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RIMS-1085

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and Suzuki's Quantum Analysis**

By

Mitsuo ABE, Noriaki IKEDA
and
Noboru NAKANISHI

July 1996

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京都大学 数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES

KYOTO UNIVERSITY, Kyoto, Japan

Operator ordering index method for multiple commutators and Suzuki's quantum analysis

Mitsuo Abe^{a)} and Noriaki Ikeda^{b)}

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-01, Japan

Noboru Nakanishi^{c)}

12-20 Asahigaoka-cho, Hirakata 578, Japan

July 1996

ABSTRACT

General formulae are presented for the multiple commutator involving a function of an operator A in terms of (multiple) commutators involving A by using "operator ordering indices". It is pointed out that this analysis essentially reproduces the quantum analysis, proposed by Suzuki very recently, in a more transparent way. Extension to the case of several operators is also made in an elegant form.

^{a)} E-mail: abe@kurims.kyoto-u.ac.jp

^{b)} E-mail: nori@kurims.kyoto-u.ac.jp

^{c)} E-mail: noboru@kurims.kyoto-u.ac.jp

I. INTRODUCTION

Some years ago,^{1,2} we encountered and solved the following problem. Let A and Φ be some operators and $f(z)$ be an analytic function of z . Without assuming $[[A, \Phi], A] = 0$, how can $[f(A), \Phi]$ be expressed in terms of $[A, \Phi]$? A general formula for expressing $[f(A), \Phi]$ in terms of $[A, \Phi]$ was given by introducing two mutually commuting operators A_L and A_R , where the indices L (= left) and R (= right) indicate operator ordering with respect to $[A, \Phi]$. We may call this analysis “operator ordering index method”.

On the other hand, very recently, Suzuki³⁻⁵ has proposed a differential analysis of $f(A)$ with respect to an operator A . His definition of operator derivative is

$$\frac{df(A)}{dA} = \frac{f(A) - f(A - \delta_A)}{\delta_A}, \quad (1.1)$$

where δ_A is a new operator commuting with A . He has made a detailed analysis about the higher derivative $(d/dA)^n f(A)$. He has called his analysis “quantum analysis”.

There is a parallelism between both analyses because both commutator and differentiation are derivations. Suzuki’s δ_A corresponds to our $A_L - A_R$ if we identify A with A_L .

The purpose of the present paper is to extend our operator ordering index method to the multiple commutator. We again find the parallelism between our analysis and Suzuki’s one. But our formulae are more transparent than his corresponding ones. This fact enables us to find the general explicit formulae for both $(d/dA)^n A^k$ ($k > 0$) and $(d/dA)^n A^{-k}$ ($k > 0$), which Suzuki could not give.

The present paper is organized as follows. In Sec. 2, we briefly review our previous analysis. In Sec. 3, we extend our operator ordering index method to the multiple commutator. In Sec. 4, the general explicit formulae are given for the cases of $f(A) = A^k$ ($k > 0$) and A^{-k} ($k > 0$). In Sec. 5, we compare Suzuki’s quantum analysis with ours. In Sec. 6, we extend our analysis to the case of a function of several variables.

II. REVIEW OF OUR PREVIOUS WORK

Quantum field theory is usually solved in the interaction picture. But such an approach is not appropriate for quantum gravity. We have therefore developed a new method for solving quantum field theory in the Heisenberg picture.⁶⁻⁸ For this purpose,

we have set up a Cauchy problem for the full-dimensional commutator between a pair of field operators by using a field equation and equal-time commutation relations. Here, we have encountered a mathematical problem, which may be stated as follows in its simplest version.¹

Let A and Φ be operators and $f(z)$ be an analytic function of z . Express $[f(A), \Phi]$ in terms of $[A, \Phi]$ without assuming the commutativity between $[A, \Phi]$ and A .

