classical electron.[10a,12] Let us consider the following. At variance with the procedures followed in the literature from Schrödinger’s till our days, we have to make recourse not to the proper time of the charge \( Q \), but rather to the proper time of the center-of-mass, i.e. to the time of the CMF.\(^\#1\) As a consequence, quantity \( \tau \) in the denominator of the 4-velocity definition, \( v^\mu \equiv \frac{dx^\mu}{d\tau} \), has to be the latter proper time. Up to now —with the exception of the above-mentioned papers by Barut et al.— in all theoretical frameworks the Lorentz factor has been assumed to be equal to \( \sqrt{1 - \frac{v^2}{c^2}} \). On the contrary, into the Lorentz factor it has to enter \( w^2 \) instead of \( v^2 \), quantity \( w \equiv \frac{p}{p^0} \) being the 3-velocity of the CM with respect to the chosen frame [\( p^0 \equiv E \) is the energy]. By adopting the correct Lorentz factor, all the formulae containing it are to be rewritten, and they get a new physical meaning. In particular, we shall show below that the new definition does actually imply\(^\#2\) the important constraint, which —holding identically for scalar particles— is often just assumed for spinning particles:

\[
p_\mu v^\mu = m,
\]

where \( m \) is the physical rest mass of the particle (and not an ad hoc mass-like quantity \( M \)).\(^\#3\)

Our choice of the proper time \( \tau \) may be supported by the following considerations:

(i) The light-like zbw —when the speed of \( Q \) is constant and equal to the speed of light in vacuum— is certainly the preferred one (among all the “a priori” possible internal motions) in the literature, and to many authors it appears the most adequate for a meaningful classical picture of the electron. In some special theoretical approaches, the light speed is even regarded as the quantum-mechanical typical speed for the zbw. In fact, the Heisenberg principle in the relativistic domain\([14]\) implies (not controllable) particle–antiparticle pair creations when the (CMF) observation involves space distances of the order of a Compton wavelength. So that \( \hbar/m \) is assumed to be the characteristic “orbital” radius and \( 2m/\hbar^2 \) the (CMF) angular frequency of the zbw —as first noticed by Schrödinger;— and the orbital motion of \( Q \) is expected to be light-like. Now, if the charge \( Q \) travels at the light speed, the proper time of \( Q \) does not exist; while the proper time of the CM (which travels at sub-luminal speeds) does exist. Adopting as time the proper time of \( Q \),

\(^\#1\) Let us recall once more that the CMF is the frame in which the kinetic impulse vanishes identically, \( \vec{p} = 0 \). For spinning particles, in general, it is not the “rest” frame, since the velocity \( \vec{v} \) is not necessarily zero in the CMF.

\(^\#2\) For all plane wave solutions \( \psi \) of the Dirac equation, we have (labelling by \( <> \) the corresponding local mean value or field density): \( p_\mu < \psi^\dagger \gamma^\mu \gamma^\nu \psi > \equiv p_\mu \psi^\dagger \gamma^\mu \psi \equiv p_\mu \vec{v} \gamma^\mu \psi = m \).

\(^\#3\) Let make just an example, recalling that Pašić\([10b]\) derived, from a lagrangian containing an extrinsic curvature, the classical equation of the motion for a rigid \( n \)-dimensional world-sheet in a curved background spacetime. Classical world-sheets describe membranes for \( n \geq 3 \), strings for \( n = 2 \), and point particles for \( n = 1 \). For the special case \( n = 1 \), he found nothing but the traditional Papapetrou equation for a classical spinning particle; also, by “quantization” of the classical theory, he actually derived the Dirac equation. In ref.\([10b]\), however, \( M \) is not the observed electron mass \( m \): and the relation between the two masses reads: \( m = M + \mu H^2 \), quantity \( \mu \) being the so-called string rigidity, while \( H \) is the second covariant derivative on the world-sheet.
as often done in the past literature, automatically excluded a light-like zbw. In our approach, by contrast, such zbw motions are not excluded. Analogous considerations may hold for Super-luminal zbw speeds, without too much problem, since the CM (which carries the energy-impulse and the “signal”) is always endowed with a subluminal motion;

(ii) The independence between the center-of-charge and the center-of-mass motion becomes evident by our definition. As a consequence the non-relativistic limit can be formulated by us in a correct, and univocal, way. Namely, by assuming the correct Lorentz factor, one can immediately see that the zitterbewegung can go on being a relativistic (in particular, light-like) motion even in the non-relativistic approximation: i.e., when $p \rightarrow 0$ (this is perhaps connected with the non-vanishing of spin in the non-relativistic limit). In fact, in the non-relativistic limit, we have to take

$$w^2 \ll 1,$$

and not necessarily

$$v^2 \ll 1$$

as usually assumed in the past literature;

