QCD Thermodynamics with Improved Actions

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The thermodynamics of the SU(3) gauge theory has been analyzed with tree level and tadpole improved Symanzik actions. A comparison with the continuum extrapolated results for the standard Wilson action shows that improved actions lead to a drastic reduction of finite cut-off effects already on lattices with temporal extent $N_\tau = 4$. Results for the pressure, the critical temperature, surface tension and latent heat are presented. First results for the thermodynamics of four-flavour QCD with an improved staggered action are also presented. They indicate similarly large improvement factors for bulk thermodynamics.

1. Why improved actions are expected to improve the QCD thermodynamics

Although the plasma phase of QCD does in many respects show strong non-perturbative properties (screening masses, poor convergence of perturbation theory,...), bulk thermodynamic observables like energy density and pressure do approach the non-interacting ideal gas limit at high temperatures ($T$) where they are expected to receive large contributions from high momentum modes, $\langle \text{avg. momentum} \rangle \sim T$. In lattice calculations $T$ is determined through the temporal extent $N_\tau$ of the lattice and the cut-off $a^{-1}$, i.e. $T \equiv 1/N_\tau a$. In the ideal gas limit the relevant momenta are therefore of the order of the cut-off, where lattice and continuum dispersion relations differ strongly from each other. Indeed this is the origin of the well-known discrepancy between the energy density of an ideal gas calculated on a finite lattice ($\epsilon(N_\tau)$) and in the continuum ($\epsilon_{SB}$),

$$\epsilon(N_\tau) = \begin{cases} 1 + \frac{30}{63} \cdot \frac{\pi^2}{N_\tau} + \mathcal{O}(N_\tau^{-4}) , \text{ Wilson} \\ 1 + c_I \cdot \frac{\pi^2}{N_\tau} + \mathcal{O}(N_\tau^{-6}) , \text{ Symanzik} \end{cases}$$

The standard Wilson action leads to nearly 50% corrections on a lattice with temporal extent $N_\tau = 4$ while the cut-off dependence is drastically reduced in the case of Symanzik improved actions which eliminate the leading $\mathcal{O}(N_\tau^{-2}) \equiv \mathcal{O}((aT)^2)$ corrections. In the case of the (1,2)-Symanzik action this is less than 2% for $N_\tau = 4$, i.e. $c_I = 0.044$. This clearly demonstrates the significance of improved actions in the ideal gas limit ($T \to \infty$).

In a first exploratory analysis we have shown that the improvement found analytically in the ideal gas limit also persists at finite temperature and does seem to be important even close to $T_c$ [1]. Here we will present further evidence that tree level and tadpole improved actions do lead to a significant reduction of finite cut-off effects even at $T_c$ [2–4].

2. SU(3) Thermodynamics

We have analyzed the thermodynamics of the SU(3) gauge theory using $\mathcal{O}(a^2)$ improved tree level and tadpole improved actions, which have been constructed by adding to the Wilson plaquette action, $S^{(1,1)}$, an appropriately weighted contribution from planar (1,2) or (2,2) loops,

$$S^{(1,2)} = \sum_{x,\mu > \nu} \frac{5}{3} \left(1 - \frac{1}{N} \text{ Re } \text{Tr} \mu \nu(x) \right)$$

$$- \frac{1}{6u_0} \left(1 - \frac{1}{2N} \text{ Re } \text{Tr} \mu \nu(x) + \mu \nu(x) \right)$$

$$S^{(2,2)} = \sum_{x,\mu > \nu} \frac{4}{3} \left(1 - \frac{1}{N} \text{ Re } \text{Tr} \mu \nu(x) \right)$$

$$- \frac{1}{48} \left(1 - \frac{1}{N} \text{ Re } \text{Tr} \mu \nu(x) \right)$$

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Table 1
Critical temperature in units of $\sqrt{\sigma}$ on lattices with temporal extent $N_{\tau} = 4$. Infinite volume extrapolations of $\beta_c$ for the Symanzik actions are based on lattices with $N_{\sigma} = 16, 24$ and $32$.

