Dynamical Wilson fermions and the problem of the chiral limit in compact lattice QED†

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Abstract

We compare the approach to the chiral transition line $\kappa_c(\beta)$ in quenched and full compact lattice QED with Wilson fermions within the confinement phase, especially in the pseudoscalar sector of the theory. We show that in the strong coupling limit ($\beta = 0$) the quenched theory is a good approximation to the full one, in contrast to the case of $\beta = 0.8$. At the larger $\beta$-value the transition in the full theory is inconsistent with the zero–mass limit of the pseudoscalar particle, thus prohibiting the definition of a chiral limit.

1 Introduction and model description

Chiral symmetry as a major concept in continuum quantum field theory has remained a problematic topic in lattice gauge theories over the years. It is well-known, that for Wilson fermions chiral symmetry is explicitly broken in QCD and QED on the lattice [1, 2]. Hopefully, it can be recovered by fine-tuning of parameters in the continuum limit. Then some line $\kappa_c(\beta)$ in the phase diagram is associated with the chiral limit of the theory. On the other hand, for non-vanishing lattice spacing only a partial restoration of chiral symmetry at $\kappa = \kappa_c(\beta)$ is possible with Wilson fermions [3, 4]. It is still an open question, how this mechanism of partial symmetry restoration should eventually be integrated into the general conception of spontaneously broken chiral symmetry. One cannot exclude that the breakdown of some other symmetry group governs the dynamics of the transitions at $\kappa_c(\beta)$ (e.g., [5]). Viewed in this light the vanishing of the pseudoscalar ‘pion’ mass $m_\pi$.

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for $\kappa \rightarrow \kappa_c(\beta)$ is a necessary but not sufficient condition for probing the chiral limit. Another point which sharpened the look on the chiral limit in QCD [6, 7] is the discussion of ‘enhanced logs’ due to quenching, demonstrating the role of dynamical fermions in chiral properties of the theory.

In this letter we are concerned with the behavior of fermionic observables close to $\kappa_c(\beta)$ in the confinement phase of compact QED. We confront the full theory with its valence approximation. Though the non-perturbative regime of compact QED is itself an interesting subject the close analogy with non-Abelian gauge theories makes it also a valuable test ground for QCD with Wilson fermions.

In the rest of the section we will introduce the model and give the main notations. Starting point is the partition function of 4$d$ compact QED which reads as follows

$$Z_{\text{QED}} = \int [dU][d\bar{\psi}d\psi] \, e^{-S_W(U, \bar{\psi}, \psi)} ,$$

where $S_W(U, \bar{\psi}, \psi)$ denotes the standard Wilson lattice action

$$S_W = S_G(U) + S_F(U, \bar{\psi}, \psi)$$

consisting of the plaquette action

$$S_G(U) = \beta \cdot \sum_{x,\mu>\nu} (1 - \cos \theta x;\mu\nu) ,$$

and the fermionic part $S_F(U, \bar{\psi}, \psi)$

$$S_F = \sum_{f=1}^{2} \sum_{x,y} \bar{\psi}_x^f \mathcal{M}_{xy} \psi_y^f$$

$$\mathcal{M}_{xy} \equiv \hat{1} - \kappa \cdot [\delta_{y,x+\hat{\mu}} \cdot (\hat{1} - \gamma_\mu) \cdot U_{x\mu} + \delta_{y,x-\hat{\mu}} \cdot (\hat{1} + \gamma_\mu) \cdot U_x^{\dagger}_{x-\hat{\mu},\mu}] ,$$

with $\beta = 1/g^2_{\text{bare}}$, and $U_{x\mu} = \exp(i\theta_{x\mu})$, $\theta_{x\mu} \in (-\pi, \pi]$ represent the link fields. The plaquette angles $\theta_{x;\mu\nu}$ in eq.(3) are given by $\theta_{x;\mu\nu} = \theta_{x;\mu} + \theta_{x+\hat{\mu},\nu} - \theta_{x+\hat{\nu},\mu} - \theta_{x;\nu}$. In the fermionic part of the action $\mathcal{M}_{xy}$ denotes Wilson’s fermionic matrix with the hopping parameter $\kappa$ and the flavor–index $f$.

The phase diagram of this model has been studied in [8]. Within the region of values $0 \leq \kappa \leq 0.30$ the existence of four phases has been established. The line $\kappa_c(\beta)$ separates both the confinement phase ($0 \leq \beta < \beta_0$ with $\beta_0 \simeq 1.01$ ) and the Coulomb phase ($\beta > \beta_0$) from two upper phases (which we called the 3rd phase at weak and the 4th phase at strong coupling). At $\beta = 0.8$ it was observed that the scalar condensate $\langle \bar{\psi}\psi \rangle$ has a discontinuity at $\kappa_c$, therefore we called the transition at this point a 1st order transition. However, the $\kappa$–dependence of the pseudoscalar observables was not studied in detail.

