SINGULAR VALUE DECOMPOSITIONS IN FOCAL PLANE TOMOGRAPHY

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The formation of focused tomograms by a planar, stationary positron camera is described by a Fredholm integral equation of the first kind

\[ \hat{f}(x,y,z) = \iiint f(x',y',z') \ h(x',y',z';x,y,z) \ dx'dy'dz' \]  

(1)

where \( f(x',y',z) \) is the true positron activity distribution within the object, \( h(x,y,z; x',y',z') \) is the camera response function and \( \hat{f}(x,y,z) \) is the measured, blurred activity distribution produced by back projection of the positron annihilations. The application of a universal cone to the data renders the response function space-invariant and the integral in (1) reduces to a three-dimensional convolution integral:

\[ \hat{f}(x,y,z) = \iiint f(x',y',z') \ h(x-x',y-y',z-z') dx' dy' dz' \]  

(2)

Integral equations of the first kind are known to belong to the class of ill-posed problems; small changes in \( \hat{f}(x,y,z) \) e.g. statistical fluctuations, can result in large changes in the solution \( f(x,y,z) \), and the inversion of the integral operation in (1) or (2) has to be done in some generalised sense (i.e. a generalised inverse).

Applying the convolution theorem in two dimensions to equation (2) gives:

\[ \hat{F}_i(k_x, k_y) = \sum_{j=1}^{N} F_j(k_x, k_y) \ H_{ij}(k_x, k_y) \ , \ i = 1, \ldots, N \]  

(3)

where \( \hat{F}_i(k_x, k_y) \) represents the two-dimensional Fourier transform of \( \hat{f}(x,y,z) \) on the plane \( z = z_i \) and the convolution in the third dimension \( z \) has been written in discrete form. The set of N equations represented by (3) is solved at a given frequency \( k_x, k_y \) by inverting the \( N \times N \) matrix \( H_{ij}(k_x, k_y) \) to give:

\[ F_\lambda(k_x, k_y) = \sum_{i=1}^{N} \hat{F}_i(k_x, k_y) \ H_{\lambda i}^{-1}(k_x, k_y) \ , \ \lambda = 1, \ldots, N \]  

(4)

Instability with respect to perturbations appears as an ill-conditioning of \( H_{ij} \) at low spatial frequencies in \( k_x, k_y \). Some of the eigenvalues, \( \lambda_n \) of \( H_{ij} \) become small, indicating that in practice the
rank $r$ of $H_{ij}$ is less than $N$. A convenient generalised inverse of such matrices is provided by the singular value decomposition $^5$ $^6$:

$$H_{ij} = \sum_{n=1}^{N} \lambda_n v_{in} v_{jn}$$  \hspace{1cm} (5)

where $v_{in}$ is an eigenvector of $H_{ij}$ and $\lambda_n$ is the corresponding eigenvalue. In forming the inverse, only those eigenvalues $\lambda_n$ greater than a lower limit $\lambda_c$ are included in the sum:

$$H_{ij}^{-1} = \sum_{n=1}^{r} \lambda_n^{-1} v_{in} v_{jn}$$  \hspace{1cm} (6)

for the $r$ eigenvalues $\lambda_n > \lambda_c$. The superscript $I$ indicates a generalised inverse, such that if $r = N$, $H_{ij}^{-1} = H_{ij}^{-1}$, the true inverse. The condition number, $\kappa(H) = \lambda_{\text{max}} / \lambda_{\text{min}}$, is a useful indication of the conditioning of a matrix; large values usually indicate a poorly-conditioned matrix. The effect of the eigenvalue cut-off on $\kappa(H)$ is shown in fig. 1 as a function of frequency. The upper curve (1) is with no cut-off: condition numbers up to $10^{10}$ are observed; the lower curve (2) corresponds to $\lambda_c = 0.8$ and indicates a more stable situation with condition numbers less than 10. In practice, the cut-off may also be applied directly to the condition number (e.g. $\kappa < 10$).

The reconstruction algorithm represented by equations (4) and (6) was tested with simulated data to investigate the dependence of the reconstructed image quality upon eigenvalue cut-off, $\lambda_c$. The results are presented in Fig. 2 for a skull and brain with tumour simulation; values of $\lambda_c$ from 0 to 2 were used, with $\lambda_c \sim 0.8$ giving the best results. The progression from a noise-dominated image (a) to one processed by a modified low-pass filter (f) is evident.