We have solved this problem in the following way. Cauchy residue theorem implies

$$\begin{aligned} [f(A), \Phi] &= \frac{1}{2\pi i} \oint dz f(z) \left[\frac{1}{z-A}, \Phi \right] \\ &= \frac{1}{2\pi i} \oint dz f(z) \frac{1}{z-A} [A, \Phi] \frac{1}{z-A}. \end{aligned} \quad (2.1)$$

Substituting the remainder-theorem formula

$$f(z) = \hat{f}^{[1]}(w, z)(z-w) + f(w) \quad (2.2)$$

with $w = A$ into (2.1), we have

$$\begin{aligned} [f(A), \Phi] &= \frac{1}{2\pi i} \oint dz \hat{f}^{[1]}(A, z) [A, \Phi] \frac{1}{z-A} \\ &\quad + \frac{f(A)}{2\pi i} \oint dz \frac{1}{z-A} [A, \Phi] \frac{1}{z-A}. \end{aligned} \quad (2.3)$$

The second term vanishes because of a double pole. In order to calculate the first term explicitly in the general form, we write^{d)}

$$\begin{aligned} A &= A_{(1)} \quad \text{if } A \text{ lies in the left of } [A, \Phi], \\ A &= A_{(2)} \quad \text{if } A \text{ lies in the right of } [A, \Phi]. \end{aligned}$$

We regard $A_{(1)}$ and $A_{(2)}$ as if they were two different operators. But, of course, they are mutually commutable. With this notation, (2.3) becomes

$$[f(A), \Phi] = \hat{f}^{[1]}(A_{(1)}, A_{(2)}) [A, \Phi] \quad (2.4)$$

because of Cauchy residue theorem.

From (2.2), the expression for $\hat{f}^{[1]}$ is

$$\hat{f}^{[1]}(A_{(1)}, A_{(2)}) = \frac{f(A_{(1)}) - f(A_{(2)})}{A_{(1)} - A_{(2)}}. \quad (2.5)$$

^{d)} We use $A_{(1)}$ and $A_{(2)}$ instead of A_L and A_R , respectively.

It is important to note that this function is *symmetric* in $A_{(1)}$ and $A_{(2)}$. If we set $A_{(1)} = A$, $A_{(1)} - A_{(2)} = \delta_A$, (2.5) becomes (1.1), but this symmetry is implicit in the latter.

III. MULTIPLE COMMUTATORS

It is rather straightforward to extend our method to the multiple commutator. For example, for a double commutator, our analysis becomes as follows.

Since

$$\begin{aligned} \left[\left[\frac{1}{z-A}, \Phi_1 \right], \Phi_2 \right] &= \left[\frac{1}{z-A} [A, \Phi_1] \frac{1}{z-A}, \Phi_2 \right] \\ &= \frac{1}{z-A} [[A, \Phi_1], \Phi_2] \frac{1}{z-A} \\ &\quad + \frac{1}{z-A} [A, \Phi_1] \frac{1}{z-A} [A, \Phi_2] \frac{1}{z-A} \\ &\quad + \frac{1}{z-A} [A, \Phi_2] \frac{1}{z-A} [A, \Phi_1] \frac{1}{z-A}, \end{aligned} \quad (3.1)$$

we have

$$\begin{aligned} [[f(A), \Phi_1], \Phi_2] &= \hat{f}^{[1]}(A_{(1)}, A_{(2)}) [[A, \Phi_1], \Phi_2] \\ &\quad + \left[\frac{1}{2\pi i} \oint dz \hat{f}^{[1]}(A, z) [A, \Phi_1] \frac{1}{z-A} [A, \Phi_2] \frac{1}{z-A} + (\Phi_1 \leftrightarrow \Phi_2) \right]. \end{aligned} \quad (3.2)$$

We rewrite the second term of (3.2) by using

$$\hat{f}^{[1]}(w_1, z) = \hat{f}^{[2]}(w_1, w_2, z)(z - w_2) + \hat{f}^{[1]}(w_1, w_2) \quad (3.3)$$

as follows:

$$\begin{aligned} \frac{1}{2\pi i} \oint dz \hat{f}^{[2]}(A_{(1)}, A_{(2)}, z) [A, \Phi_1] [A, \Phi_2] \frac{1}{z-A} \\ + \frac{\hat{f}^{[1]}(A_{(1)}, A_{(2)})}{2\pi i} \oint dz [A, \Phi_1] \frac{1}{z-A} [A, \Phi_2] \frac{1}{z-A} \\ = \hat{f}^{[2]}(A_{(1)}, A_{(2)}, A_{(3)}) [A, \Phi_1] [A, \Phi_2]. \end{aligned} \quad (3.4)$$

Here, as before, the subscript in the parentheses indicates operator ordering: $A_{(1)}$, $A_{(2)}$ and $A_{(3)}$ are mutually commutable operators; $A_{(1)}$ lies in the left of the first commutator, $A_{(2)}$ lies in between the first one and the second one, and $A_{(3)}$ lies in the right of the second one.