The fermionic observables to be discussed are the pion norm
\[ \langle \Pi \rangle = \frac{1}{4V} \cdot \langle \text{Tr} (M^{-1} \gamma_5 M^{-1} \gamma_5) \rangle_G , \quad (5) \]

being a good indicator for small eigenvalues of the fermionic matrix and the mass of the pseudoscalar particle \( m_\pi \), which is extracted from the pseudoscalar zero-momentum correlator

\[ \Gamma(\tau) = -\frac{1}{N_s^6} \cdot \sum_{\bar{x},\bar{y}} \langle \bar{\psi} \gamma_5 \psi(\tau, \bar{x}) \cdot \bar{\psi} \gamma_5 \psi(0, \bar{y}) \rangle \]

\[ \equiv \frac{1}{N_s^6} \cdot \sum_{\bar{x},\bar{y}} \left\langle \left\{ \text{Sp} (M^{-1} \gamma_5 M^{-1} \gamma_5) - \text{Sp} (M^{-1} \gamma_5) \cdot \text{Sp} (M^{-1} \gamma_5) \right\} \right\rangle_G . \quad (6) \]

In eqs.(5,6) \( \langle \rangle_G \) indicates averaging over gauge field configurations, and \( V = N_\tau \cdot N_s^3 \) is the number of sites. \( \text{Sp} \) means the trace with respect to the Dirac indices. Other observables like \( \langle \rho_{\text{mon}} \rangle \) – the density of DeGrand-Toussaint monopoles [9] – and the photon correlator have also been determined.

Formally we define the bare fermion mass parameter \( m_q \) by

\[ m_q = \frac{1}{2} \left( \frac{1}{\kappa} - \frac{1}{\kappa_c(\beta)} \right) . \quad (7) \]

For the determination of the \( \kappa_c \) see below.

In this work we discuss data from an \( 8^3 \times 16 \) lattice at two values of \( \beta \) within the confinement phase. For the production of dynamical gauge field configurations we employed a CRAY-T3D implementation of the Hybrid Monte Carlo (HMC) method. A detailed presentation of algorithmic issues, like the tuning of the HMC parameters when approaching \( \kappa_c \) is deferred to [10].

2 Effects of dynamical Wilson fermions

First, we shall discuss the behavior of the bulk observable \( \langle \Pi \rangle \) when approaching the line \( \kappa_c(\beta) \) at fixed \( \beta \) within the range \( 0 \leq \beta < \beta_0 \) (confinement phase). As in our previous work for representative \( \beta \)-values we have chosen \( \beta = 0.8 \) and the strong coupling limit \( \beta = 0 \). In the latter limit the comparison with analytical results is possible (e.g., [3, 11]).

Provided the pseudoscalar mass vanishes for \( m_q \to 0 \) it will yield the dominant contribution to the pion norm \( \langle \Pi \rangle \sim 1/m_\pi^2 \). In case of a PCAC-like relation between \( m_\pi \) and \( m_q \) the pion norm can be expressed in the following form

\[ \langle \Pi \rangle = \frac{C_0}{m_q} + C_1 , \quad m_q \to 0 , \quad (8) \]
where \( C_0 > 0 \) – up to a factor – is the subtracted chiral condensate [12] (see also [13]). \( C_1 = \langle \Pi_m \rangle \) is the contribution of the massive modes.

In Figure 1 data for the quenched and dynamical cases at \( \beta = 0 \) are presented to view. The quenched approximation follows the \( m_q \)-dependence of the full pion norm \( \langle \Pi \rangle \) very well, even quantitatively. As \( m_q \to 0^+ \) the singularity of \( \langle \Pi \rangle \) at \( \beta = 0 \) is well described in both cases by eq.(8) with values of \( C_0, C_1 \) listed in Table 1. Note that \( C_0 \) and \( C_1 \) differ somewhat for the quenched and dynamical cases and could not be forced to coincide by shifting \( \kappa_c \). The quenched and dynamical theories nevertheless exhibit the same functional dependence on \( m_q \) when \( m_q \to 0 \).

For \( \kappa \gtrsim \kappa_c \) the averages of fermionic bulk observables in the quenched approximation become poorly defined due to large fluctuations caused by ‘exceptional’ configurations [13, 14]. In the theory with dynamical fermions we also observe increasing fluctuations of, e.g., \( \Pi \) when \( \kappa \) is tuned towards \( \kappa_c \) from below. We were able to proceed to \( \kappa = 0.242 \ (m_q \sim 0.0253) \) at \( \beta = 0 \), before being faced with serious problems with the acceptance rate of the HMC method.

