String Theory and the CPT Theorem
on the World-Sheet

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ABSTRACT

We study the CPT theorem for a two-dimensional conformal field theory on an arbitrary Riemann surface. On the sphere the theorem follows from the assumption that the correlation functions have standard hermiticity properties and are invariant under the transformation $z \rightarrow 1/z$. The theorem can then be extended to higher genus surfaces by sewing. We show that, as a consequence of the CPT theorem on the world-sheet, the scattering $T$-matrix in string theory is formally hermitean at any loop order.

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Introduction and Summary

The CPT theorem is a very fundamental result which holds in any quantum field theory under very mild assumptions, notably Lorentz invariance and locality [1,2,3,4]. In this paper we consider the role played by the CPT theorem in the framework of two-dimensional conformal field theory (CFT) [5].

Our main interest is in those CFTs that enter into the construction of string theory models; accordingly, we would like to study two-dimensional CFT on a compact Riemann surface of any genus, rather than in flat two-dimensional Minkowski space-time. This means that we loose Lorentz invariance, one of the main pillars of the ordinary CPT theorem, from the very outset. Nevertheless, as we shall see, this is more than compensated for by the presence of the much more powerful conformal symmetry. Indeed, the twin assumptions of conformal invariance (more precisely, invariance under the transformation $z \to 1/z$) and hermiticity (to be formulated more precisely in section 2) leads to the identity between correlation functions on the sphere which is usually referred to as CPT invariance of two-dimensional CFT [6,7].

By using the sewing procedure of Sonoda [8] one may then try to extend the concept of world-sheet CPT (WS-CPT) invariance from the sphere to higher genus Riemann surfaces, where the global transformation $z \to 1/z$ ceases to be well-defined.

We start by considering local CFTs, i.e. theories where the difference between left- and right-moving conformal dimensions is always integer. If one further assumes modular invariance of the correlation functions defined by the sewing (plus a very mild technical requirement specified in section 3) the WS-CPT theorem can be extended to higher genus Riemann surfaces. At genus $g \geq 1$, WS-CPT invariance relates correlation functions on different Riemann surfaces, $M$ and $\tilde{M}$, that can be thought of as “mirror images” of one another.

Although the full CFT describing a consistent string theory is always local, this is not necessarily true of the various building blocks involved in the construction of the string model; important examples are free complex fermions, free chiral bosons, and the superghosts. We therefore proceed to consider the modifications needed in the
formulation of WS-CPT invariance in these specific examples. In particular, we show how ordinary WS-CPT invariance, in the sense of local, modular invariant CFT, is recovered when we put the various “constituent” CFTs together in a consistent string model and sum over the spin structures.

It is also possible to define a WS-CPT transformation acting on individual operators. Unlike the ordinary CPT transformation in Minkowski space-time, the WS-CPT transformation inverts the order of operators and therefore cannot itself be generated by any operator. In the context of string theory, the WS-CPT transformation maps the vertex operator of an incoming physical string state into the vertex operator describing the same physical string state, but outgoing, times a certain phase factor [9,10].

The CFTs we consider may, of course, be super-conformal, since any superconformal field theory is also conformal. In the case of superconformal field theories the natural aim would be to define WS-CPT on arbitrary super Riemann surfaces. Instead, we consider only ordinary Riemann surfaces, the reason being that we are mainly interested in string theory, where we always imagine the supermoduli to be integrated out. As is well known [11], this gives rise to scattering amplitudes expressed as integrals over the moduli space of ordinary Riemann surfaces, where the integrand is given by a correlation function of vertex operators with an appropriate number of Picture Changing Operators (PCOs) inserted.

When applied to the CFT underlying a given string theory, WS-CPT invariance implies that the scattering \( T \)-matrix is hermitean at tree-level away from the momentum poles, as required by unitarity. More generally, we show that the \( T \)-matrix is formally hermitean at any loop order, meaning that it is hermitean to the extent that the integral over the moduli is convergent. Thus, if we believe in unitarity of the scattering \( S \)-matrix in string theory, our result can be interpreted as an indirect demonstration that the modular integral can not be convergent in those kinematical regions where the \( T \)-matrix is required to develop an imaginary part. Indeed, unitarity becomes a consistency condition that should be imposed on whatever regularization is introduced to handle the divergences. Several regularization procedures (of varying generality) have already appeared in the literature [12,13,14,15,16].

The concept of WS-CPT invariance in two-dimensional CFT was first introduced by Moore and Seiberg as one of the fundamental assumptions underlying their axiomatic discussion of CFT [6].

Sonoda [7] extended the property of WS-CPT invariance for local and unitary CFTs to higher genus Riemann surfaces by means of sewing, and he also noticed the
connection between WS-CPT invariance and the hermiticity of the (dispersive part of
the) $T$-matrix in the context of string theory.

Compared to these authors, our point of view is somewhat different: WS-CPT
invariance on the sphere is not considered to be an axiom; instead, it is seen as a con-
sequence of the twin assumptions of conformal invariance and hermiticity. We do not
restrict ourselves to unitary CFTs; all we assume is that we have an inner product
defined on the Hilbert space, and hence a concept of hermitean conjugation of all op-
erators, such that the energy-momentum tensor of the CFT is hermitean. The inner
product is not assumed to be positive definite — states may have zero or negative norm.
Therefore we include such important non-unitary theories as the time-like component
of the space-time coordinate in string theory, and the reparametrization ghosts.

Another new feature of our paper is the discussion of world-sheet CPT for free
non-local CFTs such as complex fermions and superghosts.

The paper is organized as follows: In section 1 we briefly review the CPT theorem
for quantum field theories in $D$-dimensional Minkowski space-time. Then, in section
2, we discuss the CPT theorem for CFTs on the sphere, including the question of
hermiticity. In section 3 we extend the CPT theorem to higher genus Riemann surfaces
by means of the sewing technique, first for local CFTs, then for some important examples
of free non-local CFTs. Finally, in section 4 we apply the previous results to the CFT
underlying any given first-quantized string theory and we show that the CPT theorem
on the world-sheet implies the formal hermiticity of the $T$-matrix amplitudes to all
orders in string perturbation theory. We also provide an Appendix on mirror image
Riemann surfaces.

1. **CPT theorem in $D$ dimensional Minkowski space-time.**

In this section we briefly review the CPT theorem and CPT transformations in
even-dimensional Minkowski space-time in the context of quantum field theory, mostly
to fix our notations (see ref. [10] for more details).

The CPT theorem [1,2,3,4] asserts that any quantum field theory is invariant un-
der CPT transformations assuming that it satisfies the following very mild assumptions:
a) Lorentz invariance, b) The energy is positive definite and there exists a Poincaré-
invariant vacuum, unique up to a phase factor, c) Local commutativity, i.e. field op-
erators at space-like separations either commute or anti-commute. These assumptions
also imply the spin-statistics theorem, i.e. fields of integer (half odd integer) spin \(^1\) are quantized with respect to Bose (Fermi) statistics.

The CPT transformation actually comes in two varieties, one being the hermitean conjugate of the other:

The first, which is what Pauli \([1]\) called *strong reflection* (SR), essentially maps a quantum field \(\phi(t, \vec{x}) \rightarrow \text{phase} \phi(-t, -\vec{x})\), i.e. it reverses time and space coordinates simultaneously. It also reverses the order of operators in an operator product and therefore cannot be represented by any operator acting on the particle states. It is a symmetry of the operator algebra only.

The second, which we will consider to be the CPT transformation proper, is obtained by performing first SR and then taking the hermitean conjugate. The resulting operation clearly does not change the order of operators in an operator product but instead all \(\mathbb{C}\)-numbers are complex conjugated. It can be represented by an anti-unitary (i.e. unitary and anti-linear) operator \(\Theta\) acting on the Hilbert space of physical particle states. It will map \(|\text{“in”}\rangle\)-states into \(|\text{“out”}\rangle\)-states.

As is well known, the combination of the transformations of charge conjugation, parity and time reversal \((C + P + T)\), when defined, is equal to the combination of strong reflection and hermitean conjugation \((SR + HC)\),\(^2\) thereby justifying the name CPT for the latter transformation.

Following Lüders \([2]\), we define the SR transformation for scalar, spinor and vector fields as follows

\[
\begin{align*}
\phi(x) & \overset{\text{SR}}{\longrightarrow} \phi(-x) \\
\psi(x) & \overset{\text{SR}}{\longrightarrow} \varphi_{\text{SR}} \gamma^{D+1} \psi(-x) \quad \text{with} \quad (\varphi_{\text{SR}})^2 = -(-1)^{D/2} \\
\bar{\psi}(x) & \overset{\text{SR}}{\longrightarrow} -\varphi_{\text{SR}}^* \bar{\psi}(-x) \gamma^{D+1} \\
\phi^\mu(x) & \overset{\text{SR}}{\longrightarrow} -\phi^\mu(-x),
\end{align*}
\]

where \(\gamma^{D+1}\) is the chirality matrix. It is easy to verify that the free-field equations of motion and (anti-) commutation relations are invariant under these transformations when we also define SR to *invert the order of the operators and leave \(\mathbb{C}\)-numbers unchanged*. At this point it is obviously essential that the fields satisfy the spin-statistics relation.

\(^1\) In \(D\) dimensions a field is said to have integer (half odd integer) spin if it furnishes a representation of the Lorentz algebra \(SO(D-1,1)\) with integer (half odd integer) weights.

\(^2\) This is true provided that the phase for each transformation is chosen appropriately.
For complex fields the requirement that the free-field (anti-) commutation relations are invariant under SR only fixes the form of the SR transformations (1.1) up to an overall phase factor; but for real fields the choice of $\varphi_{SR}$ is constrained by the requirement that the SR transformation should commute with hermitean conjugation and only a sign ambiguity remains. The transformation laws given in (1.1) correspond to a choice of phase that satisfies this requirement for real fields. The dependence on the dimension that enters into the phase $\varphi_{SR}$ for the spinor field is due to the fact that the charge conjugation matrix commutes with $\gamma^{D+1}$ in dimensions $D = 4k, k \in \mathbb{N}$ but anti-commutes in dimensions $D = 2 + 4k, k \in \mathbb{N}$.

For the bosonic fields the sign is chosen to agree with what we obtain by applying the tensor transformation law to the transformation $x \rightarrow -x$. Thus a field with $N$ vector indices would transform under SR with a phase $(-1)^N$.

For a general vacuum expectation value of local field operators the statement of CPT invariance becomes

$$
\langle 0 | \Phi_1(x_1) \ldots \Phi_N(x_N) | 0 \rangle \quad (1.2)
$$

$$
\overset{\text{CPT}}{=} \langle 0 | (\Theta \Phi_1(x_1) \Theta^{-1} \ldots \Theta \Phi_N(x_N) \Theta^{-1} | 0 \rangle^* \\
= \langle 0 | (\Phi_N(x_N))^{SR} \ldots (\Phi_1(x_1))^{SR} | 0 \rangle \\
= (-1)^{J_1}(-1)^{J_2} N\langle 0 | \Phi_N(-x_N) \ldots \Phi_1(-x_1) | 0 \rangle ,
$$

where $(\Phi_i(x_i))^{SR}$ is defined by the free-field transformation laws (1.1), where $J_1$ is the number of Lorentz vector indices, $J_2$ is the number of $\gamma^{D+1} = -1$ spinor indices, and

$$
N = \begin{cases} 
  i^{N_F} & \text{if } D = 4k, k \in \mathbb{N} \\
  (-1)^{N_\downarrow} & \text{if } D = 2 + 4k, k \in \mathbb{N} 
\end{cases}
$$

(1.3)

with $N_F$ being the total number of fermions (which is even) and $N_\downarrow$ the number of covariant (lower) spinor indices ($\psi$ is contravariant and $\overline{\psi}$ is covariant). The different behavior in 2 and 4 dimensions (mod 4) is again due to the different chirality properties of the charge conjugation matrix.

