Four Quarks from Lattice to the Continuum

Petrus Pennanen

A continuum extrapolation of static four- and two-quark energies calculated in quenched SU(2) is done based on Sommer’s method of setting the scale. A model for four-quark energies with explicit gluonic degrees of freedom removed is fitted to these energies and the behavior of the parameters of the model is investigated.

1. Introduction

Our group is attempting to understand hadronic interactions from first principles by developing a potential model for multi-quark systems with explicit gluonic degrees of freedom removed ([1] and references therein). Such a model would shed light on multi-quark bound states and e.g. meson-meson scattering.

We have been simulating systems of four static quarks in quenched SU(2) at \( \beta = 2.4 \) with a \( 16^3 \times 32 \) lattice and the Wilson action. Various geometries such as rectangular, linear and tetrahedral have been considered to get a set representing the general case.

To find the lattice artifacts in energies and their parameterization, I have performed continuum extrapolations based on Sommer’s method of setting the scale [2] at \( \beta = 2.35, 2.4 (16^3 \times 32), 2.45 (20^3 \times 32), 2.5 (24^3 \times 32) \) and \( 2.55 (26^3 \times 32) \) for squares and tilted rectangles, since these geometries exhibit the largest binding energies.

2. A Model for Four-Quark Energies

The model is based on static two-quark potentials for different pairings of the four-quarks. In the case of two pairings or basis states A,B, the eigenenergies \( \lambda_i \) are obtained by diagonalising

\[
[V - \lambda_i N] \Psi_i = 0, \quad (1)
\]

with

\[
N = \begin{pmatrix}
1 & f/N_c \\
 f/N_c & 1
\end{pmatrix} \quad \text{and} \quad (2)
\]

\[
V = \begin{pmatrix}
v_{13} + v_{24} & \frac{f}{N_c} V_{AB} \\
\frac{f}{N_c} V_{BA} & v_{14} + v_{23}
\end{pmatrix},
\]

where \( v_{ij} \) represents the static two-body potential between quarks \( i \) and \( j \) and \( V_{AB} \) is from the perturbative expression \( V_{ij} = -N(N_c) T_i \cdot T_j v_{ij} \).

The normalization is chosen to give for a color singlet state \( [ij]^0 <[ij]^0|V_{ij}|[ij]^0> = v_{ij} \). The four-quark binding energies \( E_i \) are obtained by subtracting the internal energy of the basis state with the lowest energy, e.g. \( E_i = \lambda_i - (v_{13} + v_{24}) \).

A central element in the model is the phenomenological factor \( f \) appearing in the overlap of the basis states \( \langle A|B \rangle = f/N_c \) for \( SU(N_c) \).

- \( f \) is a function of the coordinates of all quarks, making the \( \frac{f}{N_c} V_{AB} \) in eq. 3 a four-body potential.
- in the weak coupling limit \( f = 1 \)
- a parameterization \( f = f_c e^{-k_A A - k_P P} b_S \), where \( A, P \) are the minimal area and perimeter, \( b_S \) the string tension and \( f_c, k_A, k_P \) are to be fitted

Perturbative calculation to \( O(\alpha^2) \) produces the two-state model of eq. 1 with \( f = 1 \) [3].

The above model using only the ground state of the two-body potential works for rectangles but fails for tetrahedrons.
3. Setting the Scale

Sommer [2] has designed a popular new way to set the scale that avoids the long-distance limit in the definition of the string tension \( b_S = \lim_{r \to \infty} F(r) \).

After calculating the force \( F(r/a) \) between two static quarks, the equation

\[
(r_0/a)^2 F(r_0/a) = c
\]

is solved with \( c = 1.65 \) for \( r_0/a \). Equating to the continuum value \( r_0 \approx 0.5 \) fm gives \( a \).

Choosing \( c = 1.65 \) produces agreement on \( r_0 = 0.49 \) fm for the Cornell [4] and Richardson [5] non-relativistic effective potentials, while other models [6,7] give \( r_0 \) from 0.44 to 0.56 fm. However, setting \( c = 2.44 \) makes three potential models agree on \( r_0 = 0.66 \) fm while others give 0.625 or 0.64 fm.