The situation changes drastically when the gauge coupling \( \beta \) is increased. By examining time histories of \( \Pi \) and other observables at \( \beta = 0.8 \) we observe the formation of metastable states in a ‘critical’ region around \( \kappa_c \) in the presence of dynamical fermions. Figure 2a illustrates a clearly double peaked distribution of \( \Pi \) close to \( \kappa_c \) for the case of dynamical fermions.

As an example concerning the behavior of other observables, the dependence of \( \langle \rho_{mon} \rangle \) on \( \kappa \) is depicted in Figure 2b. The evolution of \( \langle \rho_{mon} \rangle \) resembles the situation of the confinement–deconfinement transition at \( \beta_0 \) for a sufficiently small fixed \( \kappa \) and varying \( \beta \) [8].

Measurements of the effective photon energy extracted from plaquette–plaquette correlators for non-zero momentum confirm that in the case of dynamical fermions the system undergoes a confinement–deconfinement transition at \( \kappa_c \) (see [8]). With increasing \( \kappa \) the effective energy of the photon rapidly decreases around \( \kappa_c \) and becomes well consistent with the lattice dispersion relation for a zero–mass photon.

We used further ‘order parameters’ to determine \( \kappa_c \) and to make sure that there are no other transition points different from that within the investigated \( \kappa \)-range. For example, the variance \( \sigma^2(\Pi) \) which is a suitable parameter to locate the line \( \kappa_c(\beta) \) within the Coulomb phase [8] peaks at the same \( \kappa_c \).

In Figure 3 as counterpart of Figure 1 we confront the dependence of \( \langle \Pi \rangle \) on \( m_q \) in the quenched approximation with the corresponding data when dynamical fermions are taken into account. The general features of the valence approximation at \( \beta = 0.8 \) are the same as at \( \beta = 0 \). \( \langle \Pi \rangle \) has a singularity for \( m_q \to 0^+ \) described very well by eq.(8) (dashed line in Figure 3) with parameters \( C_0; C_1 \) given in Table 1. However, the effect of the fermionic determinant is now seen as a qualitative change in the behavior of the pion norm, which does not behave as \( \sim 1/m_q \) but rather exhibits a finite discontinuity accompanied by a metastable behavior (cf. Figure 2a) around \( \kappa_c \). In the quenched theory averages of \( \Pi \) become
statistically not well-defined for $\kappa \gtrsim \kappa_c$, as in the case of $\beta = 0$, while in the dynamical case at $\beta = 0.8$ there is no problem to go beyond $\kappa_c$ (i.e., into the 3rd phase [8]). The dependence of $\langle \Pi \rangle$ on $\kappa$ (respectively $m_q$) is not symmetric around $\kappa_c$.

To substantiate the emerging picture at $\beta = 0$ and $\beta = 0.8$ we will discuss the evolution of the pseudoscalar mass $m_\pi$ when $\kappa \to \kappa_c$. We present in Figure 4 the dependence of $m_\pi^2$ on $\kappa$ for the full and quenched theories on an $8^3 \times 16$ lattice at $\beta = 0$. The quenched data for $\kappa$ very close to $\kappa_c$ are obtained by an improved estimator of $m_\pi$ [13] in order to increase the signal-to-noise ratio. Since the existence of a massless pseudoscalar particle is predicted by strong coupling arguments [3] we extrapolate our dynamical data for $m_\pi^2$ linearly to zero in order to determine $\kappa_c(\beta = 0)$ as we have done in the quenched case [13]. In Table 1 we list the values of $\kappa_c$ obtained for the dynamical and the quenched cases on an $8^3 \times 16$ lattice. The extrapolated values of $\kappa_c$ for the quenched case are well consistent with the prediction at strong coupling [3]. In both, quenched and dynamical cases we observe the following dependence of $m_\pi^2$ on the hopping parameter when approaching $\kappa_c$ from below

$$m_\pi^2 \sim \left(1 - \frac{\kappa}{\kappa_c}\right)^2, \quad \kappa \leq \kappa_c,$$

which in this limit transforms into a PCAC–like relation between $m_\pi^2$ and the bare fermion mass $m_q$:

$$m_\pi^2 = B \cdot m_q, \quad m_q \to 0^+.$$  

The corresponding slopes $B$ for the quenched and full theories coincide within the errorbars (see Table 1). Thus, Figure 4 suggests a zero-mass pseudoscalar particle to exist. However, this is a necessary but not a sufficient prerequisite for the definition of the chiral limit.