(iii) The definition for the 4-velocity that we are going to propose [see eq.(33) in the following] does agree with the natural “classical limit” of the Dirac current. Actually, it was used in those models which (like Barut et al.’s) define velocity even at the classical level as the bilinear combination $\bar{\psi}\gamma^\mu\psi$, via a direct introduction of classical spinors $\psi$. By the new definition, we shall be able to write the translational term as $p^\mu/m$, with the physical mass in the denominator, exactly as in the Gordon decomposition, eq.(26). Quite differently, in all the theories adopting as time the proper time of $Q$, it appears in the denominator an ad-hoc variable mass $M$, which depends on the internal zbw speed $V$ (see below);

(iv) The choice of the CM proper time constitutes a natural extension of the ordinary procedure for relativistic scalar particles. In fact, for spinless particles in relativity the 4-velocity is known to be univocally defined as the derivative of 4-position with respect to the CMF proper time (which is the only one available in that case).

The most valuable reason in support of our definition turns out to be the circumstance that the old definition

$$v^\mu_{\text{old}} = (1/\sqrt{1-v^2}; \ v/\sqrt{1-v^2})$$

(32)

seems to entail a mass varying with the internal zbw speed.

But let us explicitate our new definition for $v^\mu$. The symbols which we are going to use possess the ordinary meaning; the novelty[24] is that now the Lorentz factor $d\tau/dt$ will not be equal to $\sqrt{1-v^2}$, but instead to $\sqrt{1-w^2}$. Thus we shall have:

$$v^\mu = dx^\mu/d\tau \equiv (dt/d\tau; \ dx/d\tau) \equiv (dt \ dt \ dt \ dt)$$

$$= (1/\sqrt{1-w^2}; \ v/\sqrt{1-w^2}). \quad [v \equiv dx/dt]$$

(33)
For \( w^\mu \) we can write:

\[
\begin{align*}
  w^\mu &\equiv \frac{d\xi^\mu}{d\tau} = \left( \frac{dt}{d\tau}; \frac{d\xi}{d\tau} \right) \\
  &= \left( 1/\sqrt{1-w^2}; \frac{w}{\sqrt{1-w^2}} \right); \quad [w \equiv d\xi/dt]
\end{align*}
\] (34)

and for the 4-impulse:

\[
p^\mu \equiv m w^\mu = m \left( 1/\sqrt{1-w^2}; \frac{w}{\sqrt{1-w^2}} \right).
\] (35)

[In presence of an external field such relations remain valid, provided that one makes the “minimal prescription”: \( p \rightarrow p - eA \) (in the CMF we shall have \( p - eA = 0 \) and consequently \( w = 0 \), as above).]

Let us now examine the resulting impulse–velocity scalar product, \( p_\mu v^\mu \), which has to be a Lorentz invariant, both with our \( v \) and with the old \( v_{\text{std}} \). Quantity \( p \equiv (p^0; \mathbf{p}) \) being the 4-impulse, and \( M_1, M_2 \) two relativistic invariants, we may write:

\[
p_\mu v^\mu \equiv M_1 \equiv \frac{p^0 - \mathbf{p} \cdot \mathbf{v}}{\sqrt{1-w^2}},
\] (36)

or, alternatively,

\[
p_\mu v_{\text{std}}^\mu \equiv M_2 \equiv \frac{p^0 - \mathbf{p} \cdot \mathbf{v}}{\sqrt{1-v^2}}.
\] (37)

If we refer ourselves to the CMF, we shall have \( p_{\text{CMF}} = w_{\text{CMF}} = 0 \) (but \( v_{\text{CMF}} \equiv V_{\text{CMF}} \neq 0 \)), and then

\[
M_1 = p^0_{\text{CMF}}
\] (38)

in the first case; and

\[
p^0_{\text{CMF}} = M_2 \sqrt{1-V^2_{\text{CMF}}}
\] (39)

in the second case. So, we see that the invariant \( M_1 \) is actually a constant, which —being nothing but the center-of-mass energy, \( p^0_{\text{CMF}} \)— can be identified, as we are going to prove, with the physical mass \( m \) of the particle. On the contrary, in the second case (the standard one), the center-of-mass energy results to be variable with the internal motion.

