CERN-DATA HANDLING DIVISION

DD/75/8

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25 April 1975

CM-P00059672

BICUBIC SPLINE FUNCTION APPROXIMATION OF THE SOLUTION
OF THE FAST NEUTRON TRANSPORT EQUATION

(Submitted to Computer Physics Communications)

BICUBIC SPLINE FUNCTION APPROXIMATION OF THE SOLUTION
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ABSTRACT

The numerical method of approximation of the fast neutron stationary transport equation by means of bicubic cardinal splines is investigated in order to calculate the neutron flux in the one-dimensional plane geometry. A numerical example is given.

1. INTRODUCTION

Our purpose is to approximate the solution of the fast neutron stationary transport equation by means of bicubic spline functions, for a one-dimensional plane geometry, with one group of speeds.

Hence, we have to approximate the solution of the following integro-differential equation:

$$\mu \frac{\partial \Phi(x, \mu)}{\partial x} + \frac{\Phi(x, \mu)}{\lambda(x)} = c(x) \int_{-1}^{+1} K(\mu, \mu') \Phi(x, \mu') d\mu' + Q(x, \mu) . \quad (1)$$

The unknown function (i.e. the flux Φ) is defined in the domain $x \in [0, a]$, $\mu \in [-1, +1]$, where μ is the neutron flux cosine; x represents the penetration depth of the neutrons.

The neutron velocity value corresponds to the average energy of the fission neutrons.

The neutron source is such that $Q(x, \mu) = Q$ for $x = 0$, $\mu > 0$ and $Q(x, \mu) = 0$, elsewhere, particularly for $x = a$, $\mu < 0$. $\lambda(x)$ represents the mean free path of the neutrons in the medium. We suppose that this medium is constituted by k different homogeneous materials, i.e.

$$\lambda(x) = \sum_{i=1}^k \lambda_i \chi_i(x) ,$$

where

$$\lambda_i = \frac{1}{\sigma_{ti} N_i} \quad \text{and} \quad N_i = \frac{N^0 \rho_i}{A_i}$$

$$\chi_i(x) = \begin{cases} 1 & a_{i-1} \leq x < a_i \\ 0 & \text{elsewhere} . \end{cases}$$

ρ_i : density in $[a_{i-1}, a_i[$

A_i : atomic mass of the i^{th} component

N^0 : the Avogadro number

σ_{ti} : i^{th} component total microscopic cross-section

σ_{ai} : i^{th} component absorption cross-section

σ_{fi} : i^{th} component fission cross-section

σ_{cini} : i^{th} component inelastic collision cross-section

σ_{cei} : i^{th} component elastic collision cross-section.

If $K(\mu, \mu') = \frac{1}{2}$, we will use the transport approximation cross-section, otherwise we will use the differential cross-section expanded in Legendre polynomials.

$$c(x) = \sum_{i=1}^k (v_i \sigma_{fi} + \sigma_{cei} + \sigma_{cini}) N_i \chi_i(x) ,$$

where v_i is the average number of neutrons emitted by fission in $[a_{i-1}, a_i]$.

2. APPROXIMATION SCHEME

One of the most efficient methods used to solve Eq. (1) remains the classical Carlson's S_n method [1]. In a first step, the continuous variable μ is discretized in

$$-1 = \mu_1 < \mu_2 \dots < \mu_{n-1} < \mu_n = +1 ,$$

where $n = 2\ell + 1$. (The value $\mu = 0$ will always be one knot of the mesh.)

It is assumed that the unknown function Φ varies linearly in μ in the intervals $[\mu_{j-1}, \mu_j]$ thus created. This gives:

$$\Phi(x, \mu) = \frac{(\mu - \mu_{j-1}) \Phi(x, \mu_j) + (\mu_j - \mu) \Phi(x, \mu_{j-1})}{\mu_j - \mu_{j-1}} , \quad j = 2, \dots, n . \quad (2)$$

This method obviously invites one to try to get a higher accuracy using cubic variation in the intervals instead of the linear variation.