<table>
<thead>
<tr>
<th>$\beta = 0$</th>
<th>$\kappa_c$</th>
<th>$B$</th>
<th>$C_0$</th>
<th>$C_1$</th>
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<td>4.87(5)</td>
<td>0.895(1)</td>
<td>0.88(1)</td>
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<tr>
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<td>4.91(4)</td>
<td>0.996(2)</td>
<td>0.80(1)</td>
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<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>dynamical</td>
<td>0.1832(3)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>quenched</td>
<td>0.2171(1)</td>
<td>3.42(3)</td>
<td>0.94(2)</td>
<td>0.72(1)</td>
</tr>
</tbody>
</table>

Table 1: Compilation of different parameters (see text) for the dynamical and quenched theories at $\beta = 0$ and $\beta = 0.8$ on an $8^3 \times 16$ lattice.

The corresponding behavior of $m_\pi^2$ at $\beta = 0.8$ for quenched and dynamical
fermions is plotted in Figure 5. From the inset of Figure 5 it can be seen, that as long as the quenched theory is considered the situation is fully compatible with eq.(9). Thus eq.(10) holds as in the case of $\beta = 0$. Concerning dynamical fermions, again the situation at $\beta = 0.8$ appears to be in sharp contrast to the $\beta = 0$ and to the quenched cases, as could be expected from the properties of the pion norm. By approaching the ‘critical’ value $\kappa_c(\beta)$ from below with dynamical fermions, the $\kappa$–dependence of $m^2_\pi$ is not linear anymore, i.e. is not compatible with eq.(9). Moreover, close to $\kappa_c(\beta)$ $m_\pi$ has a comparatively large finite minimal value which would imply that a zero-mass pseudoscalar particle is not contained in the spectrum of the theory at this particular coupling. Increasing $\kappa$ beyond $\kappa_c$ the pseudoscalar mass starts to rise again. Note that in the vicinity of $\kappa_c$ the dependence of the pseudoscalar mass on $\kappa$ is different for $\kappa > \kappa_c$ and $\kappa < \kappa_c$, in accordance with the discussion of the pion norm before.

3 Conclusions and discussion

We have studied the approach to $\kappa_c(\beta)$ for two $\beta$–values within the confinement phase of the compact lattice QED with Wilson fermions comparing the full theory with its quenched approximation. We have shown the importance of vacuum polarization effects due to dynamical fermions in the context of the chiral limit.

In the strong coupling limit $\beta = 0$ the main effect of dynamical fermions seems to be a renormalization of the ‘critical’ value $\kappa_c$, $\kappa_{c,\text{dyn}} \neq \kappa_{c,\text{quen}}$. The functional dependence of the studied observables on $\kappa$ (respectively $m_q$) in the limit $\kappa \to \kappa_c$ compared to the quenched approximation does not change. Our data suggest, that at $\beta = 0$ the pseudoscalar particle becomes massless when $\kappa \to \kappa_c$.

At $\beta = 0.8$ the presence of the dynamical (‘sea’) fermions drastically change the transition. There we have found a transition which cannot be associated with the zero-mass limit of a pseudoscalar particle anymore, in sharp contrast to the quenched case.

Naively, one would expect that the chiral limit could be established everywhere in the confinement phase when approaching the ‘critical’ line $\kappa_c(\beta)$. This is not the case. Therefore the question arises whether even at $\beta = 0$ the vanishing pseudoscalar mass can be interpreted as a chiral limit. An alternative scenario for the vanishing pseudoscalar mass could be the breakdown of some other symmetry than the chiral one.

Another possible conclusion from these observations is that the calculations – analytical and numerical – in the strong coupling limit $\beta = 0$ can hardly serve for the interpretation of the mechanism of the chiral transition at larger $\beta$’s (at least, in this model).

These statements need further confirmation, especially on larger lattices. Here
work is in progress [10]. It is interesting to what extent the given conclusions can be generalized to the case of QCD.

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Figure captions

**Figure 1:** The pion norm $\langle \Pi \rangle$ vs. $m_q$ at $\beta = 0$. The curved lines correspond
to eq.(8) with $C_0, C_1$ given in Table 1.

**Figure 2:** The unnormalized distribution of $\Pi$ at $\beta = 0.8$ in the vicinity of $\kappa_c$
(a) and $\langle \rho_{mon} \rangle$ in dependence of $\kappa$ at the same value of $\beta$ (b) both for the theory
with dynamical fermions.

**Figure 3:** Counterpart to Figure 1 at $\beta = 0.8$.

**Figure 4:** The $\kappa$–dependence of $m_\pi^2$ for the quenched and dynamical theories at
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**Figure 5:** The behavior of $m_\pi^2$ around $\kappa_c$ at $\beta = 0.8$. The $\kappa$-scale of the smaller
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