From (3.3) and (2.5), we obtain

$$\begin{aligned}\hat{f}^{[2]}(A_{(1)}, A_{(2)}, A_{(3)}) &= \frac{\hat{f}^{[1]}(A_{(1)}, A_{(2)}) - \hat{f}^{[1]}(A_{(1)}, A_{(3)})}{A_{(2)} - A_{(3)}} \\ &= \sum_{i=1}^3 \frac{f(A_{(i)})}{(A_{(i)} - A_{(j)})(A_{(i)} - A_{(k)})},\end{aligned}\quad (3.5)$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. It should be noted that (3.5) is *totally symmetric* in $A_{(1)}$, $A_{(2)}$ and $A_{(3)}$.

It is now straightforward to show by mathematical induction that the n -th commutator can be expressed as follows:

$$\begin{aligned}& [\cdots [f(A), \Phi_1], \Phi_2], \cdots, \Phi_n] \\ &= \hat{f}^{[1]}(A_{(1)}, A_{(2)}) [\cdots [A, \Phi_1], \Phi_2], \cdots, \Phi_n] \\ &+ \cdots \\ &+ \hat{f}^{[m]}(A_{(1)}, \cdots, A_{(m)}) \sum_{\text{all}} [m \text{ factors of admissible (multiple) commutators}] \\ &+ \cdots \\ &+ \hat{f}^{[n]}(A_{(1)}, \cdots, A_{(n)}) \sum_{\substack{\text{all permutations} \\ n!}} [A, \Phi_{i_1}] \cdots [A, \Phi_{i_n}],\end{aligned}\quad (3.6)$$

where "admissible multiple commutator" is a multiple commutator $[\cdots [A, \Phi_{i_1}], \Phi_{i_2}], \cdots, \Phi_{i_s}]$ such that $i_1 < i_2 < \cdots < i_s$.

The function $\hat{f}^{[n]}$ is recursively defined by

$$\begin{aligned}\hat{f}^{[n]}(A_{(1)}, \cdots, A_{(n+1)}) \\ = \frac{\hat{f}^{[n-1]}(A_{(1)}, \cdots, A_{(n)}) - \hat{f}^{[n-1]}(A_{(1)}, \cdots, A_{(n-1)}, A_{(n+1)})}{A_{(n)} - A_{(n+1)}}.\end{aligned}\quad (3.7)$$

It is a *totally symmetric* function, as is seen from its explicit expression

$$\hat{f}^{[n]}(A_{(1)}, \cdots, A_{(n+1)}) = \sum_{i=1}^{n+1} \frac{f(A_{(i)})}{\prod_{m=1}^n (A_{(i)} - A_{(j_m)})},\quad (3.8)$$

where $(i, j_1, \cdots, j_n)$ is a cyclic permutation of $(1, 2, \cdots, n+1)$.

The formula (3.8) is not convenient because it has operators in the denominators. It is more convenient to introduce the Feynman parametrization in the following way.

We first note that (2.5) is rewritten as

$$\hat{f}^{[1]}(A_{(1)}, A_{(2)}) = \int_0^1 d\alpha f'(\alpha A_{(1)} + (1 - \alpha)A_{(2)}),\quad (3.9)$$

where f' denotes the first derivative of f . It is easy to show by mathematical induction that

$$\hat{f}^{[n]}(A_{(1)}, \dots, A_{(n+1)}) = \int_{\Delta} d^n \alpha f^{(n)}\left(\sum_{i=1}^{n+1} \alpha_i A_{(i)}\right), \quad (3.10)$$

where $f^{(n)}$ denotes the n -th derivative of f and

$$\int_{\Delta} d^n \alpha [*] \equiv \int_0^1 d\alpha_1 \cdots \int_0^1 d\alpha_{n+1} \delta\left(1 - \sum_{i=1}^{n+1} \alpha_i\right) [*]. \quad (3.11)$$

The formula (3.10) is also *manifestly totally symmetric* in $A_{(1)}, \dots, A_{(n+1)}$.