Runs were performed for square and tilted rectangle (TR) geometries, with resulting scales shown in table 1. Here \( a \) is calculated using \( c = 1.65 \), \( a^{II} \) using \( c = 2.44 \) and \( a^{bs} \) from the string tension using a continuum value \( \sqrt{b_S} = 0.44 \) GeV.

We can see that at each \( \beta \), \( a^{bs}/a \approx 1.09 \), and becomes \( \approx 1 \) if we set \( \sqrt{b_S} = 0.478(4) \) GeV. Meanwhile \( a^{II} \) needs no such adjustment to agree with \( a^{bs} \).

So it seems \( c = 2.44 \) is a better choice than \( c = 1.65 \). I chose the former value, keeping in mind that the difference can be accidental due to the significant uncertainties in the nonrelativistic potential models and the difference of the quenched SU(2) string tension from a phenomenological value.

4. Extrapolating Two-Body Potentials...

To extrapolate, we need values of energies at different \( \beta \)'s corresponding to the same physical size. Sample two-body extrapolations (from a total of 29 potentials) are shown in fig. 1 for potentials with separation from 0.17 to 0.52 fm involved in squares. Other potentials look similar. A quadratic fit is clearly preferred.

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>( a ) [fm]</th>
<th>( a^{II} ) [fm]</th>
<th>( a^{bs} ) [fm]</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.35</td>
<td>0.1302(6)</td>
<td>0.1401(14)</td>
<td>0.1408(11)</td>
</tr>
<tr>
<td>2.35(TR)</td>
<td>0.1299(7)</td>
<td>0.1397(6)</td>
<td>0.1402(5)</td>
</tr>
<tr>
<td>2.4</td>
<td>0.1101(10)</td>
<td>0.1186(10)</td>
<td>0.1194(9)</td>
</tr>
<tr>
<td>2.4(TR)</td>
<td>0.1103(5)</td>
<td>0.1190(4)</td>
<td>0.1200(11)</td>
</tr>
<tr>
<td>2.45</td>
<td>0.0918(7)</td>
<td>0.0997(8)</td>
<td>0.0997(7)</td>
</tr>
<tr>
<td>2.45(TR)</td>
<td>0.0929(7)</td>
<td>0.1002(7)</td>
<td>0.1010(6)</td>
</tr>
<tr>
<td>2.5</td>
<td>0.0793(5)</td>
<td>0.0856(9)</td>
<td>0.0866(9)</td>
</tr>
<tr>
<td>2.5(TR)</td>
<td>0.0789(5)</td>
<td>0.0851(6)</td>
<td>0.0859(6)</td>
</tr>
<tr>
<td>2.5(W)</td>
<td>0.0793(3)</td>
<td>0.0856(6)</td>
<td>0.0864(5)</td>
</tr>
<tr>
<td>2.55</td>
<td>0.0675(5)</td>
<td>0.0730(5)</td>
<td>0.0741(7)</td>
</tr>
<tr>
<td>2.55(TR)</td>
<td>0.0673(9)</td>
<td>0.0727(10)</td>
<td>0.0734(8)</td>
</tr>
</tbody>
</table>

Table 1
Values of \( a \) for each \( \beta \). (W) is for Wuppertal.

5. ...and Four-Body Binding Energies

Binding energies for the simulated geometries can be interpolated using the \( f \)-model. I used the parameterization \( f = f_c e^{-k a^{bs} A} \), as the perimeter term was not needed. Results for squares with the length of a side from 0.17 to 0.52 fm are shown in fig. 2. Results for tilted rectangles look similar. The weaker dependence on \( (r_0/a)^2 \) shows that there are significantly less lattice artifacts than for the two-body potentials.

6. Extrapolating the Parameters of the Model

Values of the parameters were obtained by fitting \( f = f_c e^{-k a^{bs} A} \) to the energies of squares and tilted rectangles together at each \( \beta \). The normalization \( f_c \) and the constant multiplying the area, \( k_A \), are extrapolated in fig. 3.

The parameters are changing quite a lot, while \( f \) itself is more constant at different \( \beta \)'s. This can be taken as a sign of a poor parameterization.