<table>
<thead>
<tr>
<th>action</th>
<th>$N_{\sigma}$</th>
<th>$\beta_c$</th>
<th>$T_c/\sqrt{\sigma}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>standard Wilson</td>
<td>$\infty$</td>
<td>$5.69254 (24)$</td>
<td>$0.5983 (30)$</td>
</tr>
<tr>
<td>(2,2) Symanzik (tree level)</td>
<td>$24$</td>
<td>$4.3995 (2)$</td>
<td>$0.624 (4)$</td>
</tr>
<tr>
<td>(1,2) Symanzik (tree level)</td>
<td>$\infty$</td>
<td>$4.07297 (28)$</td>
<td>$0.631 (3)$</td>
</tr>
<tr>
<td>(1,2) Symanzik (tadpole)</td>
<td>$\infty$</td>
<td>$4.35228 (39)$</td>
<td>$0.635 (3)$</td>
</tr>
</tbody>
</table>

where the factor $u_0$ appearing in the definition of the (1,2)-Symanzik action denotes the tadpole improvement factor defined in terms of the self-consistently determined plaquette expectation value [7].

2.1. Bulk thermodynamics
Bulk thermodynamic quantities like the energy density ($\epsilon$) or pressure ($p$) calculated with the Wilson action on lattices of size $N_\sigma^3 \times N_{\tau}$ with $N_{\tau} = 4, 6$ and $8$ [5] show a strong cut-off dependence in the plasma phase. A first analysis with the (2,2)-Symanzik action [1] has shown that this cut-off dependence gets drastically reduced already on a $N_{\tau} = 4$ lattice. The work presented in Ref. [1] has been extended in several ways. On lattices with temporal extent $N_{\tau} = 4$ we have analyzed now also the tree-level improved (1,2)-Symanzik action, which shows even smaller cut-off distortions in the high temperature limit than the (2,2) action. In addition we have studied the influence of tadpole improvement in the case of the (1,2)-Symanzik action. In all cases we have determined the temperature scale non-perturbatively through the calculation of the string tension on symmetric ($N_\sigma^2$) lattices, $T/T_c \equiv \sqrt{\sigma(a)}(\beta_c)/\sqrt{\sigma(a)(\beta)}$. From this calculation we also obtain $T_c$ in units of $\sqrt{\sigma}$ for the different actions. These are given in Table 1 together with results for the Wilson action. In that case $T_c/\sqrt{\sigma}$ has also been calculated on lattices with $N_{\tau}$ varying between 4 and 12 and extrapolated to the continuum limit. This gave $T_c/\sqrt{\sigma} = 0.629 \pm 0.003$ [5]. On $N_{\tau} = 4$ lattices the results obtained with improved actions are thus much closer to this extrapolated value than those obtained with the Wilson action.

The pressure has been calculated by integrating differences of the action densities obtained on $16^4$ and $16^3 \times 4$ lattices (in the case of the (2,2) action we used $24^4$ and $24^3 \times 4$ lattices) following the standard procedures [1,5]. Results are shown in Figure 1.

![Figure 1](image.png)

Figure 1. Pressure of the SU(3) gauge theory on lattices with temporal extent $N_{\tau} = 4$. The solid line shows the continuum extrapolation obtained from the standard Wilson action. The dots give results from a calculation with a perfect action on a $12^3 \times 3$ lattice [6]. The arrow indicates the continuum ideal gas value.

2.2. Surface tension and latent heat
The success of improved actions for the calculation of bulk thermodynamics even at temperatures close to $T_c$ naturally leads to the question whether these actions also do lead to an improvement at $T_c$. This is, of course, a highly non-perturbative regime. However, observables like the latent heat ($\Delta \epsilon$) and the surface tension ($\sigma_I$), which characterize the discontinuities at the first
order deconfinement phase transition in a $SU(3)$
gauge theory, do depend on properties of the low
as well as the high temperature phase. As the lat-
ter is largely controlled by high momentum modes
it may be expected that some improvement does
result even from tree level improved actions.

At the $SU(3)$ deconfinement transition $\Delta \epsilon$ and
$\sigma_f$ have been studied on lattices up to temporal
extent $N_\tau = 6$ [8,9]. A strong cut-off dependence
has been found when comparing calculations for
$N_\tau = 4$ and 6. For this reason an extrapolation to
the continuum limit has so far not been possible
for these observables.

We have extracted $\sigma_f$ [3] from the probability
distribution of the absolute value of the Polyakov
loop following the analysis presented in Ref. [9].
The probability distribution at the minimum is propor-
tional to

$$ P(|L|) \sim \exp(-[f_1 V_1 + f_2 V_2 + 2\sigma_f A]/T) \quad (3) $$

where $f_i$ denotes the free energy in the phase $i,$
and $V_i$ is the volume occupied by that phase and
$A$ denotes the interface area of the finite system.
From the depth of the minimum one thus can ex-
tract the surface tension. The distribution func-
tions for the tadpole improved actions for three
different lattice sizes are shown in Figure 2.