If the points $x_1, \ldots, x_N$ are such that all separations $x_i - x_k, i \neq k$, are space-like, and if we assume the validity of the spin-statistics theorem, the following condition, sometimes called Weak Local Commutativity, holds [4]

$$
\langle 0 | \Phi_N(x_N) \ldots \Phi_1(x_1) | 0 \rangle = i^{N_F} \langle 0 | \Phi_1(x_1) \ldots \Phi_N(x_N) | 0 \rangle .
$$

(1.4)

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3 For fermions the Majorana reality condition is only compatible with the massive Dirac equation in dimensions $D = 2 + 8k$ or $D = 4 + 8k$, $k \in \mathbb{N}$. 

5
The phase appearing is just a sign, counting how many times two fermions have been transposed in the process of reversing the order of the operators. A priori this sign is $(-1)^{N_F(N_F-1)/2}$, but since $N_F$ is even, this equals $i^{N_F}$.

Combining eqs. (1.2) and (1.4) we obtain another formulation of CPT invariance:

$$\langle 0 | \Phi_1(x_1) \ldots \Phi_N(x_N) | 0 \rangle = i^{N_F} \langle 0 | (\Phi_1(x_1))^{SR} \ldots (\Phi_N(x_N))^{SR} | 0 \rangle,$$

valid when all separations are space-like.

2. CPT Theorem for two dimensional Field Theories on a genus zero surface.

In this section we specialize to the case of two dimensions, writing $x = (\tau, \sigma)$ and introducing light-cone coordinates $\sigma^\pm = \tau \pm \sigma$.

2.1 CPT for a generic two-dimensional Field Theory.

We begin by considering field theories in flat two-dimensional Minkowski space-time that satisfy the standard CPT and spin-statistics theorems outlined in section 1.

In two dimensions there is a very simple connection between the index structure of a field operator and the phase that it acquires under SR. Like in any (even) number of dimensions, a field with $N$ vector indices (whether covariant or contravariant) picks up a phase $(-1)^N$ and spinors with negative and positive chirality transform with opposite sign. Furthermore, since $\varphi_{SR} = \pm 1$ in two dimensions, it follows from eq. (1.1) that a covariant and a contravariant spinor also transform with opposite sign (unlike in four dimensions where they transform with the same sign). Let us choose $\varphi_{SR} = 1$ so that a contravariant Weyl-spinor of positive chirality transforms under SR with a plus sign.

Now, a general tensor field in $D = 2$, with any combination of vector and spinor indices, can be decomposed into component fields. Suppose we decompose the vector indices into light-cone components $\sigma_+$ and $\sigma_-$ and the Dirac spinors according to the chirality. Then, for each component field we may define the quantities

$$\Delta = N_+ - N^+ + \frac{1}{2}(n^+ - n_+), \quad \Delta = N_- - N^- + \frac{1}{2}(n^- - n_-),$$

where $N_+$ and $N^+$ ($N_-$ and $N^-$) denote the number of covariant and contravariant $\sigma_+$ ($\sigma_-$) indices, and likewise $n_+$ and $n^+$ ($n_-$ and $n^-$) denote the number of covariant and
contravariant spinor indices of positive (negative) chirality. Denoting the component field by $\Phi(\Delta, \bar{\Delta})$ we arrive at the very simple SR transformation law:

\[
\Phi(\Delta, \bar{\Delta}) (\sigma^+, \sigma^-) \xrightarrow{\text{SR}} (-1)^{N^+ + N^- + N_+ + n^- + n_+} (n_{\text{tot}}(n_{\text{tot}} - 1)/2) \Phi(\Delta, \bar{\Delta})(-\sigma^+, -\sigma^-),
\]

where $n_{\text{tot}} = n^+ + n_+$ is the total number of spinor indices. Here the first sign factor is obtained simply by putting a minus sign for each vector index, for each contravariant spinor index of negative chirality and for each covariant spinor index of positive chirality. The extra sign factor $(-1)^{n_{\text{tot}}(n_{\text{tot}} - 1)/2}$ is needed for fields with several spinor indices. This can be seen if we imagine such a field to be given by a normal-ordered product of $n_{\text{tot}}$ fermions, each having just a single spinor index. SR inverts the order of these operators, and the extra sign appears when they are permuted back into their original order.

Eq. (2.2) can be rewritten on the much simpler form

\[
\Phi(\Delta, \bar{\Delta}) (\sigma^+, \sigma^-) \xrightarrow{\text{SR}} (-1)^{\Delta - \bar{\Delta} - s} \Phi(\Delta, \bar{\Delta})(-\sigma^+, -\sigma^-),
\]

where $s = 0$ or $s = 1/2$ depending on whether $n_{\text{tot}}$ is even ($\Phi$ is bosonic) or odd ($\Phi$ is fermionic).

If we insert the transformation law (2.3) into the CPT theorem (1.2) and use the assumption (1.4) of Weak Local Commutativity to invert the order of the operators, we find that the resulting identity (1.5) assumes the following suggestive form:

\[
\langle 0 | \Phi(\Delta_1, \bar{\Delta}_1)(\sigma_1^+, \sigma_1^-) \cdots \Phi(\Delta_N, \bar{\Delta}_N)(\sigma_N^+, \sigma_N^-)|0 \rangle = (-1)^{\sum_{i=1}^{N}(\Delta_i - \bar{\Delta}_i)} \langle 0 | \Phi(\Delta_1, \bar{\Delta}_1)(-\sigma_1^+, -\sigma_1^-) \cdots \Phi(\Delta_N, \bar{\Delta}_N)(-\sigma_N^+, -\sigma_N^-)|0 \rangle.
\]

Up to this point we have not assumed that the two-dimensional field theory is conformally invariant. For a CFT, the component field $\Phi(\Delta, \bar{\Delta})$ becomes a primary conformal field of dimensions $(\Delta, \bar{\Delta})$, and the phase factors in eq. (2.4) are seen to agree exactly with what one obtains from applying the transformation law of a primary field to the transformation $(\sigma^+, \sigma^-) \rightarrow (-\sigma^+, -\sigma^-)$.

In summary, we have seen that the combined assumptions of CPT invariance and Weak Local Commutativity lead to the identity (2.4) that merely expresses the invariance under the conformal transformation $(\sigma^+, \sigma^-) \rightarrow (-\sigma^+, -\sigma^-)$. Thus, for a two-dimensional field theory that is invariant under (complexified) conformal transformations, eq. (2.4) is always valid, regardless of whether Weak Local Commutativity
holds and irrespective of the various assumptions underlying the standard CPT and spin-statistics theorems. Accordingly, inside the class of conformal field theories on the plane eq. (2.4) is a much more powerful statement than the ordinary CPT theorem. The point of view which we will take, and which will be elaborated further in the next subsection, is therefore to abandon the assumptions of Lorentz invariance, (weak) local commutativity and energy positivity, as well as the related assumption of spin-statistics, and take instead as our one fundamental assumption the invariance of the CFT under the transformation \( \sigma_\pm \to -\sigma_\pm \). In standard complex coordinates this becomes the Belavin-Polyakov-Zamolodchikov (BPZ) transformation \( z \to 1/z \) [5], which will be studied in detail in the following subsection.

Abandoning Lorentz invariance has the great advantage of liberating our discussion from the context of flat two-dimensional Minkowski space-time. Instead, we can now study the genus zero world-sheets of interest to string theory, such as the cylinder, where the space-direction has been compactified, and also surfaces with any number of external “tubes”, corresponding to the presence of many incoming and outgoing strings. On all these surfaces global Lorentz invariance ceases to be meaningful, whereas the transformation \( z \to 1/z \) remains well-defined.

Since we abandon the requirement of spin-statistics on the world-sheet, we are also free to consider reparametrization ghosts and superghosts. Finally, we may rotate to Euclidean world-sheet time, which is the natural set-up for conformal field theory and hence for the formulation of world-sheet CPT based on the assumption of conformal invariance.

Having stressed the advantages it is only fair to point out what we lose in the process. Since in eq. (2.4) the operators appear in the same order on both sides of the equality sign, the BPZ symmetry transformation \( \sigma_\pm \to -\sigma_\pm \) must be defined not to invert the order of operators. Thus, even though we may still define an anti-linear world-sheet CPT transformation by combining the BPZ transformation \( \sigma_\pm \to -\sigma_\pm \) with hermitean conjugation, the resulting transformation will invert the order of operators and hence can never be generated by any operator acting on states. Thus, the world-sheet CPT theorem for CFT is only an identity between correlation functions. This is unlike the standard CPT theorem in Minkowski space-time, where a major ingredient is the existence of an anti-unitary operator implementing the CPT transformation on the Hilbert space of particle states.
2.2 BPZ invariance in conformal field theories.

Consider a conformal field theory on a surface of genus zero. From the point of view of string theory, the interesting surfaces are those with $N$ external “tubes” representing external string states. These surfaces are conformally equivalent to the $N$-punctured sphere, where the boundary conditions describing the external string states are imposed by appropriate operators inserted at the punctures. Therefore we will consider correlation functions of primary and descendant operator fields living on the sphere.

We introduce standard holomorphic coordinates $z$ and $\bar{z}$ related to $\sigma$ and $\tau$ by

$$z = \exp\{i(\sigma + \tau)\}$$
$$\bar{z} = \exp\{i(-\sigma + \tau)\}$$

and rotate to Euclidean time $\tau \rightarrow -i\tau$. We will refer to $z$ as a global holomorphic coordinate even though, strictly speaking, $z$ is not defined at the point $z = \infty$, where we use instead the coordinate $w = 1/z$. The map $\sigma^\pm \rightarrow -\sigma^\pm$ changes sign on $\tau$ and $\sigma$ simultaneously and from a CFT point of view gives rise to the Belavin-Polyakov-Zamolodchikov (BPZ) transformation $z \rightarrow 1/z$ [5]. This transformation defines a globally holomorphic diffeomorphism on the sphere.

At the level of the operator fields, the transformation changes the coordinate system from $z$ to $w$ where $w = 1/z$, and a primary conformal field $\Phi(\Delta, \overline{\Delta})$ of conformal dimension $(\Delta, \overline{\Delta})$ transforms as:

$$\Phi_{(\Delta, \overline{\Delta})}(z = \zeta, \bar{z} = \bar{\zeta}) \overset{\text{BPZ}}{\rightarrow} \Phi_{(\Delta, \overline{\Delta})}(w = \zeta, \bar{w} = \bar{\zeta}) = (-1)^{\Delta - \overline{\Delta}} \left(\frac{1}{\zeta^2}\right)^\Delta \left(\frac{1}{\bar{\zeta}^2}\right)^{\overline{\Delta}} \Phi_{(\Delta, \overline{\Delta})}(z = 1/\zeta, \bar{z} = 1/\bar{\zeta}).$$

The BPZ transformation plays a role in CFT very similar to the role played by the SR transformation for field theories in flat Minkowski space-time: It maps an operator field located at the point $(\tau, \sigma)$ into an operator field located at the point $(-\tau, -\sigma)$; it leaves all $\mathbb{C}$-numbers unchanged and cannot be represented by any operator acting on ket-states. But as we saw in subsection 2.1 there is one important difference: Unlike the SR transformation, the BPZ transformation does not reverse the order of operators.