7. Continuum Fit of the Model

After extrapolating the two- and four-body energies the \( f \)-model was fitted to the continuum values. Both linearly and quadratically extrapolated
energies were fitted. Results of the most relevant fits and parameter extrapolations are shown in table 2.

<table>
<thead>
<tr>
<th>type</th>
<th>$f_c$</th>
<th>$k_A$</th>
<th>$\chi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>lin/quad</td>
<td>1.04(4)</td>
<td>1.04(6)</td>
<td>0.7</td>
</tr>
<tr>
<td>lin/quad, $f_c = 1$</td>
<td>1</td>
<td>0.98(3)</td>
<td>0.8</td>
</tr>
<tr>
<td>quadratic</td>
<td>1.08(8)</td>
<td>1.37(12)</td>
<td>0.4</td>
</tr>
<tr>
<td>quadratic, $f_c = 1$</td>
<td>1</td>
<td>1.26(6)</td>
<td>0.5</td>
</tr>
<tr>
<td>linear extrap.</td>
<td>1.04(3)</td>
<td>0.83(6)</td>
<td>2.6/3.3</td>
</tr>
<tr>
<td>quadratic extrap.</td>
<td>1.10(5)</td>
<td>1.09(11)</td>
<td>2.7/0.4</td>
</tr>
</tbody>
</table>

Table 2
Continuum fit and parameter extrapolation results. Fit type “lin/quad” denotes linear extrapolations for four-body energies and quadratic for two-body potentials.

Our best estimate for $k_A$ is slightly above one, while $f_c$ can be safely set to one.

In the strong coupling limit a factor $\exp(-bS_A)$, similar to $f$, appears in the off-diagonal elements of the Wilson loop matrix, $A$ being the minimal surface bounded by straight lines connecting the quarks. It has been argued that weaker couplings lead to a smaller transition area [8]. If this is the case, it is not reflected in our best estimate of $k_A$. The larger transition area pointed to by $k_A > 1$ could be explained by the finite width of flux tubes as compared to the lines in the strong coupling approximation.

Continuum fits were also performed for squares extrapolated using the original value $c = 1.65$ in equation 3. The resulting values of $f_c$ and $k_A$ for fits to all linear and quadratic data were within errors with fits to extrapolations using $c = 2.44$.

8. How to develop the model?

The failure of the simplest form of the $f$-model to predict the binding energies of the tetrahedral geometry has been proposed to be due to a dependence of the four-body ground state energy $E_4$ on the excited state of the two-body potential. This was investigated using three fuzzing levels, all with $c = 4$, producing different ground- and excited state overlaps of the two-body paths.
fuzz level 150: overlaps 97.9-98.7%, intermediate convergence of $E_4$

- fuzz level 20: overlaps 99.1-100%, best convergence of $E_4$

- fuzz level 0: overlaps 85.3-98.2%, worst convergence of $E_4$

Thus the excited state contribution to $E_4$ comes from the overlap of the ground state of a gluon field between two quarks with the excited state of a gluon field of another quark pair. In ref. [9] this is investigated further and a generalization of the model is introduced that is capable of reproducing the degenerate ground state energy of a regular tetrahedron.

It is evident that using a high fuzzing level 150 (as in Wuppertal [10]) does not necessarily lead to better ground state overlaps on a 24³ spatial lattice.

The effects of instantons are another candidate for a missing feature in our model. Instanton liquid models include neither perturbative one-gluon exchange (Coulomb part of the two-quark potential) nor confinement (linear part of said potential). They still predict the nucleon to be bound with the right mass [11]. Our model gets the energy scale only from the two-body potential and the overlap factor $f$ can hardly account for instanton effects.

Our next project is an attempt to uncover the microscopic origins of the simulation results by relating the model to flux distributions via energy sum-rules similar to those derived by C. Michael [12].

9. Acknowledgments

I warmly thank A.M. Green for support and encouragement, J. Lukkarinen and P. Laurikainen for discussions and C. Schlichter from Wuppertal for giving us their simulation results. Funding from the Finnish academy is gratefully acknowledged. Our simulations were performed at the Center for Scientific Computing in Helsinki.

REFERENCES