In Table 2 we give results for $\sigma_f$ on the largest
lattices considered. Clearly the surface tension
extracted from simulations with improved actions
on lattices with temporal extent $N_\tau = 4$ are sub-
stantially smaller than corresponding results for
the Wilson action. In fact, they are compatible
with the $N_\tau = 6$ results for the Wilson action.

The latent heat is calculated from the discon-
tinuity in $(\epsilon - 3P).$ This in turn is obtained from
the discontinuity in the various Wilson loops en-
tering the definition of the improved actions,

$$ \frac{\Delta \epsilon}{T_c^4} = \frac{1}{6} \left( \frac{N_\tau}{N_\sigma} \right)^3 \left( a \frac{d\beta}{da} \right) \left( \langle \tilde{S} \rangle_+ - \langle \tilde{S} \rangle_- \right), \quad (4) $$

with $\tilde{S} \equiv S - dS/da.$

The difference of action expectation values at
$\beta_c$ is obtained by calculating these in the two co-
existing phases at $\beta_c.$ In the time histories of the
Polyakov loop values we have cut out the trans-
ition regions in order to classify configurations
belonging to either of the two phases [8].

In order to extract the latent heat one does
still need the $\beta$-function entering the definition of
$\Delta\epsilon/T_c^4$ in Eq. 4. The necessary relation $a(\beta)$
has been obtained from a calculation of $\sqrt{\sigma a}$ (im-
proved actions) or a determination of $T_c a$ (Wilson
action). Results for $\Delta\epsilon/T_c^4$ are summarized in Ta-
ble 2. Here we also give the result obtained with
the 1-loop $\beta$-function.

3. Four-flavour QCD with an improved
staggered action

In the fermionic sector of QCD the influence
of a finite cut-off on bulk thermodynamic observ-
ables is known to be even larger than in the pure
gauge sector. For instance, in the staggered for-
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Table 2
Surface tension and latent heat for the three improved actions and the Wilson action. Results for the Wilson action are based on data from [9] using the non-perturbative $\beta$-function calculated in [5].

<table>
<thead>
<tr>
<th>action</th>
<th>$V_\sigma$</th>
<th>$N_\tau$</th>
<th>$\sigma_1/T_c^3$</th>
<th>$\Delta\epsilon/T_c^4$</th>
<th>$\left(\Delta\epsilon/T_c^4\right)_{\text{pert.}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>standard Wilson</td>
<td>$24^2 \times 36$</td>
<td>4</td>
<td>0.0300 (16)</td>
<td>2.27 (5)</td>
<td>4.06 (8)</td>
</tr>
<tr>
<td></td>
<td>$36^2 \times 48$</td>
<td>6</td>
<td>0.0164 (26)</td>
<td>1.53 (4)</td>
<td>2.39 (6)</td>
</tr>
<tr>
<td>(1,2) Symanzik (tree level)</td>
<td>$32^4$</td>
<td>4</td>
<td>0.0116 (23)</td>
<td>1.57 (12)</td>
<td>2.28 (8)</td>
</tr>
<tr>
<td>(1,2) Symanzik (tadpole)</td>
<td>$32^4$</td>
<td>4</td>
<td>0.0125 (17)</td>
<td>1.40 (9)</td>
<td>1.94 (8)</td>
</tr>
</tbody>
</table>

with

\[
A[U]_{ij} = \sum_\mu \left( U_{i,\mu} \delta_{i,j-\mu} - U_{i-\mu,\mu}^{\dagger} \delta_{i,j+\mu} \right)
\]

\[
B[U]_{ij} = \sum_\mu \left( U_{i,\mu} U_{i+\mu,\mu} U_{i+2\mu,\mu} \delta_{i,j-3\mu} - U_{i-3\mu,\mu} U_{i-2\mu,\mu} U_{i-\mu,\mu} \delta_{i,j+3\mu} \right)
\]

With this action the overall cut-off distortion of the ideal gas limit on a $16^3 \times 4$ lattices reduces to about 20%. We have performed simulations for two quark masses, $ma = 0.05$ and 0.1 [4]. Like in the pure gauge sector the improvement is visible already close to $T_c$. For instance, the energy density stays close to the ideal gas limit. Although we observe an overshooting of the ideal gas limit close to $T_c$ for the non-zero quark masses considered by us this feature does seem to disappear in the "chiral limit" [4].

4. Conclusions

Thermodynamic observables of SU(3) gauge theory and QCD studied with improved gauge and fermion actions show a drastic reduction of the cut-off dependence in the high temperature limit as well as at $T_c$. The major improvement effect is already obtained with tree level improved actions. Tadpole improvement plays a minor role even close to $T_c$.

REFERENCES