Any conformal field theory on the sphere is invariant under the transformation $z \rightarrow w = 1/z$. This implies the “Ward identity” 4

$$\langle \Phi_{\Delta_1}(z = \zeta_1) \ldots \Phi_{\Delta_N}(z = \zeta_N) \rangle \overset{\text{BPZ}}{=} \langle \Phi_{\Delta_1}(w = \zeta_1) \ldots \Phi_{\Delta_N}(w = \zeta_N) \rangle.$$

4 Here, and in most subsequent formulae, we restrict ourselves to chiral fields for ease of notation.
This equation is the statement of BPZ invariance for a conformal field theory on a genus zero Riemann surface and it holds for all fields, whether primary or descendant. For primary fields we may use eq. (2.5) to rewrite the identity (2.6) as

\[
\langle \Phi_{\Delta_1}(z = \zeta_1) \ldots \Phi_{\Delta_N}(z = \zeta_N) \rangle_{\text{BPZ}} = (-1)^{\Delta_1 + \ldots + \Delta_N} \langle \left( \frac{1}{\zeta_1} \right)^{2\Delta_1} \Phi_{\Delta_1}(z = \frac{1}{\zeta_1}) \ldots \left( \frac{1}{\zeta_N} \right)^{2\Delta_N} \Phi_{\Delta_N}(z = \frac{1}{\zeta_N}) \rangle.
\]  

(2.7)

When \( \Delta - \overline{\Delta} \) is not integer, the field \( \Phi_{(\Delta, \overline{\Delta})} \) is said to be non-local. For a non-local field the transformation (2.5) is not well-defined a priori. To make it unambiguous we have to specify the phase of the complex number \( \zeta \) and we also have to choose a certain phase for \(-1\). Likewise, in a correlation function involving non-local fields, we need to specify the phases of the various differences \( \zeta_i - \zeta_j \). We would like to make these phase choices in such a way that eqs. (2.6) and (2.7) remain valid. If we represent the explicit \(-1\) appearing in the BPZ transformation (2.5) by \( e^{-i\epsilon\pi} \), so that the overall factor in eq. (2.7) becomes \( e^{-i\epsilon\pi (\Delta_1 + \ldots + \Delta_N)} \) (where \( \epsilon \) is an odd integer), this is done by choosing

\[
\frac{\zeta_j - \zeta_i}{\zeta_i - \zeta_j} = e^{-i\epsilon\pi} \quad \text{for any pair } 1 \leq i < j \leq N.
\]

(2.8)

In this sense, the BPZ transformation and the statement of BPZ invariance can be extended to non-local CFTs.

Notice that for integer and half odd integer conformal dimension, the BPZ “Ward identity” (2.7) is nothing but eq. (2.4) transformed into \( z \)-coordinates. Indeed, in the case of half odd integer conformal dimension the choice of \( \epsilon \) merely corresponds to the sign ambiguity also found in the choice of \( \varphi_{\text{SR}} \) in eq. (1.1).

### 2.3 Hermitean Conjugation and CPT on a genus zero surface.

Just as we obtained CPT in the general \( D \)-dimensional case by composing SR with hermitean conjugation, so in the case of two-dimensional conformal field theories on the sphere we define what we call the WS-CPT transformation as the result of composing BPZ conjugation with hermitean conjugation (HC). But one should keep in mind that the WS-CPT transformation thus defined will only be a formal substitution rule: Since it inverts the order of operators it can never be implemented by any operator acting on states.

To have a concept of hermitean conjugate it is necessary to assume that an inner product is defined on the Hilbert space of the CFT, i.e. for any two states \( |\Phi_1\rangle \) and
we may form the complex number
\[ \langle \Phi_1 | \Phi_2 \rangle = \langle \Phi_2 | \Phi_1 \rangle^* . \quad (2.9) \]

The basic hermiticity property (2.9) ensures that the norm of any state is a real number. We will not assume it to be positive.

We can think of specifying the hermitean conjugate of all elementary operator fields in the conformal field theory by specifying the hermitean conjugate of the corresponding oscillators, with the further understanding that hermitean conjugation also complex conjugates all complex numbers and inverts the order of the operators.

For example, if
\[ \Phi_\Delta(z = \zeta) = \sum_n \phi_n \zeta^{-n-\Delta} \quad (2.10) \]
is a primary chiral conformal field of conformal dimension \( \Delta \), then the hermitean conjugate of this field is
\[ (\Phi_\Delta(z = \zeta))^\dagger = \left( \frac{1}{\zeta^*} \right)^{2\Delta} \tilde{\Phi}_\Delta(z = 1/\zeta^*) , \quad (2.11) \]
where \( \zeta^* \) denotes the complex conjugate of \( \zeta \) (it is sometimes convenient to think of \( \zeta \) and \( \bar{\zeta} \) as independent complex variables, so that \( \zeta^* \) and \( \bar{\zeta} \) need not be equal). The field \( \tilde{\Phi}_\Delta \) is defined by
\[ \tilde{\Phi}_\Delta(z = \zeta) \equiv \sum_n \phi^+_n \zeta^{-n-\Delta} \quad (2.12) \]
and is called the hermitean conjugate of the field \( \Phi_\Delta \). We say that a field \( \Phi_\Delta \) is hermitean (anti-hermitean) when \( \tilde{\Phi}_\Delta = +\Phi_\Delta \ (-\Phi_\Delta) \). We always require the energy-momentum tensor of the CFT to be hermitean, i.e. the mode operators satisfy \( L_n^\dagger = L_{-n} \) for all \( n \in \mathbb{Z} \). This in turn implies that \( \tilde{\Phi}_\Delta \) is a primary conformal field of the same dimension as \( \Phi_\Delta \).

The fact that HC changes the argument from \( \zeta \) to \( 1/\zeta^* \) means that an operator field situated in the vicinity of \( z = 0 \) is mapped into one situated around \( z = \infty \). For this reason, HC is only well-defined (as a map acting on the operator fields) in the case of the sphere, where we can think of \( z \) as a global complex coordinate. The peculiar behaviour of the argument is due to the fact that we are considering imaginary time on the world-sheet. Rotating back to real time, we have \( z = \zeta = \exp\{i(\tau + \sigma)\} \) and \( 1/\zeta^* = \zeta \).

In CFT we have the standard correspondence
\[ |\Phi_\Delta\rangle \equiv \lim_{\zeta \to 0} \Phi_\Delta(z = \zeta)|0\rangle \quad (2.13) \]
between states and the primary operator fields and their descendants. Therefore, if we know the vacuum expectation values (on the sphere) of arbitrary combinations of primary fields, and if we have defined the concept of hermitean conjugation of all operators, we may compute the norm of any two states as

\[
\langle \Phi_{\Delta_1} | \Phi_{\Delta_2} \rangle = \lim_{\zeta_1 \to 0} \lim_{\zeta_2 \to 0} \langle (\Phi_{\Delta_1}(z = \zeta_1))^\dagger \Phi_{\Delta_2}(z = \zeta_2) \rangle .
\]

(2.14)

From this point of view, the statement (2.9) of hermiticity becomes the requirement that

\[
\langle \Phi_{\Delta_1}(z_1) \ldots \Phi_{\Delta_N}(z_N) \rangle_{HC} = \langle (\Phi_{\Delta_N}(w = \zeta_N))^\dagger \ldots (\Phi_{\Delta_1}(w = \zeta_1))^\dagger \rangle^* \quad (2.15)
\]

for all correlation functions on the sphere.

Combining eq. (2.15) with eqs. (2.6) and (2.7) we obtain the form of the CPT theorem on a genus zero Riemann surface which we will also refer to as the World-Sheet CPT Theorem (WS-CPT) on the sphere

\[
\langle \Phi_{\Delta_1}(z = \zeta_1)(z_1) \ldots \Phi_{\Delta_N}(z = \zeta_N)(z_N) \rangle_{WS-CPT} = \langle (\Phi_{\Delta_N}(w = \zeta_N))^\dagger \ldots (\Phi_{\Delta_1}(w = \zeta_1))^\dagger \rangle^* \quad (2.16)
\]

Here the first equality sign holds for all fields whereas (a priori) the second equality sign holds only for primary fields. However, as can be seen by taking derivatives w.r.t. \( \zeta_i \), \( i = 1, \ldots, N \), the second equality sign actually holds for descendant fields as well, as long as we let \( \Delta_i \) denote the conformal dimension of the primary field from which \( \Phi_{\Delta_i} \) descends.

Like in the case of the BPZ identity (2.7), eq. (2.16) only holds for non-local fields if a proper choice of phases is made: If we represent the explicit \( -1 \) appearing in the last line by \( e^{-i\epsilon\pi} \) we have to impose the condition (2.8).

In the formulation (2.16) of the WS-CPT theorem the operator \( \Phi_{\Delta_i} \) is inserted at \( z = \zeta_i \) whereas \( \hat{\Phi}_{\Delta_i} \) is inserted at \( z = \zeta^*_i \). Thus, unless \( \zeta_i \) is real, the two operators are inserted at different points on the sphere. If we refer back to the real coordinates \((\tau, \sigma)\), in terms of which

\[
z(\tau, \sigma) = e^{\tau + i\sigma} = \zeta , \quad (2.17)
\]

difference coordinate is now given by

\[
\bar{z}(\tau, \sigma) = e^{\tau - i\sigma} = (z(\tau, \sigma))^* , \quad (2.18)
\]

We see that there is another, equivalent, way of looking at it: Namely to say that the geometrical point \((\tau, \sigma)\) remains fixed, but the complex structure is changed by the hermitean conjugation so that the holomorphic coordinate is now given by
instead of eq. (2.17).

The two points of view differ only by the diffeomorphism \((\tau, \sigma) \rightarrow (\tau, -\sigma)\) which in terms of the coordinate \(z = \zeta\) maps \(z = \zeta\) into \(z = \zeta^*\). Diffeomorphism invariance of the correlation functions ensures that

\[
\langle \Phi_{\Delta_1}(\tilde{z} = \zeta_1) \ldots \Phi_{\Delta_N}(\tilde{z} = \zeta_N) \rangle^{(\tilde{M})} = \\
\langle \Phi_{\Delta_1}(z = \zeta_1) \ldots \Phi_{\Delta_N}(z = \zeta_N) \rangle^{(M)},
\]

where the labels \((M)\) and \((\tilde{M})\) are introduced to remind us that the two correlators pertain to different complex structures. Eq. (2.19) is really just the statement that the correlation function depends only on the value taken at the operator insertion points by the global holomorphic coordinate, and not on how this coordinate is related to some underlying real coordinates.

Using eq. (2.19), the statement (2.16) can be formulated as follows

\[
\langle \Phi_{\Delta_1}(z = \zeta_1) \ldots \Phi_{\Delta_N}(z = \zeta_N) \rangle^{(M)} \overset{\text{WS-CPT}}{=} e^{-ie\pi(\Delta_1 + \ldots + \Delta_N)} \left( \langle \hat{\Phi}_{\Delta_N}(\tilde{z} = \zeta_N^*) \ldots \hat{\Phi}_{\Delta_1}(\tilde{z} = \zeta_1^*) \rangle^{(\tilde{M})} \right)^*,
\]

where now \(z = \zeta\) and \(\tilde{z} = \zeta^*\) refer to the same point \((\tau, \sigma)\) on the sphere. We have explicitly parametrized \(-1\) by the odd integer \(\epsilon\) in accordance with the phase choice (2.8). Thus, eq. (2.20) also holds for non-local theories.

In the following section we consider how to extend the formulation (2.20) of WS-CPT invariance to higher genus surfaces.

3. The World-Sheet CPT Theorem on a Genus \(g\) Surface.

In the case of Riemann surfaces of genus \(g > 0\) the transformation \(z \rightarrow w = 1/z\) is not anymore a global symmetry, i.e. does not define any globally holomorphic diffeomorphism. Thus the concept of BPZ invariance looses its meaning.\(^5\) Nor can we introduce a globally well-defined (i.e. single-valued) holomorphic coordinate \(z\), like the one employed in the previous section. This means that our definition (2.11) of the hermitean conjugate of a field operator is not readily generalized either.

\(^5\) The case \(g = 1\) is special. If we represent the torus of modular parameter \(k = \exp\{2\pi i \tau\}\) by an annulus in the complex plane, \(|k| \leq |z| \leq 1\), then the modular transformation changing sign on the two homology cycles can be represented by the globally holomorphic diffeomorphism \(z \rightarrow 1/z\). In this way the BPZ transformation can be extended to the torus.
Nevertheless, as we shall see in this section, the world-sheet CPT theorem, when written on the form

$$
\langle \Phi_{\Delta_1}(z_1 = 0) \ldots \Phi_{\Delta_N}(z_N = 0) \rangle_g^{(M)} \stackrel{\text{WS-CPT}}{=} \langle (\hat{\Phi}_{\Delta_N}(\tilde{z}_N = 0) \ldots \hat{\Phi}_{\Delta_1}(\tilde{z}_1 = 0))^{(\tilde{M})} \rangle^* ,
$$

remains valid for local, modular-invariant CFTs satisfying a very mild assumption (given by eq. (3.7) below). The correlator on the left-hand side of eq. (3.1) refers to a Riemann surface $M$ endowed with local holomorphic coordinates $z_i$, while that on the right-hand side refers to a Riemann surface $\tilde{M}$ endowed with local holomorphic coordinates $\tilde{z}_i$.