IV. SPECIAL CASES

The following generalized beta-function formula is well known:⁹

$$\int_{\Delta} d^n \alpha \frac{\prod_{i=1}^{n+1} \alpha_i^{\lambda_i - 1}}{\left(\sum_{i=1}^{n+1} \alpha_i w_i\right)^{\lambda}} = \frac{\prod_{i=1}^{n+1} \Gamma(\lambda_i)}{\Gamma(\lambda)} \cdot \frac{1}{\prod_{i=1}^{n+1} w_i^{\lambda_i}}, \quad (4.1)$$

with $\lambda \equiv \sum_{i=1}^{n+1} \lambda_i$. By using this formula, we can calculate $\hat{f}^{[n]}$ explicitly in some special cases.

First, we consider the case of

$$f(z) = z^k \quad (k > 0). \quad (4.2)$$

Evidently, $(z^{\hat{k}})^{[n]} \equiv 0$ for $n > k$. Hence we assume $n \leq k$. From (3.10), we have

$$\begin{aligned} (z^{\hat{k}})^{[n]}(A_{(1)}, \dots, A_{(n+1)}) &= \frac{k!}{(k-n)!} \int_{\Delta} d^n \alpha \left(\sum_{i=1}^{n+1} \alpha_i A_{(i)}\right)^{k-n} \\ &= k! \int_{\Delta} d^n \alpha \sum_{\substack{k_1 \geq 0, \dots, k_{n+1} \geq 0 \\ \sum_i k_i = k-n}} \prod_{i=1}^{n+1} \frac{(\alpha_i A_{(i)})^{k_i}}{k_i!} \\ &= \sum a_{k_1, \dots, k_{n+1}} \prod_{i=1}^{n+1} A_{(i)}^{k_i}, \end{aligned} \quad (4.3)$$

where

$$a_{k_1, \dots, k_{n+1}} \equiv \frac{k!}{\prod_{i=1}^{n+1} k_i!} \int_{\Delta} d^n \alpha \prod_{i=1}^{n+1} \alpha_i^{k_i}. \quad (4.4)$$

Setting $w_i = 1$, $\lambda_i = k_i + 1$ in (4.1), we obtain

$$a_{k_1, \dots, k_{n+1}} = \frac{k!}{\prod_{i=1}^{n+1} k_i!} \cdot \frac{\prod_{i=1}^{n+1} \Gamma(k_i + 1)}{\Gamma(\sum_{i=1}^{n+1} (k_i + 1))}. \quad (4.5)$$

Since

$$\sum_{i=1}^{n+1} (k_i + 1) = (k - n) + (n + 1) = k + 1, \quad (4.6)$$

(4.5) reduces to $a_{k_1, \dots, k_{n+1}} = 1$. We thus find

$$(\widehat{z^k})^{[n]}(A_{(1)}, \dots, A_{(n+1)}) = \sum_{\substack{k_1 \geq 0, \dots, k_{n+1} \geq 0 \\ \sum k_i = k - n}}^N \prod_{i=1}^{n+1} A_{(i)}^{k_i}, \quad (4.7)$$

where the number N of the terms in (4.7) is calculated by setting $A_{(i)} = 1$ in the first line of (4.3) and in (4.7):

$$N = \frac{k!}{(k - n)!n!}. \quad (4.8)$$

Next, we consider the case

$$f(z) = z^{-k} \quad (k > 0). \quad (4.9)$$

From (3.10), we have

$$(\widehat{z^{-k}})^{[n]}(A_{(1)}, \dots, A_{(n+1)}) = (-1)^n \frac{(k + n - 1)!}{(k - 1)!} \int_{\Delta} d^n \alpha \frac{1}{\left(\sum_{i=1}^{n+1} \alpha_i A_{(i)}\right)^{k+n}}. \quad (4.10)$$