Now, from eq.(35) we have

\[
p_\mu v^\mu \equiv m w_\mu v^\mu
\]

and, because of eqs.(33,34),

\[
p_\mu v^\mu \equiv m (1 - w v)/(1-w^2).
\] (40)

Since \( w \) is a vector component of the total 3-velocity \( \mathbf{v} \), due to eqs.(25), and moreover is the orthogonal projection of \( \mathbf{v} \) along the \( p \)-direction, we can write

\[ w \cdot v = w^2, \]

which, introduced into eq.(40), yields[24] the important relation:

\[
m = p_\mu v^\mu.
\] (41)
Quite differently, by use of the wrong Lorentz factor, we would have got
\[ v^\mu = (1/\sqrt{1 - v^2}; \; v/\sqrt{1 - v^2}) \]
and consequently
\[ p_\mu v^\mu \equiv m(1 - wv)/\sqrt{(1 - w^2)(1 - v^2)} \]
\[ = m\sqrt{1 - w^2}/\sqrt{1 - v^2} \neq m \].

By recourse to the correct Lorentz factor, therefore, we succeeded in showing that the noticeable constraint
\[ m = p_\mu v^\mu \], trivially valid for scalar particles, does hold for spinning particles too. Such a relation, eq.(41), would be very useful also for a hamiltonian formulation of the electron theory.[12]

Finally, we want to show that the ordinary kinematical properties of the Lorentz invariant \( v^2 \equiv v_\mu v^\mu \) do not hold any longer in the case of spinning particles, endowed with zitterbewegung. In fact, it is easy to prove that the ordinary constraint for scalar relativistic particles —quantity \( v^2 \) constant in time and equal to 1— does not hold for spinning particles endowed with zbw. Namely, if we choose as reference frame the CMF, in which \( w = 0 \), we have [cf. definition (33)]:
\[ v^\mu_{\text{CMF}} \equiv (1; V_{\text{CMF}}), \] (42)
wherefrom, it being
\[ v^2_{\text{CMF}} \equiv 1 - V^2_{\text{CMF}}, \] (43)
one can deduce[24] the following new constraints:
\[ 0 < V^2_{\text{CMF}}(\tau) < 1 \iff 0 < v^2_{\text{CMF}}(\tau) < 1 \] ("time-like")
\[ V^2_{\text{CMF}}(\tau) = 1 \iff v^2_{\text{CMF}}(\tau) = 0 \] ("light-like") (44)
\[ V^2_{\text{CMF}}(\tau) > 1 \iff v^2_{\text{CMF}}(\tau) < 0 . \] ("space-like")

Since the square of the total 4-velocity is invariant and in particular it is \( v^2_{\text{CMF}} = v^2 \), these new constraints for \( v^2 \) will be valid in any frame:
\[ 0 < v^2(\tau) < 1 \] ("time-like")
\[ v^2(\tau) = 0 \] ("light-like") (45)
\[ v^2(\tau) < 0 . \] ("space-like")

Notice explicitly that the correct application of Special Relativity to a spinning particle led us, under our hypotheses, to obtain that \( v^2 = 0 \) in the light-like case, but \( v^2 \neq +1 \) in the time-like case and \( v^2 \neq -1 \) in the space-like case.

Let us now examine the manifestation and consequences of such new constraints in a specific example: namely, in the already mentioned theoretical model by Barut–Zanghi[10a] which did implicitly adopt as time the proper time of the CMF. In this case, we get that in general it is \( v^2 \neq 1 \). And in fact one obtains[12] the remarkable relation:
\[ v^2 = 1 - \frac{\ddot{v}_\mu v^\mu}{4m^2} . \] (46)
In particular, [22] in the light-like case it is $\ddot{v}_\mu v^\mu = 4m^2$ and therefore $v^2 = 0$.

Going back to eq. (43), notice that now quantity $v^2$ is no longer related to the external speed $|w|$ of the CM, but on the contrary to the internal zitterbewegung speed $|V_{CMF}|$. Notice at last that, in general — and at variance with the scalar case — the value of $v^2$ is not constant in time any longer, but varies with $\tau$ (except when $V_{CMF}^2$ itself is constant in time).

Acknowledgements

This work is dedicated to the memory of Asim O. Barut. The authors wish to acknowledge stimulating discussions with R.J.S. Chisholm, H.E. Hernández, J. Keller, Z. Oziewicz, W.A. Rodrigues and J. Vaz. For the kind cooperation, thanks are also due to G. Andronico, M. Baldo, A. Bonasera, M. Borrometi, A. Bugini, F. Catara, L. D’Amico, G. Dimartino, M. Di Toro, G. Giuffrida, C. Kihih, L. Lo Monaco, G. Marchesini, R.L. Monaco, E.C. Oliveira, M. Pignanelli, G.M. Prosperi, R.M. Salesi, M. Sambataro, S. Sambataro, M. Scivoletto, R. Sgarlata, R. Turrisi, M.T. Vasconcelos, J.R. Zeni, and particularly I. Aragón, C. Dipietro and J.P. dos Santos. One of the authors (ER) wishes to thank Prof. J. Keller and all the Organizers for generous hospitality during this International Conference; and W.A. Rodrigues and FAPESP for a research grant.

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