The normal implementation of equation (4) is to precalculate and store $H_{ij}^{-1}(k_x,k_y)$ for all frequency pairs $(k_x,k_y)$ in order to minimise in-line calculation during reconstruction. This is particularly important if a generalised inversion is required. As a result of the space-invariance of the response function, $H_{ij}$ is a Toeplitz matrix $^7$, $H_{i-j}$ which may be extended by symmetry to a circulant form $^8$. The circulant form, although of dimension $(2N+1) \times (2N+1)$ has only $2N+1$ different elements since each row is formed by a right circular shift by one element of the previous row. In addition, although it is of larger dimension than the original Toeplitz form, a circulant is more easily inverted because it is diagonalised by the Fourier matrix, exp $\{2\pi i.j/N\}$.
with eigenvalues that are the Fourier transform of its first row. Thus, the singular value decomposition approach may be used to control noise amplification, while inversion is performed by a single Fourier transform. The eigenvalues of the circulant matrix may be either positive or negative, and the cut-off is applied to the absolute value.

The circulant form allows the flexibility of in-line calculation of the response function without appreciable increase in computer time and with minimal extra storage. Table 1 gives some typical timings, including those for the Chu and Tam method\(^1\) with a precalculated response function.

Fig. 3 shows the result of reconstructing the data from a mouse injected with 5.7 \(\mu\)Ci of \(^{18}\)F and imaged in vivo with a high resolution positron camera.\(^9\)\(^10\) A succession of planes at 10 mm intervals is shown; comparison of the reconstruction with the smoothed back-projection illustrates the improved contrast and the tomography is evident from the fact that different features (e.g. spine, kidneys, legs, etc.) are in focus on different planes.

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References


Figure Captions

1. Plot of matrix condition number $\log_{10}[\chi(H)]$ as a function of frequency. The curves are for the frequency values $k_x=k_y$ and the units are arbitrary. The upper curve (1) is for an eigenvalue limit $\lambda_c=0$, and the lower curve (2) for a limit $\lambda_c=0.8$.

2. The behaviour of the reconstruction as a function of eigenvalue cut-off, $\lambda_c$. The image is a simulation of a cylindrical skull containing two, slightly separated, hemispherical distributions ("brain"), one of which contains a small spherical "tumour" with a contrast ratio of 3:1 compared with the surrounding brain. The plane containing the tumour is shown for values of $\lambda_c$ of a) 0. b) 0.1 c) 0.8 d) 1.1 e) 1.5 f) 2.0.

3. Positron imaging in vivo of a mouse injected with 5.7 $\mu$Ci of $F^{18}$. A series of planes are shown at 10 mm intervals for both the simple back-projection and the reconstruction using the circulant form for $H_{ij}$ with an eigenvalue cut-off $\lambda_c$ of 1.0, and a 30% background subtraction. The back-projections are smoothed with a nine-point smoother before reconstruction.
<table>
<thead>
<tr>
<th>Convolution Theorem</th>
<th>Two Dimensions</th>
<th>Three Dimensions</th>
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</thead>
<tbody>
<tr>
<td>$H_{ij}$</td>
<td>Toeplitz</td>
<td>Circulant</td>
</tr>
<tr>
<td>Back Projection</td>
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<td>$\text{FFT}_1$</td>
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<td>Deconvolution</td>
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<td>3.0</td>
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<tr>
<td>$\text{FFT}_2$</td>
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<td>2.5</td>
</tr>
<tr>
<td>TOTAL</td>
<td>140.0 secs.</td>
<td>128.0 secs.</td>
</tr>
</tbody>
</table>

**TABLE I.**

Reconstruction times (in seconds) on an IBM 370/168 for a lattice of 64 x 64 x 16 and $10^6$ events. The method using the convolution theorem in three dimensions is that of Chu and Tam\(^1\), with the response function precalculated.
CONDITION NUMBER OF $\mathbf{H}$ AS A FUNCTION OF FREQUENCY

Fig. 1.