The local coordinates $z_i$ on $M$ depend holomorphically on each other (whenever the regions on which they are defined overlap). In terms of some fixed holomorphic coordinate $z$ we may write $z_i = z_i(z, z^*) = z_i(z)$. Then the local coordinate $\tilde{z}_i$ defined on $\tilde{M}$ is given by the following anti-holomorphic function of $z$

$$
\tilde{z}_i(z^*) = (z_i(z))^* .
$$

The relation between the two sets of coordinates is more easily expressed by referring back to some fixed set of real coordinates, $\xi$. In terms of these

$$
\tilde{z}_i(\xi) = (z_i(\xi))^* .
$$

Thus, the point of view underlying the formulation (3.1) of WS-CPT invariance is that $M$ and $\tilde{M}$ correspond to the same real two-dimensional manifold $\mathcal{M}$, but endowed with different complex structures $J$ and $\tilde{J}$, and $z_i = \zeta$ and $\tilde{z}_i = \zeta^*$ correspond to the same geometrical point on $\mathcal{M}$ (the same value of the coordinate $\xi$).

In the case of the sphere any two complex structures are related by a diffeomorphism and in this sense $M$ and $\tilde{M}$ are therefore equivalent. At genus $g \geq 1$ this will in general not be the case, $M$ and $\tilde{M}$ will correspond to different points in moduli space.

The Riemann surface $\tilde{M}$ is referred to as the mirror-image of $M$. A general discussion of mirror image Riemann surfaces can be found in the Appendix.

The world-sheet CPT theorem (3.1) naturally leads us to define an anti-linear two-dimensional (world-sheet) CPT transformation for the primary and descendant operator fields of a CFT by

$$
\Phi_{\Delta}(z_i = 0) \stackrel{\text{WS-CPT}}{\longrightarrow} (\hat{\Phi}_{\Delta}(\tilde{z}_i = 0))^{\text{WS-CPT}} \equiv (-1)^{\Delta} \hat{\Phi}_{\Delta}(\tilde{z}_i = 0) ,
$$

but one should keep in mind that this is a formal substitution rule, making sense only inside a correlation function, rather than a genuine CPT transformation: First of all, it
relates operator fields defined on different Riemann surfaces ($M$ and $\tilde{M}$ respectively). Second, it involves an inversion of the ordering of operators, so even when $M$ and $\tilde{M}$ are equivalent (as in the case of the sphere) it can never be represented by any operator acting on states.

Assuming the validity of the WS-CPT theorem on the sphere there is a straightforward way to extend it also to higher genus Riemann surfaces. The way to do this is just by “sewing” Riemann surfaces, using the sewing procedure for conformal field theories defined in refs. [8]. If we start from a genus $g$ surface with $N + 2$ punctures and sew together two of the legs, we obtain a genus $g + 1$ surface with $N$ punctures, and any $N$-point correlator on this genus $g + 1$ surface can be expressed as a sum over $N + 2$-point correlators on the genus $g$ surface.

Thus, assuming that the WS-CPT theorem is valid on the sphere, we may use induction to prove that it is valid at any genus. All we have to do is to show that, assuming the WS-CPT theorem to hold for the correlators at genus $g$, then it also holds for the genus $g + 1$ correlators defined from these by sewing.

We will carry out this analysis in two cases: First for a large class of local conformal field theories (to be specified more precisely in the following subsection), then for some important examples of free non-local theories. Finally, we consider some convention-dependent complications that arise for the combined theory of left- and right-moving reparametrization ghosts.

3.1 Local Conformal Field Theories

In this section we adopt Sonoda’s formulation of sewing [8] for local conformal field theories, that is, CFTs where primary fields have integer dimensions (or $\Delta - \bar{\Delta} \in \mathbb{Z}$ for non-chiral CFT). We always have in mind a CFT describing some consistent string model. We exclude the reparametrization ghosts, which need special treatment and will be discussed in subsection 3.4. The resulting CFT is always local: The vertex operators describing on-shell physical states are primary fields with $\Delta = \bar{\Delta} = 1$, and even if we go off-shell, only states satisfying the level-matching condition $L_0 - \bar{L}_0 = 0$ can propagate.

Consider a Riemann surface $M'$ with genus $g$ and $N$ punctures. Add two extra punctures, $P$ and $Q$, to this surface with local coordinates $z'$ and $w'$ vanishing at $P$ and $Q$ respectively. Let $q$ be a complex parameter with $|q| < 1$ (the sewing parameter). The standard procedure is to excise two discs around the two punctures and identify the coordinates as $z'w' = q$, obtaining in this way a genus $g + 1$ surface, $M$. We obtain
the correlators on the genus $g+1$ surface from the similar correlators on the genus $g$ surface by inserting at the points $P$ and $Q$ a complete set of conjugate local operators:

$$\langle \Phi_{\Delta_1}(z_1 = 0) \ldots \Phi_{\Delta_N}(z_N = 0) \rangle_{g+1}^{(M)} = \sum_{m,n} (M_{wz})^{-1}_{nm} \langle \Phi_{\Delta_1}(z_1 = 0) \ldots \Phi_{\Delta_N}(z_N = 0) \Phi_{m}(w = 0) \rangle_{g}^{(M')},$$

where by definition

$$(M_{wz})_{nm} = (\langle \Phi_{n}(w = 0) \Phi_{m}(z = 0) \rangle_{g=0}) , \quad (M_{zw})_{nm} = (\langle \Phi_{n}(z = 0) \Phi_{m}(w = 0) \rangle_{g=0}),$$

the coordinates $z$ and $w$ being related by $w = 1/z$. The two matrices differ by the phase factor $(-1)^{2\Delta_n}$. For the local CFTs considered in the present subsection this phase is always unity, whereas for the non-local theories considered in the next subsection it will be non-trivial.

In eq. (3.5) the $\{\Phi_{n}\}$ form a basis for the set of primary fields and their descendents. In practical applications, this set is often specified by means of an appropriate projection defined inside a larger class of fields. For example, if we imagine the CFT to describe a superstring theory in the Neveu-Schwarz Ramond formulation, then only the GSO projected fields are included in the sum. The sewing formula (3.5) is manifestly independent of the choice of basis.

A genus $g+1$ surface has 3 more moduli than a genus $g$ surface. These are the parameter $q$ and the positions of the two points $P$ and $Q$, more precisely, the values taken at these points by an appropriate holomorphic coordinate.  

Notice that on the right-hand side of the sewing formula (3.5) the local operator inserted at $P$ appears to the left of the local operator inserted at $Q$. This relative ordering of the two operators distinguishes between the points $P$ and $Q$ and can be used to define an orientation of the homology cycle $b = b_{g+1}$ formed by the sewing. Therefore, it would be more precise to denote the surface obtained from the sewing procedure (3.5) as $M_{PQ}$. The surface $M_{Q}^{P}$ can be obtained from $M_{P}^{Q}$ by interchanging the points $P$ and $Q$ on $M'$, i.e. by reversing the orientation of the homology cycle $b$ on $M$. This corresponds to a modular transformation, and $M_{P}^{Q}$ and $M_{Q}^{P}$ therefore define the same point in moduli space but different points in Teichmüller space.

If we only assume locality, there is no guarantee that the correlation functions defined by the sewing formula (3.5) will be modular invariant. Also, the correlation

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6 The two well-known exceptions are $g = 0$ and $g = 1$ where the existence of global conformal diffeomorfisms makes it possible to fix the coordinate values of $P$ and $Q$ (for $g = 0$) and either $P$ or $Q$ (for $g = 1$).
functions on a given Riemann surface will in general depend on the precise way in which the surface is sewn together from three-punctured spheres. Sonoda [8] proved that both of these problems are avoided if one further assumes i) crossing invariance and ii) that the one-point genus one correlators are modular invariant.

Crossing invariance implies that correlation functions do not depend on the order of the operator fields. From the point of view of string theory this requirement is somewhat too restrictive, since space-time fermions are described by vertex operators which satisfy Fermi statistics on the world-sheet [10]. For this reason we will not assume crossing invariance.

Instead, we take the point of view that the correlation functions on the sphere are known and that the higher-loop correlation functions are then defined recursively by the explicit sewing formula (3.5). We restrict ourselves to CFTs for which eq. (3.5) leads to modular invariant correlation functions. Obviously, the CFTs considered by Sonoda fall into this category. So does any CFT describing a consistent string theory. Such a CFT is also expected to satisfy duality, i.e. the requirement that correlation functions obtained on higher genus surfaces do not depend on the details of the sewing, except for overall numerical factors.

The absence of crossing invariance means that we have to keep track of the ordering of the operators. In particular, the precise location of the two “internal” operators \( \Phi_n \) and \( \Phi_m \) inside the correlator in eq. (3.5) is important. The locations chosen are in accordance with the sewing procedure of refs. [17,18] and with eq. (2.15) of ref. [7].

For the purpose of proving the WS-CPT theorem (3.1) for the correlation functions defined by eq. (3.5) we need one more technical assumption, which is much weaker than crossing invariance: For any pair of operators \( \Phi_n \) and \( \Phi_m \) such that \( (\mathcal{M}_{wz}^{-1})_{mn} \) is nonzero and for any set of operators \( \Phi_{\Delta_1} \ldots \Phi_{\Delta_N} \) having a nonzero correlator at genus \( g+1 \) we require that

\[
\langle \Phi_{\Delta_1}(z_1 = 0) \ldots \Phi_{\Delta_N}(z_N = 0) \Phi_n(z' = 0) \Phi_m(w' = 0) \rangle_g = \sum_{m,n} (\mathcal{M}_{wz}^{-1})_{mn} \langle \Phi_{\Delta_1}(\tilde{z}_1 = 0) \ldots \Phi_{\Delta_N}(\tilde{z}_N = 0) \Phi_m(\tilde{w}' = 0) \Phi_n(\tilde{z}' = 0) \rangle_{g'}^*,
\]

We are now ready to prove the WS-CPT theorem (3.1) at any genus. Assume it to be valid at genera \( \leq g \). Then we may rewrite the right-hand side of eq. (3.5) to obtain

\[
\sum_{m,n} (\mathcal{M}_{wz}^{-1})_{nm} \langle \Phi_{\Delta_1}(z_1 = 0) \ldots \Phi_{\Delta_N}(z_N = 0) \Phi_n(z' = 0) \Phi_m(w' = 0) \rangle_g^{(\mathcal{M}')} = \sum_{m,n} (\mathcal{M}_{wz}^{-1})_{nm} (\mathcal{M}_{wz}^{-1})_{mn} \langle \Phi_{\Delta_1}(\tilde{z}_1 = 0) \ldots \Phi_{\Delta_N}(\tilde{z}_N = 0) \Phi_m(\tilde{w}' = 0) \Phi_n(\tilde{z}' = 0) \rangle_{g'}^*,
\]

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where the assumption (3.7), as well as the inductive hypothesis, was used in the last line. We defined $\tilde{\Phi}_\Delta = (-1)^\Delta \tilde{\Phi}_\Delta$. The $\{\tilde{\Phi}_n\}$ also form a basis for the set of primary fields and their descendants, and in terms of this basis we have a matrix $(\tilde{M}_{wz})_{mn}$.