Inserting $1 = \left(\sum_{i=1}^{n+1} \alpha_i\right)^{k-1}$ into the integrand of (4.10), we obtain

$$\begin{aligned} & (\widehat{z^{-k}})^{[n]}(A_{(1)}, \dots, A_{(n+1)}) \\ &= (-1)^n (k + n - 1)! \sum_{\substack{k_1 \geq 0, \dots, k_{n+1} \geq 0 \\ \sum k_i = k - 1}} \frac{1}{\prod_{i=1}^{n+1} k_i!} \int_{\Delta} d^n \alpha \frac{\prod_{i=1}^{n+1} \alpha_i^{k_i}}{\left(\sum_{i=1}^{n+1} \alpha_i A_{(i)}\right)^{k+n}}. \end{aligned} \quad (4.11)$$

Setting $w_i = A_{(i)}$, $\lambda_i = k_i + 1$ in (4.1), we find

$$(\widehat{z^{-k}})^{[n]}(A_{(1)}, \dots, A_{(n+1)}) = (-1)^n \sum_{\substack{k_1 \geq 0, \dots, k_{n+1} \geq 0 \\ \sum k_i = k - 1}}^{N'} \frac{1}{\prod_{i=1}^{n+1} A_{(i)}^{k_i + 1}}. \quad (4.12)$$

The number N' of the terms in (4.12) is calculated by setting $A_{(i)} = 1$ in (4.10) and in (4.12):

$$N' = \frac{(k+n-1)!}{(k-1)!n!}. \quad (4.13)$$

The formula (4.7) can also be derived in the following way. To calculate $\hat{f}^{[n]}$, it is unnecessary to distinguish $\Phi_1, \dots, \Phi_n$. Hence we set $\Phi \equiv \Phi_1 = \dots = \Phi_n$. Then there is a generating function of multiple commutators:

$$e^{-t\Phi} f(A) e^{t\Phi} = \sum_{n=0}^{\infty} \frac{t^n}{n!} [\dots [f(A), \Phi]_1, \Phi]_2, \dots, \Phi]_n, \quad (4.14)$$

where subscripts of square brackets are written for sake of indicating the commutator multiplicity. Accordingly,

$$\begin{aligned} [\dots [A^k, \Phi]_1, \Phi]_2, \dots, \Phi]_n &= \left(\frac{d}{dt} \right)^n (e^{-t\Phi} A^k e^{t\Phi}) \Big|_{t=0} \\ &= \left(\frac{d}{dt} \right)^n (e^{-t\Phi} A e^{t\Phi})^k \Big|_{t=0} \\ &= \left(\frac{d}{dt} \right)^n \left(\sum_{m=0}^{\infty} \frac{t^m}{m!} [\dots [A, \Phi]_1, \Phi]_2, \dots, \Phi]_m \right)^k \Big|_{t=0} \\ &= \sum_{\substack{m_1 \geq 0, \dots, m_k \geq 0 \\ \sum_j m_j = n}} \frac{n!}{\prod_j m_j!} \prod_{j=1}^k [\dots [A, \Phi]_1, \Phi]_2, \dots, \Phi]_{m_j}. \end{aligned} \quad (4.15)$$

We can calculate $n!(\hat{z}^k)^{[n]}$ by taking out the terms in which $m_j = 0$ or 1 for all j 's, that is, in which

$$\begin{aligned} m_{k_1+1} &= m_{k_1+k_2+2} = \dots = m_{k_1+\dots+k_n+n} = 1, \\ m_j &= 0 \quad \text{otherwise,} \end{aligned} \quad (4.16)$$

where $k_1 + \dots + k_{n+1} = k - n$ [see (4.6)]. Hence we obtain (4.7).

Finally, we note that if $f(A)$ is expandable into a Taylor or Laurent series, we can write down the explicit formula of $\hat{f}^{[n]}(A_{(1)}, \dots, A_{(n+1)})$ by using (4.7) or [(4.7) and (4.12)].