However, we know that Φ has discontinuities in $\mu = 0$ [2]. Then, in order to get a close representation of Φ , we will use bilinear splines at the boundaries and bicubic elsewhere, we will also take into account the singularities ($\mu = 0$, $x = 0$, a).

First, we want to present the bilinear approximation because of its simplicity and to point out that in the computation presented here the integral term of collision is not computed by iterations but approximated as well as the left-hand side of (1).

3. APPROXIMATION BY MEANS OF BILINEAR SPLINES

Let us define the following spline S [3]:

$$S(x, \mu) = \sum_{i,j=1}^{m,n} \Phi_{i,j} \mathcal{A}_{i,j}(x, \mu) , \quad (3)$$

where

$$\Phi_{i,j} = \Phi(x_i, \mu_j) , \quad i = 1, \dots, m; \quad j = 1, \dots, n ,$$

$$x_1 = 0 < x_2 \dots < x_m = a .$$

The cardinal spline defined on the mesh $R_{i,j}$ has the value

$$\mathcal{A}_{i,j}(x_\ell, \mu_m) = \delta_{i,\ell} \delta_{j,m},$$

where $\delta_{i,j}$ is the Kronecker symbol.

Replacing Φ by the spline S in Eq. (1) we will have an approximation of Φ due to the solution of a linear system whose unknowns are $\Phi_{i,j}$.

The bilinear spline has the explicit expression:

$$\begin{aligned} \Phi(x, \mu) = & \Phi_{i-1,j-1} \frac{(x_i - x)(\mu_j - \mu)}{h_i k_j} + \Phi_{i,j-1} \frac{(x - x_{i-1})(\mu_j - \mu)}{h_i k_j} + \\ & + \Phi_{i-1,j} \frac{(\mu - \mu_{j-1})(x_i - x)}{h_i k_j} + \Phi_{i,j} \frac{(x - x_{i-1})(\mu - \mu_{j-1})}{h_i k_j} \end{aligned} \quad (4)$$

for $x_{i-1} \leq x \leq x_i$ and $\mu_{j-1} \leq \mu \leq \mu_j$, where $h_i = x_i - x_{i-1}$ and $k_j = \mu_j - \mu_{j-1}$.

As an example we take the case:

$$K \equiv \frac{1}{2}.$$

When we then apply to S the operator H

$$H = \left\{ \mu \frac{\partial}{\partial x} + \frac{1}{\lambda} - \frac{c}{2} \int_{-1}^{+1} \right\},$$

we get at the knots ($i = 2, \dots, m$; $j = 1, \dots, n$; except $i = m, j = \ell + 1$):

$$\mu_j \frac{\Phi_{i,j} - \Phi_{i-1,j}}{h_i} + \frac{\Phi_{i,j}}{\lambda_i} = \frac{c_i}{2} \sum_{\xi=2}^n \frac{\Phi_{i,\xi} + \Phi_{i,\xi-1}}{2} k_\xi + q_{i,j}, \quad (5a)$$

where $q_{i,j} = Q(x_i, \mu_j)$. Note that this leads to an integration of the collision term as a trapezoidal integration.

The remaining equations will be given by the initial conditions.

$$\begin{aligned} \Phi_{1,j} &= Q, & j &= \ell + 2, \dots, n \\ \Phi_{m,j} &= 0, & j &= 1, \dots, \ell. \end{aligned} \quad (5b)$$

In order to take into account the two singular points $\mu = 0$, $x = 0$, and $x = a$, we proceed as follows:

$$\Phi(0, 0_-) = c(0) \lambda(0) \int_{-1}^{+1} \Phi(0, \mu') d\mu'$$

and

$$\Phi(0, 0_+) = Q.$$

We then define

$$\Phi_{1,\ell+1} = \frac{\Phi(0,0_-) + Q}{2} = \frac{c_1 \lambda_1}{4} \sum_{\xi=2}^n \frac{\Phi_{1,\xi} + \Phi_{1,\xi-1}}{2} k_{\xi} + \frac{Q}{2}, \quad (5c)$$