WS-CPT invariance (3.1) and diffeomorphism invariance (2.19) ensure that

$$
(M_{wz})_{nm} = \langle \Phi_{\Delta_1}(z_1 = 0) \cdots \Phi_{\Delta_N}(z_N = 0) \rangle_{g+1}^{(M^P)} = \\
\left( \sum_{m,n} (\tilde{M}^{-1}_{zw})_{mn} \langle \tilde{\Phi}_{\Delta_N}(\tilde{z}_N = 0) \cdots \tilde{\Phi}_{\Delta_1}(\tilde{z}_1 = 0) \tilde{\Phi}_m(\tilde{w} = 0) \rangle_{g+1}^{(\tilde{M}')} \right)^* = \\
\left( \langle \tilde{\Phi}_{\Delta_N}(\tilde{z}_N = 0) \cdots \tilde{\Phi}_{\Delta_1}(\tilde{z}_1 = 0) \rangle_{g+1}^{(\tilde{M}^Q)} \right)^* .
$$

Therefore, we may rewrite eq. (3.8) in the form

$$
\langle \Phi_{\Delta_1}(z_1 = 0) \cdots \Phi_{\Delta_N}(z_N = 0) \rangle_{g+1}^{(M_P)} =
\left( \sum_{m,n} (\tilde{M}^{-1}_{zw})_{mn} \langle \tilde{\Phi}_{\Delta_N}(\tilde{z}_N = 0) \cdots \tilde{\Phi}_{\Delta_1}(\tilde{z}_1 = 0) \tilde{\Phi}_m(\tilde{w} = 0) \rangle_{g+1}^{(\tilde{M}')} \right)^* =
\left( \langle \tilde{\Phi}_{\Delta_N}(\tilde{z}_N = 0) \cdots \tilde{\Phi}_{\Delta_1}(\tilde{z}_1 = 0) \rangle_{g+1}^{(\tilde{M}^Q)} \right)^* .
$$

Here we used the sewing formula (3.5) “backwards”, keeping in mind that it is irrelevant which basis we choose for the set of local operators.

When we compare the surface $M^P_Q$ defined from $M'$ through eq. (3.5) with the surface $\tilde{M}^Q_P$ obtained from $\tilde{M}'$ by means of eq. (3.10), we first of all notice that $P$ and $Q$ have been interchanged, implying a change of sign in the homology cycle $b_{g+1}$. This will be unimportant only if the correlation functions are invariant under the corresponding modular transformation. As already mentioned, we explicitly assume the correlation functions defined by eq. (3.5) to be modular invariant.

We further notice that all local coordinates on $\tilde{M}$ are related to those on $M$ by the standard map (3.3). Thus, $\tilde{M}$ is indeed the mirror image of $M$. In particular, since $z'w' = q$, we have $\tilde{z}'\tilde{w}' = q^*$, so that $\tilde{M}$ has sewing parameter $q^*$ rather than $q$. This concludes the proof of the WS-CPT theorem for higher genus Riemann surfaces.

### 3.2 Free Non-local Conformal Field Theories

The analysis in the previous section was restricted to local conformal field theories, meaning that $\Delta - \bar{\Delta} \in \mathbb{Z}$ for all primary fields. If we relax these conditions, things get much more complicated: The operator fields become multivalued and to properly describe this we have to introduce spin structures.

We do not attempt to analyze this problem in general, instead we consider some examples of free, chiral conformal field theories that are important building blocks for
constructing superstring theories in the Neveu-Schwarz Ramond formulation: A pair of real world-sheet fermions or a single complex world-sheet fermion; equivalently, a boson with appropriately discretized momenta; and the superghosts \((\beta, \gamma)\). These CFTs are all non-local, but since they are free, they can be treated very explicitly.

We denote the spin-structure around the \(\mu\)'th \(a\)-cycle (\(b\)-cycle) by \(\alpha_\mu\) \((\beta_\mu)\), \(\mu = 1, \ldots, g\). Our conventions are that \(\alpha = 0\) denotes a Ramond sector and \(\alpha = 1/2\) denotes a Neveu-Schwarz sector. For the superghosts, \(\alpha\) and \(\beta\) can only take the values 0 and 1/2 (mod 1), but for a free complex fermion other possibilities are allowed.

The correlation functions at genus \(g\) depend on the spin structure. When we go from genus \(g\) to genus \(g + 1\) by means of sewing, an extra pair of components \(\alpha = \alpha_{g+1}\) and \(\beta = \beta_{g+1}\) are added to the vectors \(\alpha\) and \(\beta\). If we denote the spin structures of the genus \(g\) surface \(M\) by \(\alpha\) and \(\beta\), and those of the genus \(g + 1\) surface \(M'\) by \(\alpha'\) and \(\beta'\), the sewing formula (3.5) is modified as follows

\[
\langle \Phi_1(z_1 = 0) \ldots \Phi_N(z_N = 0) \rangle^{(M')}_{g+1} [\alpha] [\beta] = \sum_{m,n} \langle M^{-1}_{wz} \rangle^{(\alpha)}_{nm} \langle \Phi_1(z_1 = 0) \ldots \Phi_N(z_N = 0) \Phi_m(w' = 0) P_\beta \Phi_n(z' = 0) \rangle^{(M')}_{g} [\alpha'] [\beta'] .
\]

This is essentially the sewing formula of ref. [17] rephrased in the language of Sonoda. Notice that it differs from the sewing formula (3.5) of the previous subsection in various ways.

First of all, the operators \(\Phi_n(z' = 0)\) and \(\Phi_m(w' = 0)\) appear in the opposite relative order. Since we consider non-local theories, the two orderings differ by a non-trivial phase. The ordering in eq. (3.11) has been carefully chosen to reproduce the known formulae for the correlation functions.

The \(\{\Phi_n\}\) in eq. (3.11) form a basis for the set of all primary and descendant fields existing \(in\ the\ sector\ labelled\ by\ the\ spin\ structure\ \alpha\) (mod 1) (this is indicated by the label \((\alpha)\) appearing in the summation symbol), and the dependence on the spin structure \(\beta\) enters through the operator [17]

\[
P_\beta \equiv \exp\{2\pi i (1/2 + \beta)J_0\} ,
\]

(3.12)

where \(J_0\) is the number operator. The values of \(J_0\) carried by the operators \(\Phi_n\) are as follows,

\[
J_0 \in \mathbb{Z} + 1/2 - \alpha .
\]

(3.13)

Whether we consider a pair of real fermions, a single complex fermion or the superghost system, we may bosonize, introducing a free boson field \(\phi\); the number operator is then
given by
\[ J_0 = \oint_0 \frac{dz}{2\pi i} \partial \phi(z). \]  
(3.14)

We want to investigate to which extent the WS-CPT identity (3.1), valid for local modular-invariant CFTs satisfying the assumption (3.7), carries over to the higher genus correlation functions defined by eq. (3.11).

We start by noticing that the assumption (3.7) is satisfied for the theories under consideration in this subsection. In the bosonized formulation this follows from the fact that both the product \( \Phi_n(w' = 0)\Phi_m(z' = 0) \), as well as the product of all “external” operators, \( \Phi_{\Delta_1}(z_1 = 0) \ldots \Phi_{\Delta_N}(z_N = 0) \), carry vanishing total value of \( J_0 \) (mod 2). This is the statement of fermion number/superghost number/momentum conservation, depending on which theory we are considering.

For the same reason \( P_\beta \) commutes with \( \Phi_{\Delta_1}(z_1 = 0) \ldots \Phi_{\Delta_N}(z_N = 0) \). Even so, the presence of \( P_\beta \) in the sewing formula (3.11) influences the behaviour of the higher genus correlation functions under WS-CPT. To see exactly how, we first have to study how \( J_0 \) transforms under the WS-CPT transformation (3.4). Using the definitions (3.14) and (2.12) we obtain

\[ J_0 = \oint \frac{d\zeta}{2\pi i} \partial \phi(z = \zeta) \xrightarrow{\text{WS-CPT}} \oint \frac{d\zeta^*}{2\pi i} (-1)^{\hat{\partial} \phi(z = \zeta^*)} = -J_0^\dagger. \]  
(3.15)

The operator \( J_0 \) is hermitean for “ordinary” fermions, i.e. for fermions whose mode operators in the Neveu-Schwarz sector give rise to a Fock space with positive definite norm. The fermions describing the transverse (and possibly internal) degrees of freedom of a superstring are of this type.

Instead, \( J_0 \) is anti-hermitean \( (J_0^\dagger = -J_0) \) for the pair of real fermions, \( \psi^0 \) and \( \psi^1 \), that are related (by world-sheet supersymmetry) to the time-like and longitudinal space-time coordinate fields \( X^0 \) and \( X^1 \) (in Minkowski space-time) \([19]\).

Finally, in the superghost case \( J_0^\dagger = -J_0 + 2 \), i.e. the number operator is anti-hermitean mod 2.

In summary, the number operator \( J_0 \) has the following behaviour under WS-CPT:

\[ J_0 \xrightarrow{\text{WS-CPT}} -J_0 \quad \text{for an ordinary fermion theory,} \]

\[ J_0 \xrightarrow{\text{WS-CPT}} J_0 \quad \text{for the fermion pair } \psi^0 \text{ and } \psi^1, \]

\[ J_0 \xrightarrow{\text{WS-CPT}} J_0 - 2 \quad \text{for the superghosts.} \]

Thus, there are basically two cases to consider: In the first case \( J_0 \) is hermitean, which implies that \( P_\beta \) is invariant under the anti-linear WS-CPT transformation; in the second
case $J_0$ is anti-hermitean (mod 2), meaning that

$$P_\beta = e^{2\pi i(1/2+\beta)}J_0 \xrightarrow{\text{WS-CPT}} e^{-2\pi i(1/2+\beta)}J_0 = (-1)^{(1+2\beta)(1-2\alpha)} P_\beta .$$

Here we used the property (3.13) of the $J_0$ values carried by $\Phi_m$ (and the fact that $2\alpha, 2\beta \in \mathbb{Z}$).

The two cases also differ from each other in one more respect: If $J_0$ is hermitean, it changes sign under WS-CPT. By eq. (3.13) this implies that WS-CPT maps the operators in the sector labeled by $\alpha$ into the sector labeled by $-\alpha$, i.e. if $\{\Phi_n\}$ is a basis of the primary and descendant fields with spin structure $\alpha$, then $\{\tilde{\Phi}_n\}$ is a basis of the primary and descendant fields with spin structure $-\alpha$ (mod 1). Whereas if $J_0$ is anti-hermitean (mod 2), both $\Phi_n$ and $\tilde{\Phi}_n$ will have the same spin structure $\alpha$.

Finally we notice that, since the correlation functions pertaining to individual spin structures are not modular invariant, the distinction between the points $P \left(z' = 0\right)$ and $Q \left(w' = 0\right)$ in the sewing formula (3.11) is important.

In view of our various considerations we may now formulate the identities that replace the WS-CPT theorem (3.1) at genus $g$ for the free non-local CFTs under consideration:

Let $M_P^Q$, where $P = \{P_\mu | \mu = 1, \ldots, g\}$ and $Q = \{Q_\mu | \mu = 1, \ldots, g\}$, denote a surface obtained from the sphere by $g$ successive sewings. Then, if $J_0$ is hermitean, we have the identity

$$\langle \Phi_{\Delta_1}(z_1 = 0) \ldots \Phi_{\Delta_N}(z_N = 0) \rangle_{g}^{(M_P^Q)_{\alpha_\beta}} \xrightarrow{\text{WS-CPT}} e^{-i\epsilon \pi (\Delta_1 + \ldots + \Delta_N)} \left(\langle \tilde{\Phi}_{\Delta_N}(\tilde{z}_N = 0) \ldots \tilde{\Phi}_{\Delta_1}(\tilde{z}_1 = 0) \rangle_{g}^{(\tilde{M}_Q^P)_{-\alpha_\beta}} \right)^*. $$

Instead, for the pair of fermions describing the time- and the longitudinal space-direction, and for the superghosts, we have the identity

$$\langle \Phi_{\Delta_1}(z_1 = 0) \ldots \Phi_{\Delta_N}(z_N = 0) \rangle_{g}^{(M_P^Q)_{\alpha_\beta}} \xrightarrow{\text{WS-CPT}} (-1)^{P(\alpha, \beta)} e^{-i\epsilon \pi (\Delta_1 + \ldots + \Delta_N)} \left(\langle \tilde{\Phi}_{\Delta_N}(\tilde{z}_N = 0) \ldots \tilde{\Phi}_{\Delta_1}(\tilde{z}_1 = 0) \rangle_{g}^{(\tilde{M}_Q^P)_{\alpha_\beta}} \right)^* ,$$

where $P(\alpha, \beta) = \sum_{\mu=1}^{g} (1 - 2\alpha_\mu)(1 + 2\beta_\mu)$ so that the sign $(-1)^{P(\alpha, \beta)}$ is +1 (-1) for even (odd) spin structures.