V. COMPARISON WITH SUZUKI'S QUANTUM ANALYSIS

Suzuki³⁻⁵ starts with the Gâteaux derivation¹⁰

$$df(A) = \lim_{h \rightarrow 0} \frac{f(A + h dA) - f(A)}{h}, \quad (5.1)$$

where h is a c -number. Since $[A, dA] \neq 0$ in general, he introduces a special operator (inner derivation) δ_A , which is defined by

$$\delta_A \cdot Q \equiv [A, Q] \quad (5.2)$$

for any operator Q and by

$$[A, \delta_A] = 0. \quad (5.3)$$

With δ_A , (5.1) is rewritten into the form of

$$df(A) = f_1(A, \delta_A) \cdot dA. \quad (5.4)$$

He then defines the quantum derivative of $f(A)$ by

$$\frac{df(A)}{dA} = f_1(A, \delta_A). \quad (5.5)$$

The above consideration is extended to the n -th order derivative. One may write

$$d^n f(A) = \lim_{h \rightarrow 0} \sum_{j=0}^n \frac{(-1)^j}{h^n} \binom{n}{j} f(A + j h dA). \quad (5.6)$$

Generalizing δ_A to δ_j ($j = 1, 2, \dots, n$), which is defined by

$$\delta_j(dA)^n = (dA)^{j-1}(\delta_A dA)(dA)^{n-j}, \quad (5.7)$$

he rewrite (5.6) into the form of

$$d^n f(A) = f_n(A, \delta_1, \dots, \delta_n)(dA)^n \quad (5.8)$$

and defines

$$\frac{d^n f(A)}{dA^n} = f_n(A, \delta_1, \dots, \delta_n). \quad (5.9)$$

As stated in the Introduction, both differentiation and commutator are derivations. Suzuki's formula (5.9) is reproduced from our formula (3.6) by setting $\Phi \equiv \Phi_1 = \dots = \Phi_n$ and

$$[[A, \Phi], \Phi] = 0, \quad (5.10)$$

which corresponds to $d(dA) = 0$. With (5.10), (3.6) reduces to

$$[\cdots [[f(A), \Phi]_1, \Phi]_2, \cdots, \Phi]_n = n! \hat{f}^{[n]}(A_{(1)}, \cdots, A_{(n+1)}) [A, \Phi]^n. \quad (5.11)$$

Two formulae (5.8) and (5.11) have the following complete correspondence:

$$\begin{aligned} dA &\iff [A, \Phi], \\ d^n f(A) &\iff [\cdots [[f(A), \Phi]_1, \Phi]_2, \cdots, \Phi]_n, \\ f_n(A, \delta_1, \cdots, \delta_n) &\iff n! \hat{f}^{[n]}(A_{(1)}, \cdots, A_{(n+1)}) \end{aligned} \quad (5.12)$$

with

$$\begin{aligned} A &\iff A_{(1)}, \\ \delta_j &\iff A_{(j)} - A_{(j+1)} \quad (j = 1, 2, \cdots, n). \end{aligned} \quad (5.13)$$

Thus f_n and $n! \hat{f}^{[n]}$ are equivalent.

There is, however, an important difference between Suzuki's analysis and ours. As emphasized in Sec. 3, $\hat{f}^{[n]}$ is *totally symmetric* in $A_{(1)}, \cdots, A_{(n+1)}$. On the other hand, f_n has no such manifest symmetry in $A, \delta_1, \cdots, \delta_n$. Indeed, his expression corresponding to (3.8) is very complicated.

Another difference is that no limiting procedure is employed in our analysis. This enable us to avoid any consideration on the norm, which is cumbersome mathematical procedure.

VI. EXTENSION TO THE SEVERAL-VARIABLE CASE

So far, we have restricted our consideration to a single-variable function $f(A)$. Practically, we need to discuss the case of a several-variable function $f(A_1, \cdots, A_m)$, where $[A_i, A_j] = 0$. We can generally reduce the latter case to the former one in the following way:

$$\begin{aligned} f(A_1, \cdots, A_m) &= \frac{1}{(2\pi i)^m} \oint dz_1 \cdots \oint dz_m \frac{f(z_1, \cdots, z_m)}{\prod_{j=1}^m (z_j - A_j)} \\ &= \frac{1}{(2\pi i)^m} \oint dz_1 \cdots \oint dz_m f(z_1, \cdots, z_m) \\ &\quad \cdot (m-1)! \int_{\Delta} d^{m-1} \beta \frac{1}{\left[\sum_{j=1}^m \beta_j (z_j - A_j) \right]^m} \\ &= \int_{\Delta} d^{m-1} \beta F(A; \beta_1, \cdots, \beta_m), \end{aligned} \quad (6.1)$$

where F is a function of a *single* variable

$$A \equiv \sum_{j=1}^m \beta_j A_j, \quad (6.2)$$

defined by

$$F(A; \beta_1, \dots, \beta_m) \equiv \frac{(m-1)!}{(2\pi i)^m} \oint dz_1 \dots \oint dz_m \frac{f(z_1, \dots, z_m)}{\left(\sum_{j=1}^m \beta_j z_j - A\right)^m}. \quad (6.3)$$