(if we want to have a correct integration of the collision term k_{ℓ} must be equal to $k_{\ell+1}$)

$$\Phi(a,0_-) = 0$$

and

$$\Phi(a,0_+) = c(a) \lambda(a) \int_{-1}^{+1} \Phi(a,\mu') d\mu'.$$

We then define

$$\Phi_{m,\ell+1} = \frac{\Phi(a,0_+)}{2} = \frac{c_m \lambda_m}{4} \sum_{\xi=2}^n \frac{\Phi_{m,\xi} + \Phi_{m,\xi-1}}{2} k_{\xi}. \quad (5d)$$

We number the unknowns of the linear system (5) in the following way:

$$\begin{aligned} \Phi_{1,1} &= b_1, \dots, \Phi_{1,n} = b_n, \\ \Phi_{2,1} &= b_{n+1}, \dots, \Phi_{i,j} = b_{(i-1)n+j}, \dots, \Phi_{m,n} = b_{nm}. \end{aligned} \quad (6)$$

Having multiplied Eqs. (5a)-(5d) by h_i , we got the elements of the matrix A associated with these equations. The initial conditions give:

$$\begin{aligned} a_{j,(m-1)n+j} &= 1 \quad \text{for } i = m; j = 1, \dots, \ell \\ a_{j,j} &= 1 \quad \text{for } i = 1; j = \ell + 2, \dots, n \end{aligned} \quad (7a)$$

$$a_{(i-1)n+j,(i-2)n+j} = -\mu_j$$

$$a_{(i-1)n+j,(i-1)n+\chi} = \left(\frac{h_i}{\lambda_i} + \mu_j \right) \delta_{\chi,j} - \frac{c_i h_i}{4} (k_{\chi} + k_{\chi+1}) \quad \text{if } \chi = 2, \dots, n-1$$

$$a_{(i-1)n+j,(i-1)n+\chi} = \left(\frac{h_i}{\lambda_i} + \mu_j \right) \delta_{\chi,j} - \frac{c_i}{4} h_i k_{\chi} \quad \text{if } \chi = 1, n$$

$$\text{for } i = 2, \dots, m; j = 1, \dots, n; \text{ except for } i = m, j = \ell + 1. \quad (7b)$$

The two singular points give

$$\begin{aligned} a_{\ell+1,\chi} &= 2 \frac{h_1}{\lambda_1} \delta_{\ell+1,\chi} - \frac{c_1 h_1}{4} (k_{\chi} + k_{\chi+1}) \quad \text{if } \chi = 2, \dots, n-1 \\ a_{\ell+1,\chi} &= -\frac{c_1 h_1}{4} k_{\chi} \quad \text{if } \chi = 1, n. \end{aligned} \quad (7c)$$

$$a_{(m-1)n+\ell+1, (m-1)n+\chi} = 2 \frac{h_m}{\lambda_m} \delta_{\ell+1, \chi} - \frac{c_m h_m}{4} (k_\chi + k_{\chi+1}) \quad \text{if } \chi = 2, \dots, n = 1 \quad (7d)$$

$$a_{(m-1)n+\ell+1, (m-1)n+\chi} = - \frac{c_m h_m}{4} k_\chi \quad \text{if } \chi = 1, m.$$

The initial conditions give

$$\begin{aligned} d_{(i-1)n+j} &= Q \quad \text{for } i = 1, j = \ell + 2, \dots, n \\ d_{(i-1)n+\ell+1} &= Q h_1 / \lambda_1 \\ d_{(i-1)n+j} &= 0 \quad \text{elsewhere.} \end{aligned} \quad (7e)$$

But the matrix formed has no diagonal elements for $i = 1, j = 1, \dots, \ell$. Thus we must make some pivoting between the rows to get an easily solvable system by Gauss elimination method, for instance

$$a_{(i-2)n+j, \chi} = a_{(i-1)n+j, \chi} \quad \text{for } i = 2, \dots, m$$

and

$$a_{(m-1)n+j, \chi} = a_{j, \chi} \quad \text{for } \chi = 1, \dots, m \cdot n \quad \text{and } j = 1, \dots, \ell \quad (7f)$$

since the matrix A looks like this (for $m = n = 3$).