Both identities can be formally proven by induction, following a similar line of arguments as in the previous subsection, but starting now from the sewing formula (3.11).
Like in the case of the identity (2.20) on the sphere, the identities (3.18) and (3.19) are only valid if an appropriate $\epsilon$-dependent phase choice is made, generalizing eq. (2.8). In terms of the local holomorphic coordinates vanishing at the punctures the correct requirement is that

$$\frac{E(z_j = 0, z_i = 0)}{E(z_i = 0, z_j = 0)} = e^{-i\epsilon \pi} \quad \text{for any pair } 1 \leq i < j \leq N. \quad (3.20)$$

Here $E$ denotes the prime form on $M$, which has the short distance behaviour $E(z = \zeta_1, z = \zeta_2) = \zeta_1 - \zeta_2 + O(\zeta_1 - \zeta_2)^2$.

When we combine our various non-local CFTs into a consistent string theory and sum over the spin structures with appropriate weights, locality and modular invariance should be restored, and hence WS-CPT invariance in the sense of eq. (3.1).

In the light-cone gauge, we do not have the time-like/longitudinal fermion pair, nor the superghosts. Thus any free complex fermion (or free boson) involved will satisfy eq. (3.18). For example, we may consider the heterotic string models of refs. [20,21,22] where (apart from the space-time coordinate fields) all degrees of freedom are described by a set of free complex fermions, \{\psi\}. These models were constructed to be modular invariant and the spin structure summation coefficients $C_{\{\alpha\}}^{\{\beta\}}$ multiplying the correlation functions obtained from the sewing formula (3.11) are known explicitly. They satisfy

$$\left( C_{\{\alpha\}}^{\{\beta\}} \right)^* = C_{\{\beta\}}^{\{-\alpha\}}, \quad (3.21)$$

which is seen to be exactly the property needed in order for the string correlation functions to obey eq. (3.1) after having performed the sum over the spin-structures. 7

It can also be verified that when the sewing formula (3.11) is applied to the entire KLT model and the sum over the spin structures is performed, one recovers the sewing formula (3.5).

If we now consider the Lorentz-covariant formulation, by adding the two CFTs describing, respectively, the superghosts, and the time-like and longitudinal components of $\psi^\mu$, we see that the WS-CPT invariance of the string correlation functions are not affected: Both theories have the “anomalous” behaviour (3.19), i.e. involve an extra minus sign for odd spin structures. But this extra sign always cancels when we consider the product of the correlation functions pertaining to the two theories.

7 When a given correlation function of the KLT model is decomposed into a product of correlation functions for the constituent free fermion theories, it is of course essential to include the appropriate cocycle factors. We explicitly checked that these factors do not affect the validity of the WS-CPT theorem.
3.3 An Explicit Example: Free boson or fermion

For the free CFTs considered in the previous subsection explicit formulae are known for all correlation functions at genus $g$. Therefore we may verify the identities (3.18) and (3.19) simply by inserting the known expressions for the correlation functions involved. In this subsection we perform such a check. We consider a pair of components of the world-sheet fermion $\psi^\mu$ (which is the world-sheet super-partner of the space-time coordinate field $X^\mu$). We bosonize this pair in terms of a free boson $\phi$, which is anti-hermitean if the fermion pair corresponds to two transverse components (the case $J_0^\dagger = J_0$) and hermitean in the case of the fermion pair $\{\psi^0, \psi^1\}$ (where $J_0^\dagger = -J_0$).

The correlation function we consider is [17]

$$\langle \prod_{l=1}^{N} e^{ai_l(\phi(\zeta_l))} \rangle^{(MP_Q)}_g [\mathcal{G}] = \delta \sum_{a_{l,0}} (\det' \tilde{\partial}_0)^{-1/2} \prod_{1 \leq j < k \leq N} (E(\zeta_j, \zeta_k))^{a_j a_k} \Theta [\mathcal{G}] \left( \sum_{l=1}^{N} a_l \int \frac{\omega_i}{2\pi i} |\tau| \right).$$

The various quantities appearing in this formula can be written down explicitly if we use the Schottky parametrization, see ref. [17] for details. In the Schottky parametrization, a higher genus Riemann surface $M$ is represented by the sphere, $S^2$, endowed with the usual global holomorphic coordinate $z$, modulo a discrete symmetry group, the Schottky group $G(M)$ [17], generated by $g$ projective transformations $S_\mu(z)$, each of which can be parametrized by the multiplier $k_\mu$ together with the fixed points $\eta_\mu$ and $\xi_\mu$.

Specifying the generators $S_\mu$ (up to an overall projective transformation of the coordinate $z$) defines not only a Riemann surface but also a canonical homology basis. The modular transformation $P_\mu \leftrightarrow Q_\mu$ changing sign on all the $2g$ cycles $(a_\mu, b_\mu)$ in a canonical homology basis corresponds to the interchange $\eta_\mu \leftrightarrow \xi_\mu$, which replaces each generator $S_\mu(z)$ by the inverse map, $S_\mu^{-1}(z)$. Thus, if we associate the surface $M_P^P$ with the set of generators $\{S_\mu\}$, the surface $M_Q^Q$ can be associated with the set of generators $\{S_\mu^{-1}\}$.

WS-CPT maps the holomorphic coordinate $z$ into $\tilde{z} = z^*$ and therefore relates the Riemann surface $M$ to the “mirror image” Riemann surface $\tilde{M}$, whose Schottky group

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8 Our conventions for spin structures and $\Theta$ functions differ from those of ref. [17] and can be found in ref. [9].

9 More precisely, the points on $M$ are in one-to-one correspondence with $S^2$ minus the limit set of the Schottky group, modulo the action of $G(M)$ [23].

23
\( G(\tilde{M}) \) is generated by the transformations \( \tilde{S}_\mu(z) = \tilde{S}_\mu(z^*) \equiv (S_\mu(z))^* \) having multipliers \( k_\mu^* \) and fixed points \( \eta_\mu^* \) and \( \xi_\mu^* \). Accordingly, we may identify

\[
(\tilde{M}_P^Q) \sim \{\tilde{S}_\mu^{-1}\} \quad \text{and} \quad (\tilde{M}_Q^P) \sim \{\tilde{S}_\mu\}. \tag{3.23}
\]

Starting from eq. (3.22) (and assuming for simplicity that \( 2\alpha, 2\beta \in \mathbb{Z} \)) it is straightforward to verify the following two identities, by using the explicit expressions for all relevant quantities given in refs. [9,17]:

\[
\langle N \prod_{l=1}^{N} e^{a_l \phi(\zeta_l)} \{S_\mu\} \{\alpha/\beta\} \rangle = (-1)^{P(\alpha,\beta)} \left( \langle \prod_{l=N}^{1} e^{+i\pi \Delta_l} e^{a_l \phi(\zeta_l)} \{\tilde{S}_\mu\} \{\alpha/\beta\} \rangle \right)^*, \tag{3.24}
\]

if \( \phi \) is anti-hermitean; and

\[
\langle N \prod_{l=1}^{N} e^{a_l \phi(z_l)} \{S_\mu\} \{\alpha/\beta\} \rangle = \left( \langle \prod_{l=N}^{1} e^{+i\pi \Delta_l} e^{a_l \phi(z_l)} \{\tilde{S}_\mu\} \{\alpha/\beta\} \rangle \right)^*, \tag{3.25}
\]

if \( \phi \) is hermitean.

The overall signs encountered on the right-hand sides of these two identities are opposite to those appearing in eqs. (3.18) and (3.19). The reason for this is that in eqs. (3.24) and (3.25) we compare \( M_P^Q \) with \( \tilde{M}_P^Q \), whereas in eqs. (3.18) and (3.19) we compared \( M_Q^P \) with \( \tilde{M}_Q^P \).

Under the modular transformation \( S_\mu \to S_\mu^{-1} \) that changes sign on all canonical homology cycles, the period matrix \( \tau \) and the prime form \( E \) remain unchanged, whereas the holomorphic one-forms \( \omega_\mu \) change sign, and hence so does the argument of the \( \Theta \) function in eq. (3.22). Since the \( \Theta \) function is even (odd) for even (odd) spin structures, this produces the extra factor \( (-1)^{P(\alpha,\beta)} \) needed to recover eqs. (3.18) and (3.19), where the “anomalous” sign for odd spin structures appeared in the case where \( \phi \) was hermitean.

In a similar way one can investigate what happens for the superghosts. In this case the formula (3.22) is modified (see ref. [11]) but it can be verified that eq. (3.25) remains valid and that the modular transformation \( S_\mu \to S_\mu^{-1} \) still gives rise to a minus sign for odd spin structures. Thus we recover eq. (3.19) also in this case.

### 3.4 WS-CPT for the Reparametrization Ghosts

The reparametrization ghosts constitute an important ingredient of the full CFT underlying the Lorentz-covariant formulation of any string theory. For closed string
theories we have to consider the combined \((b, c)\) and \((\bar{b}, \bar{c})\)-system and in this case there is a small complication regarding the hermiticity property (2.15), one of the two key ingredients of WS-CPT invariance.

The problem arises when considering the basic non-vanishing correlator on the sphere, \(\langle \bar{c}_{-1} \bar{c}_0 \bar{c}_1 c_{-1} c_0 c_1 \rangle_{g=0} = 0\). Since \(c_n^\dagger = c_{-n}\) and \(\bar{c}_n^\dagger = \bar{c}_{-n}\), the operator involved is explicitly anti-hermitean. Thus, in order for the correlation functions to satisfy the hermiticity property (2.15) it is necessary to postulate that this correlator is imaginary, for example

\[
\langle \bar{c}_{-1} \bar{c}_0 \bar{c}_1 c_{-1} c_0 c_1 \rangle_{g=0} = i. 
\]

(3.26)

With this convention the discussion of subsections (2.3) and (3.1) goes through without modification and the standard formulation (3.1) of WS-CPT invariance holds.

The convention (3.26) is rather inconvenient in practical calculations. Instead, one can adopt a different convention where the basic non-vanishing correlator is real,

\[
\langle \bar{c}_{-1} \bar{c}_0 \bar{c}_1 c_{-1} c_0 c_1 \rangle_{g=0} = 1. 
\]

(3.27)

The price one has to pay is that an unusual minus sign appears in the hermiticity formula (2.15), in the statements (2.16) and (2.20) of WS-CPT invariance on the sphere, and hence also in the identity (3.9). This implies that whenever we add a loop by means of the sewing formula (3.5) an extra minus sign enters into the statement of WS-CPT invariance.