Now, we calculate the multiple commutator by substituting (6.1) into (3.6). Without loss of generality, we may confine our consideration to its last term:

$$\int_{\Delta} d^{m-1} \beta \hat{F}^{[n]}(A_{(1)}, \dots, A_{(n+1)}; \beta_1, \dots, \beta_m) \sum_{(i_1, \dots, i_n)} [A, \Phi_{i_1}] \dots [A, \Phi_{i_n}] \quad (6.4)$$

with (6.2). For simplicity, we set $\Phi \equiv \Phi_1 = \dots = \Phi_n$. Since $[A, \Phi] = \sum_j \beta_j [A_j, \Phi]$, (6.4) is rewritten as

$$n! \sum_{j_1=1}^m \dots \sum_{j_n=1}^m \hat{f}^{[n_1, \dots, n_m]}(A_{1(j_1)}, \dots; \dots; A_{m(j_n)} \dots) \prod_{k=1}^n [A_{j_k}, \Phi], \quad (6.5)$$

where n_j denotes the number of j_k 's such that $j_k = j$ and

$$\hat{f}^{[n_1, \dots, n_m]} \equiv \int_{\Delta} d^{m-1} \beta \prod_{j=1}^m \beta_j^{n_j} \cdot \hat{F}^{[n]}(A_{(1)}, \dots, A_{(n+1)}; \beta_1, \dots, \beta_m). \quad (6.6)$$

Substituting (6.3) into (6.6) and using (3.10), we have

$$\begin{aligned} \hat{f}^{[n_1, \dots, n_m]} &= \frac{(m+n-1)!}{(2\pi i)^m} \int_{\Delta} d^n \alpha \oint dz_1 \dots \oint dz_m f(z_1, \dots, z_m) \\ &\quad \cdot \int_{\Delta} d^{m-1} \beta \frac{\prod_{j=1}^m \beta_j^{n_j}}{\left[\sum_{j=1}^m \beta_j \left(z_j - \sum_{i=1}^{n+1} \alpha_i A_{j(i)}\right)\right]^{m+n}}. \end{aligned} \quad (6.7)$$

The β integration is carried out by means of (4.1). We find

$$\begin{aligned} \hat{f}^{[n_1, \dots, n_m]} &= \frac{\prod_{j=1}^m n_j!}{(2\pi i)^m} \int_{\Delta} d^n \alpha \oint dz_1 \dots \oint dz_m \frac{f(z_1, \dots, z_m)}{\prod_{j=1}^m \left[z_j - \sum_{i=1}^{n+1} \alpha_i A_{j(i)}\right]^{n_j+1}} \\ &= \int_{\Delta} d^n \alpha f^{(n_1, \dots, n_m)} \left(\sum_{i=1}^{n+1} \alpha_i A_{1(i)}, \dots, \sum_{i=1}^{n+1} \alpha_i A_{m(i)}\right), \end{aligned} \quad (6.8)$$

where $f^{(n_1, \dots, n_m)}(z_1, \dots, z_m) \equiv \partial_{z_1}^{n_1} \dots \partial_{z_m}^{n_m} f(z_1, \dots, z_m)$.

This formula is a natural generalization of (3.10) to the case of several variables, that is, the correspondence

$$\left(\frac{\partial}{\partial A_1}\right)^{n_1} \dots \left(\frac{\partial}{\partial A_m}\right)^{n_m} f \iff n! \hat{f}^{[n_1, \dots, n_m]} \quad (6.9)$$

holds.

ACKNOWLEDGMENTS

One of the authors (N. N.) would like to thank Professor M. Suzuki for informing N. N. of M. S.'s work prior to publication.

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