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4. EXISTENCE OF THE SOLUTION OF THE LINEAR SYSTEM

We want to give sufficient conditions for the solution of the linear system (5) which we can now write as

$$Ab = d,$$

where the components of the matrix A and the vectors b and d are given by (6) and (7a)-(7f).

A sufficient condition for the matrix A to be non-singular is given by

$$\left| a_{\chi, \chi} \right| \geq \sum_{\substack{\ell=1 \\ \ell \neq \chi}}^{m \cdot n} \left| a_{\chi, \ell} \right|, \quad \chi = 1, 2, \dots, m \cdot n \quad (8)$$

which means that A is diagonally dominant [4].

In our case we have

$$i) \ a_{\chi, \chi} = \mu_j + \frac{h_i}{\lambda_i} - \frac{c_i h_i}{4} (k_j + k_{j+1})$$

for $\chi = (i-1)n + j$, where $j = 1, \dots, n$ ($\Leftrightarrow \mu_j \geq 0$)

and $i = 2, \dots, m$.

$$\sum_{\substack{\ell=1 \\ \ell \neq \chi}}^{n \cdot m} \left| a_{\chi, \ell} \right| = \left| -\mu_j \right| + \frac{c_i h_i}{4} \sum_{\substack{\xi=2 \\ \xi \neq j+1}}^n (k_\xi + k_{\xi-1}).$$

Then, the condition (8) gives

$$\frac{1}{\lambda_i} \geq c_i. \quad (9)$$

$$ii) \ a_{\chi, \chi} = -\mu_j \quad \text{for } \chi = (i-2)n + j,$$

where $j = 1, \dots, n$ ($\Leftrightarrow \mu_j \leq -k_{\ell+1}$) and $i = 2, \dots, m-1$

$$\sum_{\substack{\ell=1 \\ \ell \neq \chi}}^{n \cdot m} \left| a_{\chi, \ell} \right| = \frac{c_i h_i}{4} \sum_{\substack{\xi=2 \\ \xi \neq j+1}}^n (k_\xi + k_{\xi-1}) + \left| \mu_j + \frac{h_i}{\lambda_i} - \frac{c_i h_i}{4} (k_j + k_{j+1}) \right|.$$

There exist two possibilities to verify the condition (8)

$$\alpha) \quad \sum_{\substack{\ell=1 \\ \ell \neq \chi}}^{n \cdot m} \left| a_{\chi, \ell} \right| = -\mu_j + c_i h_i - \frac{h_i}{\lambda_i} \leq -\mu_j.$$

If the condition (9) is satisfied and if

$$-\mu_j - \frac{h_i}{\lambda_i} + \frac{c_i h_i}{4} (k_j + k_{j+1}) \geq 0$$

or

$$\beta) \quad \sum_{\substack{\ell=1 \\ \ell \neq \chi}}^{n \cdot m} \left| a_{\chi, \ell} \right| = \mu_j + c_i h_i + \frac{h_i}{\lambda_i} - \frac{c_i h_i}{2} (k_j + k_{j+1}) \leq -\mu_j.$$

If

$$\mu_j + \frac{h_i}{\lambda_i} - \frac{c_i h_i}{4} (k_j + k_{j+1}) \geq 0$$

then

$$\frac{\frac{1}{\lambda_i} - \frac{c_i (k_j + k_{j+1})}{4}}{\frac{1}{\lambda_i} - \frac{c_i (k_j + k_{j+1})}{4}} \leq h_i \leq \frac{\frac{1}{\lambda_i} - \frac{2\mu_j}{c_i \left(1 - \frac{k_j + k_{j+1}}{2}\right)}}{\frac{1}{\lambda_i} - \frac{2\mu_j}{c_i \left(1 - \frac{k_j + k_{j+1}}{2}\right)}} .$$