In summary, the convention (3.27) leads to \((b, c, \bar{b}, \bar{c})\) correlation functions satisfying

\[
\langle \Phi_{\Delta_1}(z_1 = 0) \cdots \Phi_{\Delta_N}(z_N = 0) \rangle_{(M)}^{WS-CPT} = (\bar{M})^{g+1} (-1)^{\Delta_{1} + \cdots + \Delta_{N}} \left( \langle \hat{\Phi}_{\Delta_{N}}(\tilde{z}_N = 0) \cdots \hat{\Phi}_{\Delta_1}(\tilde{z}_1 = 0) \rangle_{\bar{g}}^{(\bar{M})} \right)^*. 
\]

(3.28)

4. **WS-CPT in String Theory and Space-Time Hermiticity.**

In this section we consider a generic string theory in \(D\)-dimensional Minkowski space-time, based on a CFT which is assumed to satisfy the WS-CPT theorem (3.1) (or eq. (3.28) if the reparametrization ghosts satisfy eq. (3.27)), and we study the physical meaning of the WS-CPT transformation (3.4) when applied to the vertex operator of a physical string state. We also show that WS-CPT invariance of the underlying CFT implies that the scattering \(T\)-matrix is formally hermitean.
We define the $T$-matrix element as the connected $S$-matrix element with certain normalization factors removed

$$\langle \rho_1; \ldots ; in | S | \ldots ; \rho_N; in \rangle_{\text{connected}} \over \prod_{i=1}^N (\langle \rho_i; in | \rho_i; in \rangle)^{1/2} =$$

$$i(2\pi)^D \delta^D (p_1 + \ldots - p_N) \prod_{i=1}^N (2p_i^0 V)^{-1/2} T(\rho_1; \ldots ; \ldots ; \rho_N) ,$$

where $N$ is the total number of external states, $p_i$ is the momentum of the $i$'th string state, all of them having $p_i^0 > 0$, and $V$ is the usual volume-of-the-world factor. The Minkowski metric is $\eta = \text{diag}(-1, +1, \ldots, +1)$.

The $g$-loop contribution to the $T$-matrix element is given by the Polyakov path integral. If, for the sake of being definite, we consider a heterotic superstring model, this is equivalent to the following operator formula

$$T^g(\rho_1; \ldots ; \ldots ; \rho_N) =$$

$$(-1)^{g-1} C_g \int_D \left( \prod_{I=1}^{3g-3+N} \text{d}^2 m I \right) \left( \prod_{I=1}^{3g-3+N} (\eta_{m I} | b) \prod_{i=1}^N c(z_i = 0) \right)^2 \times$$

$$\left( \prod_{A=1}^{N_{\text{PCO}}} \Pi(z_A \text{PCO} = 0) \right) \mathcal{V}^{(q_1)}(z_1 = 0) \ldots \mathcal{V}^{(q_N)}(z_N = 0) \rangle^M_g .$$

Here the integral is over the moduli space of $N$-punctured genus $g$ Riemann surfaces $M$. The $\{ m^I \}, I = 1, \ldots, 3g - 3 + N$, is a set of holomorphic coordinates on moduli space, defined on a domain $D \in \mathbb{C}^{3g-3+N}$, and $\eta_{m^I}$ is the Beltrami differential corresponding to $m^I$. We may think of $m^I$ as a global holomorphic coordinate, and of $D$ as a fundamental domain of the modular group, defined inside Teichmüller space (see also the Appendix).

By definition the correlator $\langle \ldots \rangle$ in eq. (4.2) includes the partition function and the summation over spin structures.

To each incoming string state $| \rho \rangle$ (each outgoing string state $\langle \rho |$) is associated a BRST-invariant vertex operator $\mathcal{V}^{(q)}_{\rho}(\rho)$ of conformal dimension $\Delta = \bar{\Delta} = 0$. Here $(q)$ denotes the picture (the superghost charge), and the number $N_{\text{PCO}}$ of Picture Changing Operators (PCOs) $\Pi$ is given by the integer

$$N_{\text{PCO}} = 2g - 2 - \sum_{i=1}^N q_i .$$

Since the PCOs have conformal dimension zero they do not depend on the choice of coordinate system. In eq. (4.2) they are evaluated in terms of some arbitrary local

\[10\] the details of which can be found in ref. [9].
holomorphic coordinates $z_A^{\text{PCO}}$. We assume that the PCO insertion points do not depend on the moduli (meaning that they have fixed values in a moduli-independent coordinate system).

The vertex operators are evaluated in terms of local holomorphic coordinates $z_i$, $i = 1, \ldots, N$. In ref. [9] it was shown that the vertex operator $W^{(q)}_{\rho}$ is related to $W^{(q)}_{|\rho\rangle}$ by

$$W^{(q)}_{\rho}(z_i = 0) = (-1)^{q+1} \hat{W}^{(q)}_{|\rho\rangle}(z_i = 0), \quad (4.4)$$

where for pictures of half odd integer $q$ (i.e. pictures describing space-time fermions) the phase factor $(-1)^{q+1}$ involves the choice of a sign which will not matter in what follows.

Since the on-shell vertex operators are dimension $\Delta = \bar{\Delta} = 0$ operators, we can interpret eq. (4.4) as

$$W^{(q)}_{\rho}(\tilde{z}_i = 0) = (-1)^{q+1} \left(W^{(q)}_{|\rho\rangle}(z_i = 0)\right)^{\text{WS-CPT}}. \quad (4.5)$$

Recalling that $c\bar{c}$ is anti-hermitean this is equivalent to

$$\mathcal{V}^{(q)}_{\rho}(\tilde{z}_i = 0) = (-1)^{q} \left(\mathcal{V}^{(q)}_{|\rho\rangle}(z_i = 0)\right)^{\text{WS-CPT}} \quad (4.6)$$

or

$$\mathcal{V}^{(q)}_{|\rho\rangle}(\tilde{z}_i = 0) = (-1)^{q} \left(\mathcal{V}^{(q)}_{\rho}(z_i = 0)\right)^{\text{WS-CPT}}. \quad (4.7)$$

Equation (4.5) gives the string theory interpretation of the WS-CPT transformation: Up to a picture dependent phase factor WS-CPT maps the vertex operator that describes a given incoming string state into the vertex operator that we use in the scattering formula (4.2) to describe the same string state when it is outgoing. This interpretation is consistent with the fact that WS-CPT maps $W^{(q)}_{\rho}$ into $\hat{W}^{(q)}_{|\rho\rangle}$. If the vertex operator $W^{(q)}_{|\rho\rangle}$ creates a state $|\rho\rangle = |p, \eta, \{\lambda\}\rangle$, where $p$ is the momentum, $\eta$ the “helicity” \footnote{In general, if the momentum points in the $(D - 1)$-direction, we may think of $\eta$ as the eigenvalues of the Lorentz generators $M_{12}, M_{34}, \ldots, M_{D-3,D-2}$.} and $\{\lambda\}$ a set of gauge and enumerative quantum numbers, then (as described in ref. [10]) the operator $\hat{W}^{(q)}_{|\rho\rangle}$, when acting on the conformal vacuum $|0\rangle$, creates the state $|-p, -\eta, \{-\lambda\}\rangle$ (up to a phase). This is what one expects from the fact that the new operator should describe an outgoing string state with the same quantum numbers as $|\rho\rangle$.

We now turn to consider the consequences of the WS-CPT theorem on the string scattering amplitudes. In ref. [9] it was noticed that eq. (4.4) gives the correct hermiticity
properties for the $S$-matrix elements at tree level. Indeed, unitarity requires that the tree-level $T$-matrix element is hermitean except when the momentum flowing in some intermediate channel happens to be on the mass-shell corresponding to some physical state in the theory. In field theory the imaginary part appears as a result of the $i\epsilon$-prescription present in the propagator that happens to be on-shell. In string theory the tree amplitudes are expressed by an integral over the Koba-Nielsen variables that will in general diverge but which can be treated by an appropriate analytic continuation of the invariant energy variables. The analytic continuation can be chosen so that the resulting amplitude has the correct physical poles.

The point of view of this paper is that the hermiticity of the tree-level $T$-matrix elements away from the resonances is a direct consequence of the WS-CPT theorem, as emphasized by Sonoda [7].

Indeed, we can now show that the WS-CPT theorem formulated in section 3 implies that the $T$-matrix elements, as given by eq. (4.2), are formally hermitean at any loop order, more precisely, that

$$(T^g(\rho_N; \ldots | \ldots ; \rho_1))^* = T^g(\rho_1; \ldots | \ldots ; \rho_N).$$

(4.8)

The reason why the proof is only formal is that the integral over the moduli in eq. (4.2) is not always convergent. This is very fortunate, since otherwise we would be in contradiction with unitarity. The divergent contributions to the modular integral offers a way out: When properly regularized, they should give rise to an imaginary part of the $T$-matrix, as required by $S$-matrix unitarity. Explicit regularizations accomplishing this in various cases, mainly at one-loop level, have been proposed [12,13,14,16,15].

To make the formal proof of eq. (4.8), we start from eq. (4.2), according to which

$$(T^g(\rho_N; \ldots | \ldots ; \rho_1))^* =$$

$$(-1)^{g-1} C_g^* \int_{D^*} \left( \prod_{I=1}^{3g-3+N} d^2m^I \right) \left( \prod_{I=1}^{3g-3+N} (\eta_{ml}|b) \prod_{i=1}^{N} c(z_i = 0) \right)^2 \times$$

$$\left( \prod_{A=1}^{N_{PCO}} \Pi(z_A^{PCO} = 0) \right) \chi^{(q_1)}_{\rho_1}(z_1 = 0) \cdots \chi^{(q_N)}_{\rho_N}(z_N = 0)$$

(4.9)

We can now use the WS-CPT theorem to obtain

$$(T^g(\rho_N; \ldots | \ldots ; \rho_1))^* = (-1)^{g+1} (-1)^{g-1} C_g \times$$

$$\int_{D^*} \left( \prod_{I=1}^{3g-3+N} d^2m^I \right) \langle (\chi^{(q_1)}_{\rho_1}(z_1 = 0))^{\text{WS-CPT}} \cdots (\chi^{(q_N)}_{\rho_N}(z_N = 0))^{\text{WS-CPT}} \times$$

$$\cdots$$
\[
\left( \prod_{A=N_{PCO}}^{1} \right) \left( \Pi(z_A^{PCO} = 0) \right)^{WS-CPT} \left( \prod_{I=1}^{3g-3+N} (\eta_{m^*}|b|) \prod_{i=1}^{N} c(z_i = 0) \right)^{2} \right)^{WS-CPT} \langle (M) \rangle_g.
\]

Here we assumed that the normalization constant \(C_g\) is real. This is only true if we use the convention (3.27) for the ghost correlators [9]. Therefore we also had to include the “anomalous” sign \((-1)^{g+1}\) that appears in the WS-CPT identity (3.28) for reparametrization ghosts. Since the WS-CPT transformation inverts the order of the ghost operators, this sign is cancelled when we put the ghost operators back into their original order. The PCOs are bosonic operators and can be rearranged without introducing any signs. If we further use eqs. (4.6) and (4.7), together with the fact that

\[
\left( \left( \Pi(z_A^{PCO} = 0) \right)^{WS-CPT} = -\Pi(\tilde{z}_A^{PCO} = 0) \right),
\]

we may rewrite eq. (4.10) as

\[
(T^g(\rho_0; \ldots | \ldots ; \rho_1))^* = (-1)^{\sum q_i} (-1)^{N_{PCO}} \times
\]

\[
(-1)^{g-1} C_g \int_{D^*} \left( \prod_{I=1}^{3g-3+N} \right) \left( \eta_{m^*}|b| \prod_{i=1}^{N} c(\tilde{z}_i = 0) \right) \left( \prod_{A=1}^{N_{PCO}} \Pi(\tilde{z}_A^{PCO} = 0) \right)^{2} \right)^{WS-CPT} \langle (\tilde{M}) \rangle_g.
\]

Here \((-1)^{N_{PCO}} \sum q_i = +1\) by eq. (4.3); and the PCO and reparametrization ghost factors may be moved around, so as to appear in the same places as in eq. (4.2), without introducing any extra signs.

By definition, the \(\tilde{z}_i\) and \(\tilde{z}_A^{PCO}\) are local holomorphic coordinates pertaining to the Riemann surface \(\tilde{M}\); As explained in greater detail in the appendix, a coordinate system \(\{m\}\) on moduli space, which assigns a Riemann surface \(M\) to each set of moduli \(\{m\}\), automatically defines another coordinate system on moduli space, where the mirror image Riemann surface \(\tilde{M}\) is assigned to the complex conjugate set of moduli, \(\{m^*\}\); and the complex conjugate of the Beltrami differential \(\eta_{m^*}\) on \(M\) is exactly the Beltrami differential \(\eta_{m^*}\) on \(\tilde{M}\). In summary, eq. (4.12) is nothing but the expression (4.2) for \(T^g(\rho_0; \ldots | \ldots ; \rho_N)\), merely written in terms of the coordinates \(\{m^*\}\), defined on the domain \(D^*\), rather than the \(\{m\}\), defined on the domain \(D\).

Since eq. (4.2) does not depend on which set of holomorphic coordinates we use to describe \(N\)-punctured genus \(g\) moduli space, this concludes our proof that the \(T\)-matrix is formally real at any loop order. Of course, the integral over moduli space is
not always convergent, and then the above proof breaks down. Indeed, we see that in order to recover unitarity it is essential that the integral over the moduli diverges in the kinematical regions where the $T$-matrix is required to develop an imaginary part. To handle this divergence, a regularization of the integral is needed, and it is exactly the failure of the regularization to be invariant under complex conjugation that restores the $S$-matrix unitarity.

**Appendix A: Mirror Image Riemann Surfaces**

In this appendix we give some details concerning mirror image Riemann surfaces.

### A.1 Moduli Space Generalities

A Riemann surface $M$ can be defined as a compact, real, oriented two-dimensional manifold, $\mathcal{M}$, endowed with a complex structure $J$. We may write $M = (\mathcal{M}, J)$ for short. A complex structure is a tensor field of rank $(1, 1)$, i.e. in terms of some real coordinate system $\xi^r$, $r = 1, 2$, it has real components $J^s_r(\xi)$. These are required to satisfy

$$J^s_r(\xi)J^s_r(\xi) = -\delta^s_r.$$  

(A.1)

Given a complex structure we may define complex, holomorphic coordinates as the solutions to Beltrami’s equation

$$J^s_r(\xi) \frac{\partial z}{\partial \xi^s} = i \frac{\partial z}{\partial \xi^r}.$$  

(A.2)

Without specifying any boundary conditions, eq. (A.2) does not determine the holomorphic coordinate $z$ uniquely. But it is rather easy to verify that if $z_1(\xi)$ and $z_2(\xi)$ are two solutions, then one depends holomorphically on the other, i.e.

$$\frac{\partial z_1}{\partial z_2^*} = 0.$$  

(A.3)

Thus, Beltrami’s equation specifies the holomorphic coordinate up to conformal coordinate transformations.

It is easy to see that the complex structure $J$, when evaluated in any coordinate system $z$ solving Beltrami’s equation, is simply given by the conformally invariant expressions

$$J_z^z = i \quad J_{\bar{z}}^{z} = -i \quad J_z^{\bar{z}} = J_{\bar{z}}^z = 0.$$  

(A.4)
Moduli space is the set of complex structures modulo diffeomorphisms. That is, two complex structures $I$ and $J$ are considered equivalent from the point of view of moduli space if and only if there exists a diffeomorphism $\Phi: \mathcal{M} \to \mathcal{M}$ such that (in terms of the coordinate system $\xi$)

$$\frac{\partial \Phi' (\xi)}{\partial \xi^r} I^s_r (\Phi(\xi)) = J^s_r (\xi) \frac{\partial \Phi^s (\xi)}{\partial \xi^{s'}} ,$$

or, in a convenient shorthand notation, $I^\Phi \sim J$.

As always, a diffeomorphism is required to be differentiable and invertible. If we consider compact Riemann surfaces with $N$ punctures, i.e. with marked points $P_i$, $i = 1, \ldots, N$, then the diffeomorphisms $\Phi$ are also required to keep these points fixed.

The inequivalent complex structures on a genus $g$ surface with $N$ punctures span a moduli space of real dimension $6g - 6 + 2N$. Local coordinates on moduli space can be introduced by assigning (in a differentiable way) a complex structure $J_m$ to each point $m = m^a$ belonging to some open subset of $\mathbb{R}^{6g - 6 + 2N}$,

$$m^a \to J^s_r (m^a; \xi) = (J_m)_r^s (\xi) ,$$

in such a way that $J_m$ and $J_{m'}$ are inequivalent unless $m = m'$.

If the complex structure $J$ appearing in Beltrami’s equation (A.2) depends on $m$, then so will the complex coordinate $z = z_m$ solving this equation. By differentiating eq. (A.2) with respect to $m^a$ (keeping $\xi$ fixed) we find the relation

$$-2 i (\eta_{m^a})_r^s (m; \xi) \frac{\partial z_m (\xi)}{\partial \xi^s} + J^s_r (m; \xi) \frac{\partial}{\partial \xi^s} \frac{\partial z_m (\xi)}{\partial m^a} - i \frac{\partial}{\partial \xi^r} \frac{\partial z_m (\xi)}{\partial m^a} = 0 \quad (A.7)$$

between the so-called quasiconformal vector field $\partial z_m (\xi)/\partial m^a$, measuring the change in the conformal coordinate, and the Beltrami differential

$$(\eta_{m^a})_r^s (m; \xi) \equiv \frac{-1}{2i} \frac{\partial}{\partial m^a} J^s_r (m; \xi)$$

that parametrizes the change in the conformal structure. If we evaluate eq. (A.7) in the holomorphic coordinates $z = z_m$, it reduces to

$$(\eta_{m^a})_z (z, \bar{z}) = - \frac{\partial}{\partial \bar{z}} \left( \frac{\partial z_m}{\partial m^a} (z, \bar{z}) \right) .$$

In string theory it is customary to introduce holomorphic coordinates on moduli space, i.e. choose $3g - 3 + N$ complex parameters $m^I$, with complex conjugates $m^* I$, such that

$$\frac{\partial z_m (\xi)}{\partial m^* I} = 0 . \quad (A.10)$$
Then we obtain the correct integration measure on moduli space by inserting into the path integral over the right-moving reparametrization ghosts $b = b_{zz}$ and $c = c^z$ the $3g - 3 + N$ conformally invariant factors

$$ (\eta_m \mid b) = \int \frac{d^2 z}{\pi} (\eta_m) \overline{z} (z, \overline{z}) b_{zz} (z) , \quad (A.11) $$

with similar factors for the left-movers.

So far we have only considered local coordinates on moduli space. To have a global description it is convenient to consider instead Teichmüller space, which is the space of complex structures modulo diffeomorphisms continuously connected to the identity. Unlike moduli space, Teichmüller space is actually a complex manifold (in the strict mathematical sense); moreover, it is simply connected and can be covered by a single global set of coordinates, which may even be chosen to be everywhere holomorphic [24]. Thus, the points of Teichmüller space are in one-to-one correspondence with the points of the open domain $\mathcal{O} \subset \mathbb{C}^{3g - 3 + N}$ where this coordinate is defined. By definition each point in Teichmüller space corresponds to a set of complex structures related to each other (like in eq. (A.5)) by diffeomorphisms continuously connected to the identity. Two different points $m$ and $m'$ in Teichmüller space will correspond to the same point in moduli space whenever the complex structures pertaining to one point are related to those pertaining to the other point by means of a diffeomorphism that is not continuously connected to the identity. The group of such diffeomorphisms (modulo the group of diffeomorphisms continuously connected to the identity) is called the group of modular transformations. It acts on the points of Teichmüller space and this allows us to identify moduli space with any subset $\mathcal{D}$ of $\mathcal{O}$ which is a fundamental domain of the group of modular transformations. \(^{12}\)

### A.2 Mirror image Riemann surfaces.

By taking the complex conjugate of the Beltrami equation (A.2) we see that if $z(\xi)$ is a holomorphic coordinate pertaining to the complex structure $J$, then

$$ \tilde{z}(\xi) \equiv (z(\xi))^* $$

(A.12)

is a holomorphic coordinate pertaining to the complex structure $\tilde{J}$ defined by

$$ \tilde{J}_r^s (\xi) \equiv -(J_r^s (\xi))^* = -J_r^s (\xi) , \quad (A.13) $$

\(^{12}\) The group of modular transformations also acts on the homology cycles, and it is often very convenient to describe a modular transformation by its action on a canonical homology basis. However, this is only a partial description, because there exist modular transformations (the so-called Torelli subgroup) which act trivially on the homology basis.
where the last equality sign follows from the fact that the $J_r^s$ are real-valued functions.

We call the complex structure $\tilde{J}$ the \textit{mirror image} of $J$, and the Riemann surface $\tilde{M} = (\mathcal{M}, \tilde{J})$ the \textit{mirror image} of the Riemann surface $M = (\mathcal{M}, J)$. By construction all holomorphic coordinates on $M$ are anti-holomorphic functions of the holomorphic coordinates on $\tilde{M}$. The two Riemann surfaces describe different points in moduli space unless $J \cong \tilde{J}$, i.e. unless the two complex structures are related by a diffeomorphism $\Phi$, as in eq. (A.5). Obviously, $M \to \tilde{M}$ is a one-to-one map of moduli space onto itself, since performed twice it is just the identity map.

By multiplying both sides of eq. (A.5) with minus one we see that if the complex structures $I$ and $J$ are related by the diffeomorphism $\Phi$, so are the complex structures $\tilde{I}$ and $\tilde{J}$, and vice versa. In short,

$$ I \cong J \iff \tilde{I} \cong \tilde{J}. \quad (A.14) $$

This means that, given a local coordinate $m \to J_m$ on moduli space, as described in the previous subsection, we can define another equally good local coordinate by the map $m \to \tilde{J}_m$.

When we consider the dependence of the complex structure on a holomorphic set of moduli, we have

$$ J_r^s(\xi) = J_r^s(m^I, m^{*I}; \xi). \quad (A.15) $$

Since $J_r^s$ is real-valued it has to depend on both $m^I$ and $m^{*I}$. But then, since $\tilde{J}$ is defined from $J$ by means of complex conjugation, as in eq. (A.13), it is a function of $m^{*I}$ and $m^I$, rather than $m^I$ and $m^{*I}$:

$$ \tilde{J}_r^s(\xi) = -(J_r^s)^*(m^{*I}, m^I; \xi) \equiv \tilde{J}_r^s(m^{*I}, m^I; \xi). \quad (A.16) $$

In a convenient shorthand notation, the map (A.15) is represented by $m \to J_m$ and the map (A.16) by $m^* \to \tilde{J}_{m^*}$.

In summary, the situation is the following: Given a local holomorphic coordinate system $\mathcal{D} \ni m \to J_m$, then the map $J \to \tilde{J}$ relating a Riemann surface to its mirror image automatically defines a new local coordinate system $\mathcal{D}^* \ni m^* \to \tilde{J}_{m^*}$. If the Riemann surface $M$ is described by the point $m^I \in \mathcal{D}$ in the first set of coordinates, then the mirror image $\tilde{M}$ is described by the point $m^{*I} \in \mathcal{D}^*$ in the second set of coordinates.

By taking the complex conjugate of eq. (A.8) we find

$$ ((\eta_{m^I})_r^s(m^I, m^{*I}; \xi))^* = \left(\frac{-1}{2i} \frac{\partial}{\partial m^I} J_r^s(m^I, m^{*I}; \xi)\right)^* = \frac{-1}{2i} \frac{\partial}{\partial m^I} \tilde{J}_r^s(m^{*I}, m^I; \xi) = (\eta_{m^{*I}})_r^s(m^{*I}, m^I; \xi). \quad (A.17) $$

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which expresses the fact that the complex conjugate of the Beltrami differential pertaining to $m^I$ in the coordinate system $\{m^I\}$ is the Beltrami differential pertaining to $m^*^I$ in the coordinate system $\{m^*^I\}$.

So, if we imagine a family of local coordinates $\{m_1^I\}, \{m_2^I\}, \ldots$, which covers exactly one fundamental domain of the modular group inside Teichmüller space, then $\{m_1^*^I\}, \{m_2^*^I\}, \ldots$ is another family of local coordinates which will also cover exactly one fundamental domain. This follows from the fact that the map $M \to \tilde{M}$ relating a Riemann surface to its mirror image is a one-to-one map of moduli space onto itself.

References

G. Lüders and B. Zumino, Phys.Rev. 106 (1957) 345;