The range in which h_i can vary is different from zero if $c_i \leq 1/\lambda_i$ as can easily be verified. We can deduce from this that condition (8) implies

$$c_i \leq \frac{1}{\lambda_i} \quad \text{and} \quad h_i \leq \frac{\frac{1}{\lambda_i} - \frac{2\mu_j}{c_i \left(1 - \frac{k_j + k_{j+1}}{2}\right)}}{\frac{1}{\lambda_i} - \frac{2\mu_j}{c_i \left(1 - \frac{k_j + k_{j+1}}{2}\right)}} \quad (10)$$

For equal mesh sizes, the second inequality becomes

$$c_i h \leq \frac{2k}{1 - k + (1/\lambda_i c_i)} .$$

iii) The remaining cases are obviously diagonally dominant if the condition (9) is satisfied.

Hence the two inequalities (10) must be verified if we want the matrix A to be diagonally dominant. The first inequality corresponds to

$$\sigma_{a_i} \geq (v_i - 1) \sigma_{f_i} .$$

Then, if the neutron flux diverges in a given medium we cannot guarantee a well-conditioned linear system. It is obvious that the greater eigenvalue of a supercritical system will be greater than 1 [1].

5. CONSERVATION RELATIONS

It is well known that the convergence of the numerical process which solves the transport equation does not guarantee an exact conservation relation. On the contrary, a numerical method operating on the conservation relations is not generally convergent. This inconvenience recalled, we write the neutron flux conservation relation verified by the solution Φ .

Integrating the transport equation with respect to μ , we get

$$\frac{dJ(x)}{dx} + \Phi(x) \left(\frac{1}{\lambda(x)} - c(x) \right) = Q(x) .$$

Integrating once more:

$$J(x_i) - J(x_{i-1}) = \int_{x_{i-1}}^{x_i} \left(c_i - \frac{1}{\lambda_i} \right) \Phi(x) dx + \int_{x_{i-1}}^{x_i} Q(x) dx, \quad (11)$$

where the current is

$$J(x) = \int_{-1}^{+1} \mu' \Phi(x, \mu') d\mu'$$

and the integrated flux is

$$\Phi(x) = \int_{-1}^{+1} \Phi(x, \mu') d\mu'$$

and

$$Q(x) = \int_{-1}^{+1} q(x, \mu') d\mu'.$$

Replacing Φ by the bilinear spline we get for (11) the following relation

$$\begin{aligned} \sum_{j=2}^n \left\{ (\Phi_{i,j} - \Phi_{i-1,j}) (2\mu_j + \mu_{j-1}) + (\Phi_{i,j-1} - \Phi_{i-1,j-1}) (\mu_j + 2\mu_{j-1}) \right\} \frac{k_j}{3!} = \\ = \left(c_i - \frac{1}{\lambda_i} \right) \sum_{j=2}^n (\Phi_{i-1,j} + \Phi_{i,j} + \Phi_{i-1,j-1} + \Phi_{i,j-1}) \frac{h_i k_j}{4} + Q_i \end{aligned} \quad (12)$$

for $i = 2, \dots, m$.

6. BICUBIC APPROXIMATION OF THE TRANSPORT EQUATION

6.1 The matrix A

In the same way as in the bilinear case, we get after the replacement of Φ in Eq. (1) by the bicubic spline S [5] the system of equations:

$$\begin{aligned} \frac{\Phi_{i,j}}{\lambda_i} + \mu_j \left[\frac{h_i}{2} M_{i,j} + \frac{(\Phi_{i,j} - \Phi_{i-1,j})}{h_i} - \frac{(M_{i,j} - M_{i-1,j})}{h_i} \frac{h_i^2}{6} \right] = \\ = \frac{c_i}{2} \sum_{\xi=2}^n \frac{k_\xi}{2} (\Phi_{i,\xi-1} + \Phi_{i,\xi}) - \frac{k_\xi^3}{4!} (N_{i,\xi} + N_{i,\xi-1}) + q_{i,j} \end{aligned} \quad (13)$$

for $i = 2, \dots, m; j = 1, \dots, n$ (except $i = m, j = n + 1$).

The $M_{i,j}$, $N_{i,j}$ represent the "moments" of the spline. In this form, we point out that the integration of the collision term is equivalent to a Simpson's quadrature. The accuracy of this rule is sufficient since the cross-sections are not known with a very high precision.

The moments are determined by means of continuity relations at the limits of a mesh:

$$\begin{aligned} \frac{\Phi_{i+1,j} - \Phi_{i,j}}{h_{i+1}} - M_{i,j} \frac{h_{i+1}}{2} - \frac{(M_{i+1,j} - M_{i,j})}{h_{i+1}} \frac{h_{i+1}^2}{6} = \\ = \frac{\Phi_{i,j} - \Phi_{i-1,j}}{h_i} + M_{i,j} \frac{h_i}{2} - \frac{(M_{i,j} - M_{i-1,j})}{h_i} \frac{h_i^2}{6} \end{aligned} \quad (14)$$

for $i = 2, \dots, m-1; j = 1, \dots, n$.

Equivalent relations may be found for the $N_{i,j}$ "moments". The remaining equations are given by the initial conditions. Special care is taken for the two singular points $i = 1, m; j = \ell + 1$, in the same way as in the preceding section.

The values are assigned to $M_{i,j}$ and $N_{i,j}$ in order to get a parabolic or linear approximation at the boundaries; for instance, $j = 1, \dots, n$.

parabolic $M_{1,j} = M_{2,j}$ (or $M_{1,j} = 0$ linear)

parabolic $M_{m,j} = M_{m-1,j}$ (or $M_{m,j} = 0$ linear)

6.2 The conservation relations

The conservation relations (11) will be written in the form:

$$\begin{aligned} J(x_i) - J(x_{i-1}) = \left(c_i - \frac{1}{\lambda_i}\right) \sum_{j=2}^n (\Phi_{i,j} + \Phi_{i-1,j} + \Phi_{i,j-1} + \Phi_{i-1,j-1}) \frac{h_i k_j}{4} - \\ - (M_{i,j} + M_{i-1,j} + M_{i,j-1} + M_{i-1,j-1}) \frac{h_i^3 k_j}{2 \times 4!} - \\ - (N_{i,j} + N_{i-1,j} + N_{i,j-1} + N_{i-1,j-1}) \frac{h_i k_j^3}{2 \times 4!} + \\ + (g_{i,j} + g_{i-1,j} + g_{i-1,j-1} + g_{i,j-1}) \frac{h_i^3 k_j^3}{(4!)^2} + Q_i \end{aligned} \quad (15)$$

for $i = 2, \dots, m$,

where

$$\begin{aligned} J(x_i) = \sum_{j=2}^n \left[\Phi_{i,j} (2\mu_j + \mu_{j-1}) + \Phi_{i,j-1} (\mu + 2\mu_{j-1}) \right] \frac{k_j}{3!} - \\ - \left[N_{i,j} (8\mu_j + 7\mu_{j-1}) + N_{i,j-1} (7\mu_j + 8\mu_{j-1}) \right] \frac{k_j^3}{360}. \end{aligned}$$

The $g_{i,j}$ are given by a linear system expressing the continuity of the cross derivatives. The boundary conditions are

$$g_{1,j} = g_{2,j} ; \quad g_{m,j} = g_{m-1,j} \quad \text{for } j = 1, \dots, n$$

$$g_{i,1} = g_{i,2} ; \quad g_{i,n} = g_{i,n-1} \quad \text{for } i = 1, \dots, m$$

and

$$\begin{aligned} & \frac{(\phi_{i+1,j+1} - \phi_{i,j+1} + \phi_{i+1,j} - \phi_{i,j})}{h_{i+1}k_{j+1}} + \\ & + \frac{h_{i+1}}{6k_{j+1}} \left[2(M_{i,j} - M_{i,j+1}) + (M_{i+1,j} - M_{i+1,j+1}) \right] + \\ & + \frac{k_{j+1}}{6h_{i+1}} \left[2(N_{i,j} - N_{i+1,j}) + (N_{i,j+1} - N_{i+1,j+1}) \right] + \\ & + \frac{k_{j+1}h_{i+1}}{36} \left[4g_{i,j} + 2(g_{i+1,j} + g_{i,j+1}) + g_{i+1,j+1} \right] = \\ & = \frac{(\phi_{i,j-1} - \phi_{i-1,j-1} + \phi_{i,j} - \phi_{i-1,j})}{h_i k_j} + \tag{16} \\ & + \frac{k_j}{6h_i} \left[2(N_{i-1,j-1} - N_{i,j-1}) + (N_{i-1,j} - N_{i,j}) \right] + \\ & + \frac{h_i}{6k_j} \left[2(M_{i-1,j-1} - M_{i-1,j}) + (M_{i,j-1} - M_{i,j}) \right] + \\ & + \frac{h_i k_j}{36} \left[4g_{i-1,j-1} + 2(g_{i,j-1} + g_{i-1,j}) + g_{i,j} \right], \\ & \text{for } i = 2, \dots, m-1; j = 2, \dots, n-1. \end{aligned}$$

7. NUMERICAL EXAMPLE

Let us study the fast neutron flux in U_{235} . The source is

$$Q(0,\mu) = 1 \quad \text{for } \mu > 0.$$

The numerical data are:

$$\sigma_t = 4.86 \text{ barns (transport approximation)}$$

$$\sigma_f = 1.31 \text{ barns}$$

$$\sigma_c = 3.492 \text{ barns}$$

$$\nu = 2.67 \text{ neutrons ,}$$

i.e. for monoenergetic neutrons of 1.35 MeV and $N = 0.048 \times 10^{24} \text{ n/cm}^3$, $a = 3 \text{ cm}$.

Using the bicubic approximation, we obtain the results noted in Figs. 1 and 2. with 9 directions and a step $h = 0.5$ cm and 11 directions and a step $h = 0.75$ cm .

We are clearly in an unfavourable case, since

$$\sigma_c + v\sigma_f > \sigma_t .$$

In spite of this, as noted in Fig. 3, the results given for the integrated flux $\Phi(x)$ and the current $J(x)$, closely agree for the two different mesh sizes. Moreover, in Fig. 4, the absolute errors on the two conservation relations are presented and seem to lead to a sufficient accuracy.

8. CONCLUSION

A method has been presented to approximate the fast neutron transport equation in the stationary case and in the one-dimensional plane geometry. This method uses bicubic spline functions to represent the neutron flux. No use of iterative methods has been made in the computation of the collision term; it has been approximated as well as the right-hand side of the transport equation.

It has been pointed out that in the case of the bilinear spline approximation, the existence of the solution of the linear system resulting from the discretization of the transport equation is related to two inequalities on the cross-sections and on the mesh sizes.

In the numerical examples provided, one can see that the conservation relations resulting from the integration of the transport equation are sufficiently accurate even for a supercritical medium.

Acknowledgement

I am deeply indebted to Dr. Benno Schorr for reading the manuscript and making helpful comments.

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Figure captions

- Fig. 1 : Neutron flux for 9 directions and a step $h = 0.5$ cm.
- Fig. 2 : Neutron flux for 11 directions and a step $h = 0.75$ cm.
- Fig. 3 : Integrated flux and current.
- Fig. 4 : Conservation relations.

Fig. 1

Fig. 2

Fig. 3

Fig. 4